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ON SOME OPEN PROBLEMS CONNECTED WITH THE DISCONTINUITY OF CLOSED AND DARBOUX FUNCTIONS

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ABSTRACT. In this paper we prove some theorems and signal some open problems connected with the possibility of the construction of discontinuous and closed Darboux function.

Klee V. L. and Utz W. R. in [6] showed that if $f : X \to Y$ (where X and Y are arbitrary metric spaces) is connected and compact and x_0 is a local connectedness point of X, then x_0 is a point of continuity of f. Generalizations of this result are contained in papers [2, 11, 12]. T. R. Hamlett in paper [3] proved that if f is a closed, connected and monotone transformation defined on a T_3-space, assuming its values in some compact T_1-space, then f is continuous. H. Pawlak in [8] showed that if $f : \mathcal{R}^n \to \mathcal{R}^m$ is a closed function, then f is continuous if and only if the image of each segment is a connected set. Similar problems are investigated in [9] and [10]. Some results which are connected with the continuity of connected (Darboux) functions defined and assuming their values in some topological spaces are contained in [5].

Questions connected with the possibility of constructing discontinuous and closed Darboux functions arise from the problems discussed in the above papers. These problems will be the object of considerations in the present paper. Unfortunately, not all the questions that can be formulated when this subject

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is analyzed can be answered by us. Therefore, besides the theorems, we shall signal open problems whose solutions can constitute a very interesting complement to mathematicians' knowledge of the continuity (and discontinuity) of Darboux transformations.

We shall use the standard notions and notations. In particular, by $\mathcal{I}$ we denote the unit interval with the natural topology $\mathcal{T}_0$, and by $\mathcal{N}$ we shall denote the set of all positive integers.

We say that $f : X \to Y$ (where X and Y are arbitrary topological spaces) is a *Darboux (closed) function* if $f(C)$ is connected (closed) for each connected (closed) set $C \subset X$.

We say that $f : X \to Y$ is *nowhere constant* at x_0 if, for any neighbourhood V of x_0, $f(V)$ contains at least two elements.

The closure (in the space X) and cardinality of a set A will be denoted by $cl_X(A)$ and $card(A)$, respectively.

The smallest cardinal number $\mathbf{m}$ such that each open cover of a space X has an open refinement of cardinality $\leq\mathbf{m}$ is called the *Lindelöf number* of the space X and is denoted by $l(X)$.

A topological space X is called *paracompact* if X is a Hausdorff space and each open cover of X has a locally finite open refinement.

Let X be a connected topological space. We say that a is an *exploding point* of X relative to $x \in X$ if $\{x\}$ is a component of $X \setminus \{a\}$ and there exist open sets U and V such that $x \in U$, $a \in V$ and $U \cap V = \emptyset$.

A pair (Y, ξ), where Y is a topological space and $\xi : X \to Y$ is a homeomorphic embedding of X in Y such that $\xi(X)$ is a dense set in Y, is said an *extension* of the space X. In the sequel, by an extension of X we shall mean not only the pair (Y, ξ) but also the topological space Y in which X can be embedded as a dense subspace. Let Y be an extension of X; the set $Y \setminus \xi(X)$ is called the *remainder* of the extension Y.

Let X_1,Y_1 be subspaces of the topological spaces X and Y, respectively, and let $f_1 : X_1 \to Y_1$. We say that a function $f : X \to Y$ is a *d- extension* of f_1 if $f_1(x) = f(x)$ for each $x \in X_1$.

The results included in the papers cited before imply that each closed Darboux function $f : \mathcal{I} \to \mathcal{I}$ is continuous. The theorem below will show that the natural topology of the segment $[0,1]$ can be extended to a topology T so that there exist discontinuous closed functions $g : \mathcal{I} \to ([0,1],T)$ which are Darboux functions.

Theorem 1. *There exists a topology T defined in $[0,1]$ finer than the natural topology of a line $\mathcal{T}_0$ such that $([0,1],T)$ is a connected and Hausdorff space which possesses the Lindelöf number $l([0,1],T) = \aleph_0$, such that any closed and Darboux function $f : \mathcal{I} \to \mathcal{I}$ considered as a function $f : \mathcal{I} \to ([0,1],T)$ is closed, Darboux and discontinuous at each point where it is nowhere constant.*
One can choose the topology T in the way that any closed Darboux function $g : \mathcal{I} \to ([0,1],T)$, such that $g(\mathcal{I}) = [0,1]$, is discontinuous.

Proof: Let T be the topology of type of Hashimoto ([4]) generated by the base:

$$\boldsymbol{B} = \{U \setminus A : U \in \mathcal{T}_0 \wedge card(A) \leq \aleph_0\}.$$

Of course, T is finer than $\mathcal{T}_0$, and so, $([0,1],T)$ is a Hausdorff space.

Infer now that:

each interval $K \subset [0,1]$ is a connected subset of the space $([0,1],T)$.

From the above we have:

- $([0,1]),T)$ is a connected space,
- each Darboux function $f : \mathcal{I} \to \mathcal{I}$ considered as a function assuming its values in $([0,1],T)$ is a Darboux function, too.

We shall now show that

$$(1) \qquad l([0,1],T) = \aleph_0.$$

So let $R = \{V_s\}_{s \in S}$ be a family consisting of sets from T such that $[0,1] = \bigcup_{s \in S} V_s$. Then, for any $s \in S$, there exist families of sets $\{U_h\}_{h\in H_s} \in \mathcal{T}_0$ and $\{A_h\}_{h\in H_s}$ such that $card(A_h) \le \aleph_0$ for $h \in H_s$ and $V_s = \bigcup_{h\in H_s}(U_h \setminus A_h)$ $(s \in S)$. From the compactness of $\mathcal{I}$ we infer that there exist $h_i \in H_{s_i}$ $(i = 1,\dots,n)$ such that

$$(2) \qquad \bigcup_{i=1}^{n} U_{h_i} = [0,1].$$

The countable sets A_{h_i} are paired with the sets $U_{h_i}(i = 1,\dots,n)$.

Remark that $U_{h_i} \setminus A_{h_i} \subset V_{s_i} \in R$ $(i = 1,\dots,n)$, which, according to (2), means that

$$(3) \qquad card([0,1] \setminus \bigcup_{i=1}^{n} V_{s_i}) \le \aleph_0.$$

From the last observation and the fact that R covers $[0,1]$ it follows that there exists a sequence $\{V_{s_k}\}_{k=1}^{\infty}$ such that $[0,1] \setminus \bigcup_{i=1}^{n} V_{s_i} \subset \bigcup_{k=1}^{\infty} V_{s_k}$. This means that $l([0,1],T) \le \aleph_0$. On the other hand, the family $\{[0,1] \setminus \{\frac{1}{2n} : n = 1,2,\dots\} \cup \{(\frac{1}{2n+1}, \frac{1}{2n-1}) : n = 1,2,\dots\} \subset T$ is a cover of $[0,1]$ and does not contain a finite subcover; consequently, equality (1) takes place.

Let now $f : \mathcal{I} \to \mathcal{I}$ be an arbitrary closed Darboux function (so, f is a continuous and closed function).

Hence, by the above observations, it is easy to see that $f : \mathcal{I} \to ([0,1],T)$ is a closed Darboux function. So, suppose that $x_0 \in [0,1]$ is a point such that f is nowhere constant at x_0. This means that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = x_0$ and $f(x_n) \neq f(x_0)$ (for any $n = 1,2,\dots$). Consider the set $V = [0,1] \setminus \{f(x_n) : n = 1,2,\dots\}$. Thus V is a $T-$neighbourhood of $f(x_0)$ such that, for any $\mathcal{T}_0$- neighbourhood U of x_0, $f(U) \setminus V \neq \emptyset$. This remark proves that $f : \mathcal{I} \to ([0,1],T)$ is discontinuous at x_0.

Now we shall show the second part of this theorem. First, we remark that $([0,1],T)$ is not regular. Indeed, if $A = \{\frac{1}{n} : n = 1,2,\dots\}$, then A is a $T-$closed set and $0 \notin A$. It is not difficult to see that, for any $T-$open sets U and V such that $0 \in U$ and $A \subset V$, we have $U \cap V \neq \emptyset$. The above fact proves that, by the well-known theorem of Michael [1] ([7]), the second part of the theorem is true.

Note first that this theorem cannot be strengthened by replacing the equality $l([0,1],T) = \aleph_0$ with the inequality $l([0,1],T) < \aleph_0$. For then $([0,1],T)$ would be a compact space, and thus ([1], Corollary 3. 1. 14, p. 169) $T = \mathcal{T}_0$. However, the following problems remain unsolved:

Problem 1. *Can one introduce in the interval* $[0,1]$ *a topology* T *(finer than the natural topology of the segment) in such a way that* $([0,1],T)$ *is a topological space "close to a compact space" (for example, paracompact) such that any closed Darboux function* $f : \mathcal{I} \to \mathcal{I}$ *considered as a function* $f : \mathcal{I} \to ([0,1],T)$ *is a closed and discontinuous Darboux function?*

One can also raise the question in a somewhat weaker version:

Problem 2. *Can one find, for any closed Darboux function* $f : \mathcal{I} \to \mathcal{I}$*, a topology* T *such that* $([0,1],T)$ *is a space "close to a compact space" (for example, paracompact) such that* $f : \mathcal{I} \to ([0,1],T)$ *is a closed and discontinuous Darboux function?*

Problem 3. *Is it possible to define a topology* T *such that* $([0,1],T)$ *is a* T_i*−space for* $i > 2$ *and a theorem analogous to Theorem 1 holds (with the eventual omission of the requirement that* $l([0,1],T) = \aleph_0$*)?*

One can also formulate a very general question:

Problem 4. *Do there exist a paracompact (or normal) space* X *and a closed and discontinuous Darboux function* $f : \mathcal{I} \to X$*?*

[1] Paracompact is an invariant of closed and continuous functions.

In the case when the domain of the transformations considered is not equal to $\mathcal{I}$, the above question is answered in the affirmative, which is stated in Theorem 3.

Previously, we were dealing with the situation where the functions under consideration were defined on the unit interval $\mathcal{I}$. At present, we shall examine the case when the transformations considered take values in $\mathcal{I}$.

Theorem 2. *Let X be a continuum having an extension X^* with a one-point remainder $\{x_0\}$, such that X^* has an exploding point with respect to x_0. Then there exists a closed Darboux function $f : X^* \to \mathcal{I}$ which is discontinuous at x_0.*

Proof:. By the letter a we denote an exploding point of X^* with respect to x_0. Then, let U be a neighbourhood in X^* of x_0 such that $a \notin cl_{X^*}(U)$. Denote $U^* = (cl_{X^*}U) \setminus \{x_0\}$. Let ξ be a homeomorphic embedding of X in X^*. Thus the set $F = \xi^{-1}(U^*) = \xi^{-1}\left(cl_{\xi(X)}(U)\right)$ is closed in X. Moreover, let $x \in \xi^{-1}(a)$. Of course, $x \notin F$. There exists ([1],Theorem 3.1.9) a continuous function $\eta : X \to \mathcal{I}$ such that $\eta(x) = 0$ and $\eta(z) = 1$ for $z \in F$. Define $f : X^* \to \mathcal{I}$ in the following way:

$$f(t) = \begin{cases} \eta(\xi^{-1}(t)) & if \quad t \neq x_0, \\ 0 & if \quad t = x_0. \end{cases}$$

It is not difficult to see that

$$f_{|\xi(X)} \textit{ is continuous} \tag{4}$$

Now, we shall show that f is a closed function.

Indeed. Let K be a closed set in X^*. Then $K \cap \xi(X)$ is a compact subset of the space $\xi(X)$, so the set $f(K \cap \xi(X)) = f_{|\xi(X)}(K \cap \xi(X))$ is, according to (4), compact in $\mathcal{I}$. The above fact proves that the set

$$f(K) = \begin{cases} f(K \cap \xi(X)) & if \quad x_0 \notin K, \\ f(K \cap \xi(X)) \cup \{0\} & if \quad x_0 \in K \end{cases}$$

is closed in $\mathcal{I}$.

Now, we shall show that f is a Darboux function.

Indeed. Let C be a connected subset of X^*. Consider two cases:

1^o. $x_0 \notin C$. Therefore C is a connected subset of the subspace $\xi(X)$; so, by (4), $f(C) = f_{|\xi(X)}(C)$ is a connected set.

2^o. $x_0 \in C$. Then either $C = \{x_0\}$ (and so, $f(C) = \{0\}$ is connected) or $a \in C$. Of course, we consider only the case when C is different from a singleton. Thus there exists an element $p \in U^* \cap C$. Therefore $f(p) = 1$.

We shall show that

$$f(C) = [0,1]. \tag{5}$$

Conversely, assume that there exists $\alpha \in (0,1)$ such that $f^{-1}(\alpha) \cap C = \emptyset$. Put $A = \{x \in C \setminus \{x_0\} : f(x) < \alpha\}$, $B_1 = \{x \in C \setminus \{x_0\} : f(x) > \alpha\}$ and $B = B_1 \cup \{x_0\}$. Then $C = A \cup B$ where A and B are nonempty separated sets in X^*, which contradicts the connectedness of C. The contradiction obtained proves (5), and thus the fact that f is a Darboux function.

Of course, f is discontinuous at x_0.

Note that the assumption of the compactness of the space X (used in the proof essentially) does not allow one to obtain an extension of X^* being a Hausdorff space. Indeed, if X^* were a T_2-space, then $\xi(X)$ would be a closed subset of the space X^*, thus $\xi(X)$ would not be a dense set in X^*, which contradicts the supposition that X^* is an extension of X.

The following questions arise from this observation.

Problem 5. *Under what thypothesis (weaker than compactness) concerning a space X can one prove a theorem analogous to Theorem 2?*

Problem 6. *Assume that X is a connected and locally compact space. Under what additional assumptions concerning a space X does there exist a connected Alexandroff compactification X^* such that a theorem analogous to Theorem 2 holds?*

The next problem we are going to consider in this paper concerns the possibility of a d-extension of a homeomorphism to a closed and discontinuous Darboux transformation.
At present, we shall prove the following theorem:

Theorem 3. *Let X be a non-singleton, locally connected metrizable continuum and let $x_0 \in X$. Then there exist a locally connected continuum X_1 and a locally connected and connected paracompact space X_2, such that X is a subspace of X_1 and X_2 and, for any homeomorphism $h : X \to X$, there exists a d-extension $h^* : X_1 \to X_2$ of h such that h^* is a closed and Darboux function discontinuous at x_0.*

Proof:. Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence of elements of X such that $x_n \neq x_0$ $(n \in \mathcal{N})$ and $\lim_{n\to\infty} x_n = x_0$.

Put $X_1 = X \cup (\{x_n : n \in \mathcal{N}\} \times (0,1])$. Define the topology T_1 in X_1 generated by the neighbourhood system:

$$B_1(x) = \{\{x_n\} \times ((x' - \tfrac{1}{m}, x' + \tfrac{1}{m}) \cap (0,1]) : m \in \mathcal{N}\}$$
$$\text{if } x = (x_n, x') \in \{x_n\} \times (0,1] \textit{ for some } n \in \mathcal{N},$$
$$B_1(x) = \{K(x, \tfrac{1}{m}) \cup \textstyle\bigcup_{x_n \in K(x,\frac{1}{m})}(\{x_n\} \times (0, \tfrac{1}{m})) : m \geq m_x\} \text{ if }$$
$$x \in X \setminus \{x_0\},$$
$$B_1(x) = \{K(x, \tfrac{1}{m}) \cup \textstyle\bigcup_{x_n \in K(x,\frac{1}{m})}(\{x_n\} \times (0,1]) : m \in \mathcal{N}\} \text{ if }$$
$$x = x_0$$

where $K(x,\varepsilon)$ denotes the open ball in X and, for any $x \in X \setminus \{x_0\}$, the symbol m_x denotes a positive integer such that $\frac{1}{m_x} < \rho(x, x_0)$ (by the letter ρ we denote the metric in X).

Let $X_2 = X \cup \{(h(x_n) : n \in \mathcal{N}\} \times (0,1])$. Define the topology T_2 in X_2 generated by the neighbourhood system:

$$B_2(x) = \{\{h(x_n)\}\} \times ((x' - \tfrac{1}{m}, x' + \tfrac{1}{m}) \cap (0,1]) : m \in \mathcal{N}\}$$
$$\text{if } x = (h(x_n), x') \in \{h(x_n)\} \times (0,1] \textit{ for some } n \in \mathcal{N},$$
$$B_2 = \{K(h(x), \tfrac{1}{m}) \cup \textstyle\bigcup_{h(x_n) \in K(h(x),\frac{1}{m})}(\{h(x_n)\} \times (0, \tfrac{1}{m})) : m \in$$
$$\mathcal{N}\} \text{ if } x \in X.$$

To simplify the notation, we shall write X_i instead of (X_i, T_i) $(i = 1, 2)$. It is not difficult to see that X_i $(i = 1, 2)$ is a connected and Hausdorff space.

Now, we shall prove that X_1 is compact. So, let $\{V_t\}_{t\in T}$ be an open cover of X_1. Thus there exists a finite sequence $t_1, t_2, \dots, t_p \in T$ such that $X \subset \bigcup_{i=1}^{p} V_{t_i}$. Assume, for instance, that $x_0 \in V_{t_1}$. There exists an integer N such that $\{x_n\} \times (0,1] \subset V_{t_1}$, for $n \geq N$. Let $\{n_1, \dots, n_s\} = \{n \in \mathcal{N} : n < N\}$. Let k_i (for $i = 1, \dots, s$) be a positive integer such that $\{x_{n_i}\} \times (0, \frac{1}{k_i}) \subset \bigcup_{j=1}^{p} V_{t_j}$ and $k = \max\{k_i : i = 1, \dots, s\}$. Hence $X_1 \setminus \bigcup_{j=1}^{p} V_{t_j} \subset \bigcup_{i=1}^{s}\{x_{n_i}\} \times [\frac{1}{k}, 1]$. Of course, the set $\bigcup_{i=1}^{s}\{x_{n_i}\} \times [\frac{1}{k}, 1]$ is compact in X_1, so there exists $\{t_{p+1}, t_{p+2}, \dots, t_{p+q}\} \subset T$ such that $\bigcup_{i=1}^{p}\{x_{n_i}\} \times [\frac{1}{k}, 1] \subset \bigcup_{j=1}^{q} V_{t_{p+j}}$. This means that $\{V_{t_1}, \dots, V_{t_{p+q}}\}$ is a finite subcover of the cover $\{V_t\}_{t\in T}$.

At present, we shall show that X_2 is a paracompact space. So, let $\{W_s\}_{s\in S}$ be an arbitrary open cover of X_2. Thus there exists a finite sequence $s_1, \dots, s_l \in S$ such that $X \subset \bigcup_{j=1}^{l} W_{s_j}$. Assume, for instance, that $h(x_0) \in W_{s_1}$. Let m_0 be a positive integer such that $K(h(x_0), \frac{1}{m_0}) \cup \bigcup_{x_n \in K(x_0, \frac{1}{m_0})}\{h(x_n)\} \times (0, \frac{1}{m_0}) \subset W_{s_1}$. Let $\{h(x_{n_1}), \dots, h(x_{n_t})\} = \{h(x_n) : n \in \mathcal{N}\} \setminus K(h(x_0), \frac{1}{m_0})$. Let α_i (for a fixed $i \in \{1, \dots, t\}$) be a positive real number such that $(h(x_{n_i}), \alpha_i) \in \bigcup_{j=1}^{l} W_{s_j}$ and, moreover, let s_y (for any $y \in X_2$) be a fixed element from S such that $y \in W_{s_y}$. Then, for $\beta = (h(x_n), p) \in (\{h(x_n)\} \times (0,1]) \setminus \bigcup_{j=1}^{l} W_{s_j}$, assume that:

$$\Xi_\beta =$$

$$\begin{cases} \{h(x_n)\} \times (\frac{1}{2m_0}, 1]) \cap W_{s_\beta} & if \;\; h(x_n) \in K(h(x_0), \frac{1}{m_0}), \\ \{h(x_n)\} \times (\frac{\alpha_i}{2}, 1]) \cap W_{s_\beta} & if \;\; h(x_n) = h(x_{n_i}) \; (i \in \{1, \dots, t\}). \end{cases}$$

Thus the family $\{\Xi_\beta\}$ of open sets is a refinement of $\{W_s\}_{s\in S}$. Let n_0 be a fixed positive integer and let

$$I_{n_0} = \begin{cases} \{h(x_{n_0})\} \times [\frac{1}{m_0}, 1] & if \;\; h(x_{n_0}) \in K(h(x_0), \frac{1}{m_0}), \\ \{h(x_{n_0})\} \times [\alpha_i, 1] & if \;\; h(x_n) = h(x_{n_i}) \; (i \in \{1, \dots, t\}). \end{cases}$$

Then there exists a finite sequence $\{z_{p_1}^{n_0}, \dots, z_{p_{q_{n_0}}}^{n_0}\}$ consisting of elements of the segment I_{n_0}, such that $I_{n_0} \subset \bigcup_{j=1}^{l} W_{s_j} \cup$

$\bigcup_{i=1}^{q_{n_0}} \Xi_{z_{p_i}^{n_0}}$. It is easy to see that $\{W_{s_j} : j = 1, \dots, l\} \cup \bigcup_{n=1}^{\infty} \{\Xi_{z_{p_i}^n} : i = 1, \dots, q_n\}$ is a locally finite open (cover of X_2) refinement of $\{W_s\}_{s \in S}$.

Moreover, one can notice that X_1 and X_2 are locally connected spaces.

Define a function $h^* : X_1 \to X_2$ by letting:

$$h^*(x) = \begin{cases} h(x) & if \quad x \in X, \\ (h(x_n), \alpha_x) & if \quad x = (x_n, \alpha_x) \in \{x_n\} \times (0,1]. \end{cases}$$

We remark that h^* is a Darboux function. Indeed. Let C be an arbitrary connected set in X_1. In the case when $C \cap X = \emptyset$, $h^*(C)$ is, of course, connected. So, let $C \cap X \neq \emptyset$. Thus $C \cap X$ is a connected set. Let $\{k_n\}$ be a sequence of positive integers such that $(\{x_{k_n}\} \times (0,1]) \cap C \neq \emptyset$. (It can happen that the set of all these k_n is empty). Therefore $(\{x_{k_n}\} \times (0,1]) \cap C = \{x_{k_n}\} \cap (0, \alpha_{k_n} >$ $(n = 1, 2, \dots)$, where by the symbol $(0, \alpha_{k_n} >$ we understand an interval open or closed on the right. Then $h^*(C) = h(C \cap X) \cup \bigcup_{k_n} (\{h(x_{k_n})\} \times (0, \alpha_{k_n} >$ is connected.

Now, we shall show tht h^* is a closed function.

Let K be an arbitrary closed set in X_1 and suppose that $cl_{X_2}(h^*(K)) \setminus h^*(K) \neq \emptyset$. Let $z_0 \in cl_{X_2}(h^*(K)) \setminus h^*(K)$ and let $\{z_\sigma\}_{\sigma \in \Sigma} \subset h^*(K)$ be a net such that $z_0 = \lim_{\sigma \in \Sigma} z_\sigma$. Moreover, let $h^*(t_0) = z_0$ and $h^*(t_\sigma) = z_\sigma$ for $\sigma \in \Sigma$ and $t_\sigma \in K$.

Consider the following cases:

1^o. There exists $\sigma_0 \in \Sigma$ such that $z_\sigma \in X$ for any $\sigma \geq \sigma_0$. Then $t_\sigma \in X$ $(\sigma \geq \sigma_0)$ and $t_0 \in \lim_{\sigma \in \Sigma} t_\sigma$; cosequently, $z_0 = h^*(t_0) \in h^*(K)$, which is impossible.

2^o. There exists a subnet $\{z_\sigma\}_{\sigma \in \Sigma'} \subset X_2 \setminus X$.

Thus there can happen two cases:

2a). There exists $\sigma_0' \in \Sigma'$ such that $z_\sigma \in \{h(x_{n_0})\} \cap (0,1]$ for $\sigma \geq \sigma_0'$, $\sigma \in \Sigma'$, and for some $n_0 \in \mathcal{N}$. Thus $z_0 \in (\{h(x_n)\} \times (0,1]) \cup \{h(x_n)\}$, which means that $t_0 \in \lim_{\sigma \in \Sigma'} t_\sigma$, thus $t_0 \in K$ and, similarly as above, we can show that $z_0 \in h^*(K)$.

2b). Assume that 2a) does not hold. Then $z_0 = h^*(x_0)$ and, as is easy to see, $t_0 \in \lim_{\sigma \in \Sigma'} t_\sigma$, so $z_0 \in h^*(K)$.

The contradictions obtained prove that h^* is a closed function. Of course, h^* is discontinuous at x_0.

In connection with this theorem, it seems interesting to pose:

Problem 7. *Can one prove a theorem analogous to Theorem 3 in such a way that X_2 is "closer to compactness" (for example: locally compact), with an eventual weakening of the requirements concerning the space X_1?*

One can also consider some "weaker problem":

Problem 8. *Do there exist, for a non-singleton locally connected continuum X and any homeomorphism $h : X \to X$, spaces X_1,X_2 "close to compactness" such that X is a subspace of X_1 and X_2 and there exists a d-extension $h^* : X_1 \to X_2$ of the function h such that h^* is a discontinuous and closed Darboux function?*

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