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STRATIFIABILITY OF FREE ABELIAN TOPOLOGICAL GROUPS

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ABSTRACT. It is proved that if X is a stratifiable T_1 space then the free Abelian topological group of X is also stratifiable. This implies, in particular, that every stratifiable space can be embedded into an Abelian stratifiable group as a closed subspace.

The purpose of this work is to gain insight into the relation between the properties of a topological group and those of a subspace generating this group. More specifically, we will be concerned with the question of what properties of a topological space are inherited by its free Abelian topological group. This problem is closely related to the still more general problem of embedding a space with certain topological properties into a topological group having the same or some other properties.

The free Abelian topological group $A(X)$ of a completely regular T_1 space X is the free Abelian algebraic group of X with the stgrongest group topology such that induces the original topology of X. In other words, any continuous map of X to an arbitrary Abelian topological group G can be extended to a continuous homomorphism of $A(X)$ to G. A free

Abelian topological group is a convenient and important object: every Abelian topological group algebraically generated by X or by its continuous image is a continuous homomorphic image of $A(X)$. Thanks to this solving many problems concerning the properties of topological groups and embedding into topological groups may be reduced to consideration of free groups. However, free Abelian (and non-Abelian) topological groups are never metrizable, except for the trivial discrete case, though one can easily embed any metric space (X, ρ) into a metric topological group; to do this, it is sufficient to consider the Graev extension of ρ to the free group of X [1].

Here we deal with stratifiability, which is one of the most popular generalizations of metrizability. Stratifiable spaces were introduced by Ceder [2] as M_3-spaces; the term 'stratifiable space' was proposed by Borges [3]. One of the reasons why the class of stratifiable spaces is important and useful is its invariance under many operations: in particular, this class is hereditary, countably productive, and preserved under closed mappings. At the same time stratifiable spaces are not too unlike metrizable spaces. For example, every stratifiable space is paracompact and has a G_δ-diagonal (see [4]).

The principal result of this work is that the free Abelian topological group of a stratifiable T_1 space is stratifiable. This allows any stratifiable T_1 space to be embedded into an Abelian stratifiable T_0 group as a closed subspace. As far as I know, the possibilty of embedding a stratifiable space into an arbitrary, not Abelian, stratifiable group as an arbitrary, not closed, subspace has been unknown yet.

I don't know the answer to Arhangel'skii's question [5] if the free (non-Abelian) topological group of a stratifiable space is stratifiable. Free groups of metric compacta are stratifiable [6], but for non-compact metrizable spaces the problem is open.

Free Abelian topological groups were introduced and first investigated by Markov [7, 8]. Algebraically the free Abelian group $A(X)$ of X is the set of words

$$\boldsymbol{g} = \varepsilon_1 g_1 + \cdots + \varepsilon_n g_n,$$

where n is a positive integer or zero (in the latter case word $\boldsymbol{g}$ is empty), $\varepsilon_i = \pm 1$, and $g_i \in X$ for $i = 1, \ldots, n$. So, every nonempty word is the sum of letters, or the elements of the alphabet $X \cup -X$, where $-X$ is a homeomorphic copy of X such that $X \cap -X = \varnothing$. To make words distinguishable from letters, words are given in boldface. For example, $\boldsymbol{g} = \varepsilon_1 g_1 + \cdots + \varepsilon_n g_n$ represents $\boldsymbol{g}$ as the sum of letters $\varepsilon_i g_i$, while $\boldsymbol{g} = \boldsymbol{u} + \boldsymbol{v}$ represents this word as the sum of words $\boldsymbol{u}$ and $\boldsymbol{v}$. The zero of $A(X)$, the empty word, is denoted $\mathbf{0}$.

When we speak about a word we mean that its letters are numbered in some fixed way, and by a letter of a word we mean not only an element of the alphabet $X \cup -X$ but also its position in the word. For example, if $i \neq j$, the letters $\varepsilon_i g_i$ and $\varepsilon_j g_j$ are different in $\boldsymbol{g}$, even if $\varepsilon_i g_i = \varepsilon_j g_j$.

Let $\varepsilon_1 g_1 + \cdots + \varepsilon_n g_n$ be the reduced form of a word $\boldsymbol{g}$. The number n is the length of $\boldsymbol{g}$ denoted $l(\boldsymbol{g})$. We use the designation $A_n(X)$ for the set of all words in $A(X)$ the length of which does not exceed n.

The symbol $\mathbb{N}^+$ stands for the set of all positive integers, and $\mathbb{N}$ for the set of all nonnegative integers. Letters i, j, k, l, m, n, p, q, r, s, and t are positive integers. The closure of a set A is denoted as $\overline{A}$, and the integral part of a number a as $[a]$.

Let $\mathcal{A}$ be a family of subsets of set X and $A \subset X$. By the *star* of $\mathcal{A}$ about A we mean

$$\mathrm{St}_{\mathcal{A}}(A) = \{U \in \mathcal{A} : U \cap A \neq \varnothing\}.$$

In other works the star of $\mathcal{A}$ about A is often defined as the union of the elements of $\mathrm{St}_{\mathcal{A}}(A)$ defined as above.

Let γ_1 and γ_2 be covers of a set X. The relation $\gamma_1 \prec \gamma_2$ means that γ_1 is a refinement of γ_2. The *composition* of γ_1 and γ_2 is the cover

$$\gamma_1 \circ \gamma_2 = \{\cup \mathrm{St}_{\gamma_2}(U) : U \in \gamma_1\}.$$

An open cover γ of a space X is said to be *normal* if there exists a sequence $\{\gamma(n) : n \in \mathbb{N}^+\}$ of open covers such that $\gamma(1) \circ \gamma(1) \prec \gamma$ and $\gamma(n+1) \circ \gamma(n+1) \prec \gamma(n)$ for any $n \in \mathbb{N}^+$. Clearly, for every normal cover γ and for every k there exists an open cover γ' such that

$$\underbrace{\gamma' \circ \cdots \circ \gamma'}_{k \text{ times}} \prec \gamma.$$

In particular, every open cover of a paracompact space has this property, since all of them are normal (see [9, Theorem 5.1.12]).

The following remark is a modification of the well-known fact; we include its proof for completeness.

Remark 1. Let $\{\gamma_n : n \in \mathbb{N}^+\}$ be a sequence of covers such that for every n

$$\underbrace{\gamma_{n+1} \circ \cdots \circ \gamma_{n+1}}_{5 \text{ times}} \prec \gamma_n.$$

Suppose that $n, m, i_1, \ldots, i_m \in \mathbb{N}^+$, $i_j > n$ for $j \leqslant m$, and for every $p \leqslant m$ there exists not more than one $q \leqslant m$, $q \neq p$, for which $i_p = i_q$. Then

$$\gamma_{i_1} \circ \gamma_{i_2} \circ \cdots \circ \gamma_{i_m} \prec \gamma_n.$$

Proof: This is evident if $m = 1$. Assume that $m > 1$ and for smaller m the assertion is proved. Let $p \leqslant q \leqslant m$ and i_p and i_q be the smallest number(s) among $i_1, \ldots, i_m$. We will suppose for definiteness that $i_p \leqslant i_q$. By the induction hypothesis $\gamma_{i_1} \circ \cdots \circ \gamma_{i_{p-1}} \prec \gamma_{i_p}$, $\gamma_{i_{p+1}} \circ \cdots \circ \gamma_{i_{q-1}} \prec \gamma_{i_p}$, and $\gamma_{i_{q+1}} \circ \cdots \circ \gamma_{i_m} \prec \gamma_{i_p}$. Therefore,

$$\begin{aligned}\gamma_{i_1} \circ \cdots \circ \gamma_{i_m} &= \gamma_{i_1} \circ \cdots \circ \gamma_{i_{p-1}} \circ \gamma_{i_p} \circ \gamma_{i_{p+1}} \circ \cdots \circ \gamma_{i_{q-1}} \circ \gamma_{i_q} \circ \\ &\quad \gamma_{i_{q+1}} \circ \cdots \circ \gamma_{i_m} \prec \gamma_{i_p} \circ \gamma_{i_p} \circ \gamma_{i_p} \circ \gamma_{i_q} \circ \gamma_{i_p},\end{aligned}$$

and the last cover is a refinement of γ_n by the conditions. □

For covers γ_1 and γ_2 of a set X

$$\gamma_1 \wedge \gamma_2 = \{U_1 \cap U_2 : U_1 \in \gamma_1, U_2 \in \gamma_2, U_1 \cap U_2 \neq \varnothing\}.$$

Clearly, $\gamma_1 \wedge \gamma_2 \prec \gamma_i$ for $i = 1, 2$.

All covers and neighborhoods are presupposed to be open below.

For a pseudometric d on X, a nonnegative number ε, and $x \in X$

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

A space X is called *semi-stratifiable* if there is a function G which assigns an open set $G(n, H)$ to each $n \in \mathbb{N}^+$ and a closed set $H \subset X$ such that $G(n, H) \supset H$ and the following conditions are met:

(i) $H = \cap_n (n, H)$;

(ii) $H \subset K \implies G(n, H) \subset G(n, K)$.

If also

(iii) $H = \cap_n \overline{G(n, H)}$, then X is *stratifiable* (see [4]).

A space X is *monotonically normal* (see [4]) if to each pair (H, K) of disjoint closed subsets of X, one can assign an open set $D(H, K)$ such that

(i) $H \subset D(H, K) \subset \overline{D(H, K)} \subset X \setminus K$;

(ii) if $H \subset H'$ and $K' \subset K$ then $D(H, K) \subset D(H', K')$.

We will use the following characterization of monotonically normal spaces:

Proposition 1. (see [4, Theorem 5.19]) *A space X is monotonically normal if and only if to each open set $V \subset X$ and $x \in V$, one can assign an open set $U(V, x)$ containing x in such a way that for any $x, y \in X$ and their neighborhoods V, W*

$$U(V, x) \cap U(W, y) \neq \varnothing \quad \textit{implies} \quad x \in W \textit{ or } y \in V.$$

A space X is a *σ-space* if X has a σ-discrete network (see [4]). In the following reasoning we will need

Proposition 2. *If X is a paracompact σ-space then $A(X)$ is also a σ-space.*

This fact is, probably, known; in any event, Arhangel'skii proved in [5] that if X is a paracompact σ-space then $F(X)$ is a paracompact σ-space. His proof is valid for Abelian groups almost without changes.

Proposition 3. *Let X be an arbitrary completely regular T_1 space. For every sequence $\Gamma = \{\gamma(n) : n \in \mathbb{N}^+\}$ of open covers of X put*

$$U(\Gamma) = \bigcup_{n\in\mathbb{N}^+} \{x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n :$$
$$x_i, y_i \in O_i \text{ for some } O_i \in \gamma(i)\}.$$

The family

$$\Big\{U(\Gamma) : \Gamma = \{\gamma(n) : n \in \mathbb{N}^+\},\ \gamma(n) \text{ is a normal cover of } X\Big\}$$

constitutes a base at zero of the topology of the free Abelian topological group $A(X)$.

Proof: The proposition can be proved following the standard line of reasoning (see, for example, [11–13]); for shortness we reduce the description of the free Abelian group topology to one of the earlier descriptions of the free non-Abelian group topology.

Let $F(X)$ be the free topological group of X in the Markov sense and

$$i_n : (X \cup X^{-1})^n \to F(X)$$

be the natural map defined by $(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) \mapsto x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, where $n \in \mathbb{N}^+$, X^{-1} is a disjoint homeomorphic copy of X, $x_i \in X$, and $\varepsilon_i = \pm 1$. For each n define map

$$j_n : (X \cup X^{-1})^{2n} \to F(X)$$

by the rule $(\boldsymbol{x}, \boldsymbol{y}) \mapsto i_n(\boldsymbol{x}) \cdot i_n(\boldsymbol{y})^{-1}$ for $\boldsymbol{x}, \boldsymbol{y} \in (X \cup X^{-1})^n$. Let $\mathcal{U}_n$ denote the universal uniformity of $(X \cup X^{-1})^n$, i.e., the finest uniformity that induces the original topology of $(X \cup X^{-1})^n$, and $\mathcal{R}$ the family of all sequences $E = \{U_n : n \in \mathbb{N}^+\}$ such that $U_n \in \mathcal{U}_n$ whenever $n \in \mathbb{N}^+$. For $k \in \mathbb{N}^+$ and $E \in \mathcal{R}$, where $E = \{U_n : n \in \mathbb{N}^+\}$, put

$$V_k(E) = \cup\Big\{j_{\pi(1)}(U_{\pi(1)}) \cdot \ldots \cdot j_{\pi(k)}(U_{\pi(k)}) : \pi \in S_k\Big\}$$

(S_k is the set of all permutations of $\{1, \ldots, k\}$). Finally, put

$$V(E) = \bigcup_{k \in \mathbb{N}^+} V_k(E).$$

Tkačenko proved [11] that the family

$$\{V(E) : E \in \mathcal{R}\}$$

is a neighborhood base of the unity in $F(X)$.

Group $A(X)$ is the quotient group of $F(X)$ modulo the commutator subgroup of $F(X)$ [8]. The natural homomorphism h of $F(X)$ onto $A(X)$ maps $j_n(U_n)$ to

$$\begin{aligned} & h(j_n(U_n)) \\ = \; & \{h\Big(x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} y_n^{-\delta_n} \ldots y_1^{-\delta_1}\Big) : \\ & x_i, y_i \in X,\ \varepsilon_i, \delta_i = \pm 1,\ (x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}, y_1^{\delta_1}, \ldots, y_n^{\delta_n}) \in \mathcal{U}_n\} \\ = \; & \{\varepsilon_1 x_1 - \delta_1 y_1 + \cdots + \varepsilon_n x_n - \delta_n y_n : \\ & x_i, y_i \in X,\ \varepsilon_i, \delta_i = \pm 1,\ (x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}, y_1^{\delta_1}, \ldots, y_n^{\delta_n}) \in \mathcal{U}_n\}, \end{aligned}$$

and $V_k(E)$ to

$$\begin{aligned} & h(V_k(E)) \\ = \; & \cup\Big\{h(j_{\pi(1)}(U_{\pi(1)})) + \cdots + h(j_{\pi(k)}(U_{\pi(k)})) : \pi \in S_k\Big\} \\ = \; & h(j_1(U_1)) + \cdots + h(j_k(U_k)) \end{aligned}$$

$$= \Big\{\varepsilon_{1,1}x_{1,1} - \delta_{1,1}y_{1,1} + \varepsilon_{1,2}x_{1,2} - \delta_{1,2}y_{1,2} + \varepsilon_{2,2}x_{2,2} - \delta_{2,2}y_{2,2} + \dots + \varepsilon_{1,k}x_{1,k} - \delta_{1,k}y_{1,k} + \varepsilon_{2,k}x_{2,k} - \delta_{2,k}y_{2,k} + \dots + \varepsilon_{k,k}x_{k,k} - \delta_{k,k}y_{k,k} : x_{i,j}, y_{i,j} \in X,\ \varepsilon_{i,j}, \delta_{i,j} = \pm 1, (x_{1,j}{}^{\varepsilon_{1,j}}, x_{2,j}{}^{\varepsilon_{2,j}}, \dots, x_{j,j}{}^{\varepsilon_{j,j}}, y_{1,j}{}^{\delta_{1,j}}, y_{2,j}{}^{\delta_{2,j}}, \dots, y_{j,j}{}^{\delta_{j,j}}) \in U_j \text{ for } i,j = 1,\dots,k\Big\}.$$

Denote the universal uniformity of $X \cup -X$ as $\mathcal{U}$. Put

$$\mathcal{R}' = \Big\{E' = \{U'_n : n \in \mathbb{N}^+,\ U'_n \in \mathcal{U}\}\Big\}$$

and

$$U(E') = \bigcup_{n \in \mathbb{N}^+} \{\varepsilon_1 x_1 - \delta_1 y_1 + \varepsilon_2 x_2 - \delta_2 y_2 + \dots + \varepsilon_n x_n - \delta_n y_n : (\varepsilon_i x_i, \delta_i y_i) \in U'_i\}$$

for $E' = \{U'_n : n \in \mathbb{N}^+\} \in \mathcal{R}'$. Evidently, for each $E \in \mathcal{R}$ there exists $E' \in \mathcal{R}'$ and for every $E' \in \mathcal{R}'$ there exists $E \in \mathcal{R}$ such that

$$U(E') = h(V(E)) = \bigcup_{n \in \mathbb{N}^+} h(V_n(E));$$

it is sufficient to identify X^{-1} with $-X$ and consider elements of $\mathcal{U}$ produced as projections of members of E and, accordingly, elements of $\mathcal{U}_n$ produced as powers of members of E'. Therefore, the family

$$\{U(E') : E' \in \mathcal{R}'\}$$

is a base at zero of the topology of $A(X)$. Obviously, the family $\mathcal{R}'$ may be replaced by

$$\mathcal{R}'' = \Big\{E'' = \{U''_n : n \in \mathbb{N}^+,\ U''_n \in \mathcal{V}\}\Big\},$$

where $\mathcal{V}$ is the universal uniformity of X, not $X \cup -X$. The family

$$\left\{\cup\{V \times V : V \in \gamma\} : \gamma \text{ is a normal cover of } X\right\}$$

generates the universal uniformity $\mathcal{V}$ (see [9, Exercise 8.1.C(b)]). This completes the proof. $\square$

Theorem. *If X is a stratifiable T_1 space then the free Abelian topological group $A(X)$ of X is stratifiable.*

Proof: Every σ-space is semi-stratifiable (see [4, Theorem 5.9]), and every semi-stratifiable monotonically normal space is stratifiable (see [4, Theorem 5.16]). Therefore, all that we need is to prove that $A(X)$ is monotonically normal and apply Proposition 2.

Every open neighborhood of an arbitrary word $\boldsymbol{g}$ from $A(X)$ is representable as $\boldsymbol{g} + O$, where O is an open neighborhood of zero in $A(X)$. According to Proposition 1, we must assign an open neighborhood $W(U, \boldsymbol{g})$ of zero to each open neighborhood U of zero and $\boldsymbol{g} \in A(X)$ in such a way that for any $\boldsymbol{g}, \boldsymbol{h} \in A(X)$ and open neighborhoods U, V of zero

$$(\boldsymbol{g} + W(U, \boldsymbol{g})) \cap (\boldsymbol{h} + W(V, \boldsymbol{h})) \neq \varnothing$$

implies

$$\boldsymbol{g} \in \boldsymbol{h} + V \text{ or } \boldsymbol{h} \in \boldsymbol{g} + U,$$

or, which is the same,

$$\boldsymbol{g} - \boldsymbol{h} \in W(V, \boldsymbol{h}) - W(U, \boldsymbol{g})$$

implies

$$\boldsymbol{g} - \boldsymbol{h} \in V \text{ or } \boldsymbol{g} - \boldsymbol{h} \in -U.$$

By Proposition 3 the family

$$\left\{U(\Gamma) : \Gamma = \{\gamma(n) : n \in \mathbb{N}^+\},\ \gamma(n) \text{ is a normal cover of } X\right\},$$

where

$$U(\Gamma) = \bigcup_{n \in \mathbb{N}^+} \{x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n :$$

$$x_i, y_i \in O_i \text{ for some } O_i \in \gamma(i)\},$$

constitutes a base at zero of the topology of $A(X)$. It is clear that $U(\Gamma) = -U(\Gamma)$ and

$$(1)\quad U(\Gamma) = \bigcup_{n\in\mathbb{N}^+} \{x_{k_1} - y_{k_1} + x_{k_2} - y_{k_2} + \dots + x_{k_n} - y_{k_n} :$$

$$x_{k_i}, y_{k_i} \in O_{k_i} \text{ for some } O_{k_i} \in \gamma(k_i) \text{ and } k_i \neq k_j \text{ for } i \neq j\}.$$

The space X is stratifiable and therefore paracompact (see [4, Theorem 5.7]), hence all its covers are normal. Besides, without the loss of generality we may only consider sequences $\Gamma = \{\gamma(n) : n \in \mathbb{N}^+\}$ of covers such that for all n

$$(2)\qquad \gamma(n+1) \circ \gamma(n+1) \circ \gamma(n+1) \prec \gamma(n).$$

Put

$$\mathcal{G} = \Big\{\Gamma = \{\gamma(n) : n \in \mathbb{N}^+\},\ \gamma(n) \text{ is an open cover of } X,$$

$$\gamma(n+1) \circ \gamma(n+1) \circ \gamma(n+1) \prec \gamma(n) \text{ for } n \in \mathbb{N}^+\Big\}.$$

According to what has been said above, the family $\{U(\Gamma) : \Gamma \in \mathcal{G}\}$ is a base at zero of the topology of $A(X)$.

Principal Lemma. *There exist maps*

$$f : A(X) \times \mathcal{G} \to \mathbb{N}$$

and

$$F : A(X) \times \mathcal{G} \to \mathcal{G}$$

such that if

(i) $\boldsymbol{g}, \boldsymbol{h} \in A(X)$, $\Gamma_{\boldsymbol{g}}, \Gamma_{\boldsymbol{h}} \in \mathcal{G}$,
(ii) $f(\boldsymbol{h}, \Gamma_{\boldsymbol{h}}) \leqslant f(\boldsymbol{g}, \Gamma_{\boldsymbol{g}})$, *and*
(iii) $\boldsymbol{g} - \boldsymbol{h} \in U(F(\boldsymbol{g}, \Gamma_{\boldsymbol{g}})) + U(F(\boldsymbol{h}, \Gamma_{\boldsymbol{h}}))$

then $\boldsymbol{g} - \boldsymbol{h} \in 2U(\Gamma_{\boldsymbol{h}})$.

Monotone normality of $A(X)$ easily follows from this lemma. Indeed: suppose, the Principal Lemma is proved. For every open neighborhood U of zero fix $\Gamma_U \in \mathcal{G}$ such that $2U(\Gamma_U) \subset U$. Assign $W(U, \boldsymbol{g}) = U(F(\boldsymbol{g}, \Gamma_U))$ to each neighborhood U of zero and $\boldsymbol{g} \in A(X)$. Then by the Principal Lemma we have that condition

$$\boldsymbol{g} - \boldsymbol{h} \in W(U, \boldsymbol{g}) + W(V, \boldsymbol{h}) = U(F(\boldsymbol{g}, \Gamma_U)) + U(F(\boldsymbol{h}, \Gamma_V))$$

implies

$$\begin{aligned} &\boldsymbol{g} - \boldsymbol{h} \in 2U(\Gamma_V), \quad \text{if } f(\boldsymbol{h}, \Gamma_V) \leqslant f(\boldsymbol{g}, \Gamma_U), \text{or} \\ &\boldsymbol{h} - \boldsymbol{g} \in 2U(\Gamma_U), \quad \text{otherwise.} \end{aligned}$$

As $U(\Gamma) = -U(\Gamma)$ for any $\Gamma \in \mathcal{G}$, we eventually have that

$$\boldsymbol{g} - \boldsymbol{h} \in W(V, \boldsymbol{h}) - W(U, \boldsymbol{g}) \quad \text{implies} \quad \boldsymbol{g} - \boldsymbol{h} \in V \text{ or } \boldsymbol{g} - \boldsymbol{h} \in -U,$$

as required.

Proof of the Principal Lemma: Since X is stratifiable it has a G_δ-diagonal (see [4]). As X is paracompact, there is a continuous metric ρ on X (see [9, Exercise 5.5.7]). In addition there is a stratification in the Heath sense on X [14], i.e., a sequence

$$\mathcal{M} = \{\mathcal{M}_n : n \in \mathbb{N}^+\}$$

of open covers of X such that for any n

$$\mathcal{M}_n = \{M_n(x) : x \in X\},$$

where $M_n(x)$ is an open neighborhood of x and where

> for any point x_0 from X and for any its neighborhood U_{x_0} there exists $m \in \mathbb{N}^+$ such that $x_0 \notin \overline{\cup \{M_m(x) : x \notin U_{x_0}\}}$.

Clearly, this condition is preserved with decreasing open neighborhoods $M_n(x)$ for $x \in X$, therefore, we may, without the loss of generality, assume that for every n from $\mathbb{N}^+$ and x from X

($1^{\mathcal{M}}$) $\underbrace{\mathcal{M}_{n+1} \circ \cdots \circ \mathcal{M}_{n+1}}_{5 \text{ times}} \prec \mathcal{M}_n$ (here we use the paracompactness of X);

($2^{\mathcal{M}}$) $M_{n+1}(x) \subset M_n(x)$;

($3^{\mathcal{M}}$) $M_n(x) \subset B_\rho(x, 1/2^{n+2})$.

Construct the map

$$f : A(X) \times \mathcal{G} \to \mathbb{N}.$$

Let $\boldsymbol{u} \in A(X)$ and $\Gamma_{\boldsymbol{u}} = \{\gamma_{\boldsymbol{u}}(n) : n \in \mathbb{N}^+\} \in \mathcal{G}$. If $\boldsymbol{u} = \mathbf{0}$ put $f(\boldsymbol{u}, \Gamma_{\boldsymbol{u}}) = 0$. Suppose that $\boldsymbol{u} \neq \mathbf{0}$, i.e., $l(\boldsymbol{u}) > 0$. Let

$$\boldsymbol{u} = \varepsilon_1 u_1 + \cdots + \varepsilon_{l(\boldsymbol{u})} u_{l(\boldsymbol{u})}, \qquad \varepsilon_i = \pm 1,\ u_i \in X,$$

be the reduced form of the word $\boldsymbol{u}$. Put

$$d_{\boldsymbol{u}} = \min\{\rho(u_i, u_j) : \varepsilon_i \neq \varepsilon_j\},$$

if $\varepsilon_i \neq \varepsilon_j$ for some i and j, and $d_{\boldsymbol{u}} = 4$ otherwise. For every $i \leqslant l(\boldsymbol{u})$ fix a neighborhood

$$U_{\boldsymbol{u}}(u_i) \in \gamma_{\boldsymbol{u}}(l(\boldsymbol{u}) + 1)$$

of u_i. Choose n'_{u_i} such that

$$u_i \notin \overline{\cup \{M_{n'_{u_i}}(x) : x \notin U_{\boldsymbol{u}}(u_i)\}}.$$

Put

$$V_{\boldsymbol{u}}(u_i) = M_{n'_{u_i}}(u_i) \setminus \overline{\cup \{M_{n'_{u_i}}(x) : x \notin U_{\boldsymbol{u}}(u_i)\}}$$

and choose n_{u_i} for which

$$u_i \notin \overline{\cup \{M_{n_{u_i}}(x) : x \notin V_{\boldsymbol{u}}(u_i)\}}.$$

Put

$$n_{\boldsymbol{u}} = \max\{n_{u_1}, \ldots, n_{u_{l(\boldsymbol{u})}}, [\log_2(4/d_{\boldsymbol{u}})] + 1\};$$

clearly, $n_{\boldsymbol{u}}$ is a positive integer. Finally, put

$$f(\boldsymbol{u}, \Gamma_{\boldsymbol{u}}) = n_{\boldsymbol{u}}.$$

Map f is constructed. The objects we defined to do this will be used in what follows. We will also need some other related objects: for every $i \leqslant l(\boldsymbol{u})$ put

$$W_{\boldsymbol{u}}(u_i) = M_{n_{\boldsymbol{u}}}(u_i) \setminus \overline{\cup \{M_{n_{\boldsymbol{u}}}(x) : x \notin V_{\boldsymbol{u}}(u_i)\}}$$

and choose m_{u_i} for which

$$u_i \notin \overline{\cup \{M_{m_{u_i}}(x) : x \notin W_{\boldsymbol{u}}(u_i)\}}.$$

Put

$$m_{\boldsymbol{u}} = \max\{m_{u_1}, \dots, m_{u_{l(\boldsymbol{u})}}, n_{\boldsymbol{u}}\}$$

and

$$O_{\boldsymbol{u}}(u_i) = M_{m_{\boldsymbol{u}}}(u_i) \setminus \overline{\cup \{M_{m_{\boldsymbol{u}}}(x) : x \notin W_{\boldsymbol{u}}(u_i)\}}.$$

Construct the map

$$F : A(X) \times \mathcal{G} \to \mathcal{G}.$$

Let $\boldsymbol{u} \in A(X)$, $\boldsymbol{u} = \varepsilon_1 u_1 + \dots + \varepsilon_{l(\boldsymbol{u})} u_{l(\boldsymbol{u})}$ be the reduced form of $\boldsymbol{u}$, where $\varepsilon_i = \pm 1$, $u_i \in X$, and $\Gamma_{\boldsymbol{u}} = \{\gamma_{\boldsymbol{u}}(n) : n \in N\} \in \mathcal{G}$. Now we will, by induction, construct the sequence

$$\{\theta_{\boldsymbol{u}}(n) : n \in \mathbb{N}\}$$

of open covers of X such that

$$F(\boldsymbol{u}, \Gamma_{\boldsymbol{u}}) = \Theta_{\boldsymbol{u}} = \{\theta_{\boldsymbol{u}}(n) : n \in \mathbb{N}^+\}.$$

Simultaneously, we will define some auxiliary objects frequently referred to below.

Put $\theta_{\boldsymbol{u}}(0) = \zeta_{\boldsymbol{u}}(0) = \xi_{\boldsymbol{u}}(0) = \{X\}$ and $c_{\boldsymbol{u}}(X, 0) = 0$.

Let $k > 0$. We assume that for all nonnegative integers $r < k$, covers $\theta_{\boldsymbol{u}}(r)$ and some auxiliary objects (locally finite covers $\zeta_{\boldsymbol{u}}(r)$, covers $\xi_{\boldsymbol{u}}(r)$, and numbers $c_{\boldsymbol{u}}(O, r)$ for $O \in \zeta_{\boldsymbol{u}}(r)$) are defined. Construct $\theta_{\boldsymbol{u}}(k)$ and related auxiliary objects.

First define the auxiliary cover $\zeta_{\boldsymbol{u}}(k)$.

For every x from X fix some neighborhood

$$U_{\boldsymbol{u}}(x, k) \in \gamma_{\boldsymbol{u}}(k+1)$$

and choose

$$m_{\boldsymbol{u}}(x, k) > m_{\boldsymbol{u}}$$

for which

$$x \notin \overline{\cup \{M_{m_{\boldsymbol{u}}(x,k)}(y) : y \notin U_{\boldsymbol{u}}(x,k)\}}.$$

Put

$$V_{\boldsymbol{u}}(x,k) = M_{m_{\boldsymbol{u}}(x,k)}(x) \setminus \overline{\cup \{M_{m_{\boldsymbol{u}}(x,k)}(y) : y \notin U_{\boldsymbol{u}}(x,k)\}}.$$

Clearly, $V_{\boldsymbol{u}}(x,k) \subset U_{\boldsymbol{u}}(x,k)$, hence, the cover $\{V_{\boldsymbol{u}}(x,k) : x \in X\}$ is a refinement of $\gamma_{\boldsymbol{u}}(k+1)$. Let $\zeta_{\boldsymbol{u}}(k)$ be a cover of X such that the following conditions hold:

(1^ζ) $\zeta_{\boldsymbol{u}}(k)$ is locally finite;
(2^ζ) $\zeta_{\boldsymbol{u}}(k)$ is a refinement of the cover $\{V_{\boldsymbol{u}}(x,k) : x \in X\}$ (and, therefore, of the cover $\gamma_{\boldsymbol{u}}(k+1)$);
(3^ζ) if $\boldsymbol{u} \neq \boldsymbol{0}$ then for any $i \leqslant l(\boldsymbol{u})$

$$\cup \operatorname{St}_{\zeta_{\boldsymbol{u}}(k)}(V_{\boldsymbol{u}}(u_i)) \subset U_{\boldsymbol{u}}(u_i)$$

(the sets $V_{\boldsymbol{u}}(u_i)$ and $U_{\boldsymbol{u}}(u_i)$ were defined when constructing map f). A cover satisfying these conditions exists because space X is paracompact and the number of sets $V_{\boldsymbol{u}}(u_i)$ is finite.

Next we will construct one more auxiliary cover $\xi_{\boldsymbol{u}}(k)$.

For each element O of cover $\zeta_{\boldsymbol{u}}(k)$ fix one of sets $V_{\boldsymbol{u}}(x,k)$ defined above so that $O \subset V_{\boldsymbol{u}}(x,k)$, and put

$$c_{\boldsymbol{u}}(O,k) = m_{\boldsymbol{u}}(x,k).$$

For every x from X put

$$c_{\boldsymbol{u}}(x,k) = \max\Big\{c_{\boldsymbol{u}}(O,r) : O \in \operatorname{St}_{\zeta_{\boldsymbol{u}}(r)}(\{x\}),\ r \leqslant k\Big\}.$$

This number is defined correctly, because all covers $\zeta_{\boldsymbol{u}}(r)$ are locally finite.

Let $\xi_{\boldsymbol{u}}(k)$ denote some open cover of X satisfying the condition

(3) $\xi_{\boldsymbol{u}}(k) \circ \xi_{\boldsymbol{u}}(k) \circ \xi_{\boldsymbol{u}}(k) \prec \zeta_{\boldsymbol{u}}(k) \wedge \xi_{\boldsymbol{u}}(k-1) \wedge \theta_{\boldsymbol{u}}(k-1)$.

For each i denote the ith prime number as p_i:

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \ldots .$$

For every point x of X fix some its neighborhood

$$W_{\boldsymbol{u}}(x,k) \in \xi_{\boldsymbol{u}}(k)$$

and choose an integer $\mu_{\boldsymbol{u}}(x,k)$ for which

(1^μ) $\mu_{\boldsymbol{u}}(x,k) = p_k^{r(x)}$ for some positive integer $r(x)$;
(2^μ) $\mu_{\boldsymbol{u}}(x,k) \geqslant c_{\boldsymbol{u}}(x,k)$ (and, therefore, $\mu_{\boldsymbol{u}}(x,k) > m_{\boldsymbol{u}}$);
(3^μ) $x \notin \overline{\bigcup \{M_{\mu_{\boldsymbol{u}}(x,k)}(y) : y \notin W_{\boldsymbol{u}}(x,k)\}}$.

Put

$$O_{\boldsymbol{u}}(x,k) = M_{\mu_{\boldsymbol{u}}(x,k)}(x) \setminus \overline{\bigcup \{M_{\mu_{\boldsymbol{u}}(x,k)}(y) : y \notin W_{\boldsymbol{u}}(x,k)\}}$$

and

$$\theta_{\boldsymbol{u}}(k) = \{O_{\boldsymbol{u}}(x,k) : x \in X\}.$$

Conditions (1^μ) and (2^μ) imply that

$$\theta_{\boldsymbol{u}}(k) \prec \mathcal{M}_{p_k^r} \prec \mathcal{M}_{m_{\boldsymbol{u}}} \tag{4}$$

where r is the minimal number $r(x)$ defining $\mu_{\boldsymbol{u}}(x,k)$ (see (1^μ)) for all x.

Put

$$\Theta_{\boldsymbol{u}} = \{\theta_{\boldsymbol{u}}(n) : n \in \mathbb{N}^+\}$$

and

$$F(\boldsymbol{u}, \Gamma_{\boldsymbol{u}}) = \Theta_{\boldsymbol{u}}.$$

The map F is constructed.

It remains to show that the maps f and F satisfy the conditions of the lemma.

Let $\boldsymbol{g}, \boldsymbol{h} \in A(X)$ and

$$\begin{aligned} \boldsymbol{g} &= \varepsilon_1 g_1 + \cdots + \varepsilon_{l(\boldsymbol{g})} g_{l(\boldsymbol{g})}, \\ \boldsymbol{h} &= \delta_1 h_1 + \cdots + \delta_{l(\boldsymbol{h})} h_{l(\boldsymbol{h})}, \end{aligned}$$

where $\varepsilon_i, \delta_i = \pm 1$ and $g_i, h_i \in X$. Let

$$\begin{aligned} \Gamma_{\boldsymbol{g}} &= \{\gamma_{\boldsymbol{g}}(n) : n \in \mathbb{N}^+\} \in \mathcal{G}, \\ \Gamma_{\boldsymbol{h}} &= \{\gamma_{\boldsymbol{h}}(n) : n \in \mathbb{N}^+\} \in \mathcal{G}. \end{aligned}$$

Suppose that

$$f(g,\Gamma_g) = n_g \geqslant n_h = f(h,\Gamma_h). \tag{5}$$

Let

$$u_i, u_i' \in U_i \text{ for some } U_i \in \theta_g(i), \quad i = 1,\dots,p, \tag{6}$$

$$v_j, v_j' \in V_j \text{ for some } V_j \in \theta_h(j), \quad j = 1,\dots,q, \tag{7}$$

and let (∗) stand for

(∗)

$$\begin{array}{rcl}
u_{k_{i_0+1}} & - & u'_{k_{i_0+1}} + u_{k_{i_0+2}} - u'_{k_{i_0+2}} + \dots \\
& + & u_{k_{i_1-1}} - u'_{k_{i_1-1}} + u_{k_{i_1}} - u'_{k_{i_1}} \\
+\ v_{l_{j_0+1}} & - & v'_{l_{j_0+1}} + v_{l_{j_0+2}} - v'_{l_{j_0+2}} + \dots \\
& + & v_{l_{j_1-1}} - v'_{l_{j_1-1}} + v_{l_{j_1}} - v'_{l_{j_1}} \\
+\ u_{k_{i_1+1}} & - & u'_{k_{i_1+1}} + u_{k_{i_1+2}} - u'_{k_{i_1+2}} + \dots \\
& + & u_{k_{i_2-1}} - u'_{k_{i_2-1}} + u_{k_{i_2}} - u'_{k_{i_2}} \\
+\ v_{l_{j_1+1}} & - & v'_{l_{j_1+1}} + v_{l_{j_1+2}} - v'_{l_{j_1+2}} + \dots \\
& + & v_{l_{j_2-1}} - v'_{l_{j_2-1}} + v_{l_{j_2}} - v'_{l_{j_2}} \\
\dots & & \\
+u_{k_{i_{\nu-1}+1}} & - & u'_{k_{i_{\nu-1}+1}} + \dots + u_{k_{i_\nu-1}} - u'_{k_{i_\nu-1}} + u_{k_{i_\nu}} - u'_{k_{i_\nu}} \\
+v_{l_{j_{\nu-1}+1}} & - & v'_{l_{j_{\nu-1}+1}} + \dots + v_{l_{j_\nu-1}} - v'_{l_{j_\nu-1}} + v_{l_{j_\nu}} - v'_{l_{j_\nu}},
\end{array}$$

where

(1*) k_α are different positive integers not exceeding p, and l_α are different positive integers not exceeding q, and

(2*) every odd letter in the record (∗), except for the first one, is cancelled with the letter nearest on the left, i.e.,

$$\begin{aligned}
u'_{k_i} &= u_{k_{i+1}} \quad \text{for} \quad i = i_0+1, i_0+2, \ldots, i_1-1, i_1+1, i_1+2, \\
&\qquad\qquad \ldots, i_{\nu-1}-1, i_{\nu-1}+1, i_{\nu-1}+2, \ldots, i_\nu-1, \\
v'_{l_j} &= u_{k_{j+1}} \quad \text{for} \quad j = j_0+1, j_0+2, \ldots, j_1-1, j_1+1, j_1+2, \\
&\qquad\qquad \ldots, j_{\nu-1}-1, j_{\nu-1}+1, j_{\nu-1}+2, \ldots, j_\nu-1, \\
u'_{k_{i_{s+1}}} &= v_{l_{j_s+1}} \quad \text{for} \quad s = 0, 1, \ldots, \nu-1, \text{ and} \\
v'_{l_{j_t}} &= u_{k_{i_t+1}} \quad \text{for} \quad t = 1, 2, \ldots, \nu-1.
\end{aligned}$$

Condition (2^*) implies that $(*) = u_{k_{i_0+1}} - v'_{l_{j_\nu}}$; in particular, the length of the reduced form of $(*)$ is 2 or 0.

Put for convenience

$$I = \{i_0+1, i_0+2, \ldots, i_\nu\} \quad \text{and} \quad J = \{j_0+1, j_0+2, \ldots, j_\nu\}.$$

Lemma 1. *For any $i, j \leqslant l(\boldsymbol{h})$, $i \neq j$,*

$$-\delta_i h_i - \delta_j h_j \neq (*).$$

Proof: By (4) for all $s \in I$ and $t \in J$

$$\theta_{\boldsymbol{g}}(k_s) \prec \mathcal{M}_{p_{k_s}^{r_s}} \quad \text{and} \quad \theta_{\boldsymbol{h}}(l_t) \prec \mathcal{M}_{p_{l_t}^{r_t}},$$

where p_{k_s} and p_{l_t} are the k_sth and l_tth prime numbers, respectively, and (see (1^μ), (2^μ), (4), and (5))

$$\begin{aligned}
p_{k_s}^{r_s} &\geqslant m_{\boldsymbol{g}} \geqslant n_{\boldsymbol{g}} \geqslant n_{\boldsymbol{h}} \geqslant [\log_2(4/d_{\boldsymbol{h}})] + 1 \geqslant \log_2(4/d_{\boldsymbol{h}}), \\
p_{l_t}^{r_t} &\geqslant m_{\boldsymbol{h}} \geqslant n_{\boldsymbol{h}} \geqslant [\log_2(4/d_{\boldsymbol{h}})] + 1 \geqslant \log_2(4/d_{\boldsymbol{h}}).
\end{aligned}$$

Conditions ($3^{\mathcal{M}}$), (4), (6), and (7) imply that for any $s \in I$

$$\rho(u_{k_s}, u'_{k_s}) \leqslant 1/2^{p_{k_s}^{r_s}+1},$$

and for any $t \in J$

$$\rho(v_{l_t}, v'_{l_t}) \leqslant 1/2^{p_{l_t}^{r_t}+1}.$$

All numbers $p_{k_s}^{r_s}$ are different from each other, because they are powers of different prime numbers, and all $p_{l_t}^{r_t}$ are also different for the same reason. We therefore have

$$\sum_{s\in I}\rho(u_{k_s},u'_{k_s})\leqslant 1/2^{\min\{p_{k_s}^{r_s}\}}$$

and

$$\sum_{t\in J}\rho(v_{l_t},v'_{l_t})\leqslant 1/2^{\min\{p_{l_t}^{r_t}\}}.$$

The fact that all $p_{k_s}^{r_s}$ are not smaller than $\log_2(4/d_{\boldsymbol{h}})$ implies that $2^{\min\{p_{k_s}^{r_s}\}}\geqslant 4/d_{\boldsymbol{h}}$, therefore,

$$\sum_{s\in I}\rho(u_{k_s},u'_{k_s})\leqslant d_{\boldsymbol{h}}/4.$$

Similarly,

$$\sum_{t\in J}\rho(v_{l_t},v'_{l_t})\leqslant d_{\boldsymbol{h}}/4.$$

Hence

$$\sum_{s\in I}\rho(u_{k_s},u'_{k_s})+\sum_{t\in J}\rho(v_{l_t},v'_{l_t})\leqslant d_{\boldsymbol{h}}/2. \tag{8}$$

Assume that $-\delta_i h_i-\delta_j h_j=(*)$. Then, obviously, $\delta_i\neq\delta_j$. Suppose for definiteness that $\delta_i=-1$ and $\delta_j=1$, i.e., $h_i-h_j=(*)$. Then $h_i=u_{k_{i_0+1}}$ and $h_j=v'_{l_{j_\nu}}$. By the definition of $d_{\boldsymbol{u}}$ we have $d_{\boldsymbol{h}}/2<\rho(h_i,h_j)$. Condition (2*) immediately implies that

$$\rho(h_i,h_j)=\rho(u_{k_{i_0+1}},v'_{l_{j_\nu}})\leqslant\sum_{s\leqslant p}\rho(u_{k_s},u'_{k_s})+\sum_{t\leqslant q}\rho(v_{l_t},v'_{l_t}),$$

in contradiction with (8). The lemma is proved. □

Lemma 2. *Let $i,j\leqslant l(\boldsymbol{g})$, $i\neq j$, and*

$$\varepsilon_i g_i+\varepsilon_j g_j=(*).$$

Then there exist l equal to l_t for some $t\in J$ and $U_{g_ig_j}\in\gamma_{\boldsymbol{h}}(l)$ such that

$$g_i,g_j\in U_{g_ig_j}.$$

Proof: Clearly, $\varepsilon_i = -\varepsilon_j$; we will presume for definiteness that $\varepsilon_i = -\varepsilon_j = 1$, i.e., $g_i - g_j = (*)$. If $g_i - g_j = \mathbf{0}$ then the assertion of the lemma is trivially true. Let $g_i - g_j \neq \mathbf{0}$. Then $(*) \neq \mathbf{0}$ and by condition (2^*) $g_i = u_{k_{i_0+1}}$, $g_j = v'_{l_{j_v}}$.

For every $s \in I$, $u_{k_s}, u'_{k_s} \in U_{k_s} \in \theta_g(k_s)$, and for every $t \in J$, $v_{l_t}, v'_{l_t} \in V_{l_t} \in \theta_h(l_t)$ (see (6) and (7)). By construction, covers $\theta_g(k_s)$ have the form

$$\{O_g(x, k_s) : x \in X\},$$

where

$$O_g(x, k_s) = M_{\mu_g(x,k_s)}(x) \setminus \overline{\cup \{M_{\mu_g(x,k_s)}(y) : y \notin W_g(x, k_s)\}}.$$

For every $s \in I$ fix $x_s \in X$ such that

$$u_{k_s}, u'_{k_s} \in O_g(x_s, k_s) \subset M_{\mu_g(x_s,k_s)}(x_s). \tag{9}$$

In a similar way, for every $t \in J$ fix $y_t \in X$ such that

$$v_{l_t}, v'_{l_t} \in O_h(y_t, l_t) \subset M_{\mu_h(y_t,l_t)}(y_t). \tag{10}$$

Conditions $(3^{\mathcal{M}})$, (1^{μ}), (9), and (10) imply that for any $s \in I$

$$\rho(u_{k_s}, u'_{k_s}) \leqslant 1/2^{p_{k_s}^{r(x_s)}+1},$$

and for any $t \in J$

$$\rho(v_{l_t}, v'_{l_t}) \leqslant 1/2^{p_{l_t}^{r(y_t)}+1},$$

where $p_{k_s}^{r(x_s)} = \mu_g(x_s, k_s)$ and $p_{l_t}^{r(y_t)} = \mu_h(y_t, l_t)$. By construction, for all $s \in I$ $\mu_g(x_s, k_s) \geqslant m_g \geqslant n_g$ (see (2^{μ})). Applying the same reasoning as with Lemma 1 shows that inequalities $\mu_h(y_t, l_t) \geqslant n_g$ cannot hold for all $t \in J$, therefore, there exists $t^* \in J$ for which

$$\mu_h(y_{t^*}, l_{t^*}) < n_g.$$

Put

$$\mathrm{A} = \{\alpha \in J : \ \mu_h(y_\alpha, l_\alpha) \leqslant m_g\}.$$

We have $t^* \in A$, because $n_g \leqslant m_g$ and $\mu_h(y_{t^*}, l_{t^*}) < n_g$. It follows that the set A is nonempty.

Assume that $A = \{\alpha_1, \ldots, \alpha_n\}$ and, if $n > 1$, $\alpha_i < \alpha_{i+1}$ for $i < n$. For every natural i such that $1 \leqslant i < n$ (if exists) consider the fragment $\boldsymbol{w}_i$ that comprises all letters positioned strictly in between $-v'_{l_{\alpha_i}}$ and $v_{l_{\alpha_{i+1}}}$. Suppose that the word $\boldsymbol{w}_i$ is nonempty. By (2*) the first letter of $\boldsymbol{w}_i$ is cancelled with letter $-v'_{l_{\alpha_i}}$ of word $(*)$, the third letter of $\boldsymbol{w}_i$ is cancelled with the second letter of $\boldsymbol{w}_i$, the fifth one is cancelled with the fourth, and so on (see (2*)); finally, letter $v_{l_{\alpha_{i+1}}}$ of the word $(*)$ is cancelled with the last letter of $\boldsymbol{w}_i$. Thus,

$$\boldsymbol{w}_i = v'_{l_{\alpha_i}} - v_{l_{\alpha_{i+1}}},$$

and the word $\boldsymbol{w}_i$ is representable as the sum of pairs of letters having the form $u_{k_s} - u'_{k_s}$ and $v_{l_t} - v'_{l_t}$, where

$$\begin{aligned} u_{k_s}, u'_{k_s} &\in O_g(x_s, k_s) \subset M_{\mu_g(x_s,k_s)}(x_s), \\ v_{l_t}, v'_{l_t} &\in O_h(y_t, l_t) \subset M_{\mu_h(y_t,l_t)}(y_t), \end{aligned}$$

and the corresponding numbers $\mu_g(x_s, k_s)$ and $\mu_h(y_t, l_t)$ are greater than m_g (the former because $\mu_g(x_s, k_s) > m_g$, see (2^μ), and the latter because any α from J such that $\alpha_i < \alpha < \alpha_{i+1}$ does not belong to A and by the definition of A). Note that for different s, numbers $\mu_g(x_s, k_s)$ are also different, for they are powers of different prime numbers p_{k_s} (see (1^μ) and (1^*)), and the same is true for numbers of the form $\mu_h(y_t, l_t)$. Remark 1 and condition $(1^{\mathcal{M}})$ imply that for some $y_i^* \in X$

$$v'_{l_{\alpha_i}}, v_{l_{\alpha_{i+1}}} \in M_{m_g}(y_i^*). \tag{11}$$

By the definition of A

$$\mu_h(y_{\alpha_i}, l_{\alpha_i}) \leqslant m_g, \quad \mu_h(y_{\alpha_{i+1}}, l_{\alpha_{i+1}}) \leqslant m_g. \tag{12}$$

Since

$$O_h(y_{\alpha_k}, l_{\alpha_k}) = \tag{13}$$

$$M_{\mu_h(y_{\alpha_k},l_{\alpha_k})}(y_{\alpha_k}) \setminus \overline{\cup \{M_{\mu_h(y_{\alpha_k},l_{\alpha_k})}(y) : y \notin W_h(y_{\alpha_k}, l_{\alpha_k})\}}$$

and

$$v'_{l_{\alpha_i}} \in O_h(y_{\alpha_i}, l_{\alpha_i}) \cap M_{mg}(y_i^*), \tag{14}$$

$$v_{l_{\alpha_{i+1}}} \in O_h(y_{\alpha_{i+1}}, l_{\alpha_{i+1}}) \cap M_{mg}(y_i^*)$$

(see (10) and (11)), we have

$$y_i^* \in W_h(y_{\alpha_i}, l_{\alpha_i}) \cap W_h(y_{\alpha_{i+1}}, l_{\alpha_{i+1}}). \tag{15}$$

Indeed, for example, let $y_i^* \notin W_h(y_{\alpha_i}, l_{\alpha_i})$. Then by (13)

$$M_{\mu_h(y_{\alpha_i},l_{\alpha_i})}(y_i^*) \cap O_h(y_{\alpha_i}, l_{\alpha_i}) = \varnothing,$$

hence by ($2^{\mathcal{M}}$) and (12)

$$M_{mg}(y_i^*) \cap O_h(y_{\alpha_i}, l_{\alpha_i}) = \varnothing,$$

which contradicts (14).

If the word w_i is empty then $\alpha_{i+1} = \alpha_i + 1$ and $v'_{l_{\alpha_i}} = v_{l_{\alpha_{i+1}}}$. Put

$$y_i^* = v'_{l_{\alpha_i}} = v_{l_{\alpha_{i+1}}}.$$

Then $y_i^* = v'_{l_{\alpha_i}} \in O_h(y_{\alpha_i}, l_{a_i}) =$

$$M_{\mu_h(y_{\alpha_i},l_{\alpha_i})}(y_{\alpha_i}) \setminus \overline{\cup \{M_{\mu_h(y_{\alpha_i},l_{\alpha_i})}(y) : y \notin W_h(y_{\alpha_i}, l_{\alpha_i})\}} \subset W_h(y_{\alpha_i}, l_{\alpha_i}),$$

and, similarly,

$$y_i^* = v_{l_{\alpha_{i+1}}} \in W_h(y_{\alpha_{i+1}}, l_{\alpha_{i+1}});$$

therefore

$$y_i^* \in W_h(y_{\alpha_i}, l_{\alpha_i}) \cap W_h(y_{\alpha_{i+1}}, l_{\alpha_{i+1}}).$$

Now consider the fragment w_0 of (∗) comprising letters to the left of letter $v_{l_{\alpha_1}}$ in (∗); clearly, w_0 is nonempty. The first letter of w_0 is $g_i = u_{k_{i_0+1}}$, the third one is cancelled with the second, the fifth with the fourth, and so on (see (2^*)); letter

$v_{l_{\alpha_1}}$ of the word $(*)$ is cancelled with the last letter of $\boldsymbol{w}_0$. Thus,

$$\boldsymbol{w}_0 = g_i - v_{l_{\alpha_1}},$$

and the word $\boldsymbol{w}_0$ is representable as the sum of pairs of letters having the form $u_{k_s} - u'_{k_s}$ and $v_{l_t} - v'_{l_t}$, where

$$\begin{aligned} u_{k_s}, u'_{k_s} &\in O_g(x_s, k_s) \subset M_{\mu_g(x_s,k_s)}(x_s), \\ v_{l_t}, v'_{l_t} &\in O_h(y_t, l_t) \subset M_{\mu_h(y_t,l_t)}(y_t), \end{aligned}$$

and the corresponding numbers $\mu_g(x_s, k_s)$ and $\mu_h(y_t, l_t)$ are greater than m_g. For different s, numbers $\mu_g(x_s, k_s)$ are different, and for different t, numbers $\mu_h(y_t, l_t)$ are different, for they are powers of different prime numbers. Remark 1 and condition $(1^{\mathcal{M}})$ imply that for some $y_0^* \in X$ we have

$$g_i, v_{l_{\alpha_1}} \in M_{m_g}(y_0^*). \tag{16}$$

As $\mu_h(y_{\alpha_1}, l_{\alpha_1}) \leqslant m_g$ and by (13) we have $y_0^* \in W_h(y_{\alpha_1}, l_{\alpha_1})$; this can be shown similarly to (15). On the other hand,

$$g_i \in O_g(g_i) = M_{m_g}(g_i) \setminus \overline{\cup \{M_{m_g}(x) : x \notin W_g(g_i)\}}$$

(sets $O_g(g_i)$ and $W_g(g_i)$ were defined just after constructing the map f) implies that

$$y_0^* \in W_g(g_i).$$

It follows that

$$y_0^* \in W_g(g_i) \cap W_h(y_{\alpha_1}, l_{\alpha_1}). \tag{17}$$

In a similar way we can find $y_n^* \in W_h(y_{\alpha_n}, l_{\alpha_n}) \cap W_g(g_j)$: consider the fragment $\boldsymbol{w}_n$ of $(*)$ consisting of letters to the right of the letter $-v'_{l_{\alpha_n}}$. Assume that $\boldsymbol{w}_n \neq \boldsymbol{0}$. Then the first letter of $\boldsymbol{w}_n$ is cancelled with the letter $-v'_{l_{\alpha_n}}$ of the word $(*)$, the third one with the second, and so on (see (2^*)); the last letter of $\boldsymbol{w}_n$ is $-g_j = -v'_{l_{j_\nu}}$. Thus,

$$\boldsymbol{w}_n = v'_{l_{\alpha_n}} - g_j,$$

and the word $\boldsymbol{w}_n$ is representable as the sum of pairs of letters having the form $u_{k_s} - u'_{k_s}$ and $v_{l_t} - v'_{l_t}$, where

$$\begin{aligned} u_{k_s}, u'_{k_s} &\in O_{\boldsymbol{g}}(x_s, k_s) \subset M_{\mu_{\boldsymbol{g}}(x_s,k_s)}(x_s), \\ v_{l_t}, v'_{l_t} &\in O_{\boldsymbol{h}}(y_t, l_t) \subset M_{\mu_{\boldsymbol{h}}(y_t,l_t)}(y_t), \end{aligned}$$

and the corresponding numbers $\mu_{\boldsymbol{g}}(x_s, k_s)$ and $\mu_{\boldsymbol{h}}(y_t, l_t)$ are greater than $m_{\boldsymbol{g}}$. For different s, numbers $\mu_{\boldsymbol{g}}(x_s, k_s)$ are different, and for different t, numbers $\mu_{\boldsymbol{h}}(y_t, l_t)$ are different. Remark 1 and condition $(1^{\mathcal{M}})$ imply that for some $y_n^* \in X$

$$v'_{l_{\alpha_n}}, g_j \in M_{m_{\boldsymbol{g}}}(y_n^*).$$

As $\mu_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}) \leqslant m_{\boldsymbol{g}}$ and by (13) we have $y_n^* \in W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n})$. On the other hand, it follows from

$$g_j \in O_{\boldsymbol{g}}(g_j) = M_{m_{\boldsymbol{g}}}(g_j) \setminus \overline{\cup \{M_{m_{\boldsymbol{g}}}(x) : x \notin W_{\boldsymbol{g}}(g_j)\}}$$

that

$$y_n^* \in W_{\boldsymbol{g}}(g_j).$$

Hence

$$y_n^* \in W_{\boldsymbol{g}}(g_j) \cap W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}).$$

If word $\boldsymbol{w}_n$ is empty, we have $v'_{l_{\alpha_n}} = g_j$. Then put

$$y_n^* = v'_{l_{\alpha_n}} = g_j;$$

clearly,

$$y_n^* = g_j \in W_{\boldsymbol{g}}(g_j),$$

and

$$y_n^* = v'_{l_{\alpha_n}} \in O_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}) \subset W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}).$$

Thus,

$$\begin{aligned} &y_0^* \in W_{\boldsymbol{g}}(g_i) \cap W_{\boldsymbol{h}}(y_{\alpha_1}, l_{\alpha_1}), \\ &y_i^* \in W_{\boldsymbol{h}}(y_{\alpha_i}, l_{\alpha_i}) \cap W_{\boldsymbol{h}}(y_{\alpha_{i+1}}, l_{\alpha_{i+1}}) \text{ for } i = 1, \ldots, n-1, \end{aligned}$$

and

$$y_n^* \in W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}) \cap W_{\boldsymbol{g}}(g_j).$$

By the definition of $W_g(g_k)$ we have $W_g(g_k) \subset M_{n_g}(g_k)$, therefore, $y_0^* \in M_{n_g}(g_i)$ and $y_n^* \in M_{n_g}(g_j)$. In addition,

$$W_h(y_{\alpha_k}, l_{\alpha_k}) \in \xi_h(l_{\alpha_k}) \qquad \text{for } k = 1, \dots, n.$$

As by (1*) all numbers l_{α_k} are different, it follows from condition (3) that

$$W_h(y_{\alpha_1}, l_{\alpha_1}) \cup \dots \cup W_h(y_{\alpha_n}, l_{\alpha_n}) \subset O^* \tag{18}$$

for some $O^* \in \zeta_h(l)$, where $l = \min\{l_{\alpha_1}, \dots, l_{\alpha_n}\}$. Clearly,

$$y_{\alpha_1}, \dots, y_{\alpha_n} \in O^*. \tag{19}$$

We remind that t^* has been defined by

$$\mu_h(y_{t^*}, l_{t^*}) < n_g,$$

and $t^* \in \mathrm{A}$. By (19) we have $y_{t^*} \in O^*$. It follows from conditions $l_{t^*} \geqslant l$, $\mu_h(y_{t^*}, l_{t^*}) \geqslant c_h(y_{t^*}, l_{t^*})$ (see (2^μ)), and

$$c_h(y_{t^*}, l_{t^*}) = \max\{c_h(O, t) : O \in \mathrm{St}_{\zeta_h(t)}(\{y_{t^*}\}), t \leqslant l_{t^*}\}$$

that

$$c_h(y_{t^*}, l_{t^*}) \geqslant c_h(O^*, l).$$

By the definition of $c_h(O^*, l)$ there exists $x \in X$ such that

$$\begin{gathered} O^* \subset V_h(x, l) = \\ M_{m_h(x,l)}(x) \setminus \overline{\cup \{M_{m_h(x,l)}(y) : y \notin U_h(x, l)\}}, \\ c_h(O^*, l) = m_h(x, l). \end{gathered} \tag{20}$$

We have

$$m_h(x, l) \leqslant c_h(y_{t^*}, l_{t^*}) \leqslant \mu_h(y_{t^*}, l_{t^*}) < n_g,$$

and by condition ($2^{\mathcal{M}}$), $M_{m_h(x,l)}(g_i) \supset M_{n_g}(g_i)$. Since by the definition, $W_g(g_i) \subset M_{n_g}(g_i)$,

$$y_0^* \in M_{n_g}(g_i) \subset M_{m_h(x,l)}(g_i) \tag{21}$$

cf. (17)). Besides, by (17) and (18) $y_0^* \in O^* \subset V_h(x, l)$, therefore, by (20) $g_i \in U_h(x, l)$ (otherwise $M_{m_h(x,l)}(g_i) \cap V_h(x, l) =$

$\varnothing$, in contradiction with (21)). Analogously, $g_j \in U_{\boldsymbol{h}}(x,l)$. As $U_{\boldsymbol{h}}(x,l) \in \gamma_{\boldsymbol{h}}(l+1) \prec \gamma_{\boldsymbol{h}}(l)$, the lemma is proved. □

Lemma 3. *If $i \leqslant l(\boldsymbol{g})$, $j \leqslant l(\boldsymbol{h})$, and*

$$\varepsilon_i g_i - \delta_j h_j = (*),$$

then there exist l equal to l_t for some $t \leqslant q$ or to $l(\boldsymbol{h})$ and $U_{g_i h_j} \in \gamma_{\boldsymbol{h}}(l)$ such that $g_i, h_j \in U_{g_i h_j}$.

Proof: It is clear that $\varepsilon_i = \delta_j$. For definiteness, we will assume that $\varepsilon_i = \delta_j = 1$, i.e., $g_i - h_j = (*)$. If $g_i - h_j = \mathbf{0}$ then the assertion of the lemma is trivially true. Let $g_i - h_j \neq \mathbf{0}$. Then $(*) \neq \mathbf{0}$ and by (2*) $g_i = u_{k_{i_0+1}}$, $h_j = v'_{l_{j_\nu}}$.

For every $s \in I$ $u_{k_s}, u'_{k_s} \in U_{k_s} \in \theta_{\boldsymbol{g}}(k_s)$, and for every $t \in J$ $v_{l_t}, v'_{l_t} \in V_{l_t} \in \theta_{\boldsymbol{h}}(l_t)$ (see (6) and (7)). As in Lemma 2, for every $s \in I$ fix $x_s \in X$ such that

$$u_{k_s}, u'_{k_s} \in O_{\boldsymbol{g}}(x_s, k_s) \subset M_{\mu_{\boldsymbol{g}}(x_s,k_s)}(x_s),$$

and for every $t \in J$ fix $y_t \in X$ such that

$$v_{l_t}, v'_{l_t} \in O_{\boldsymbol{h}}(y_t, l_t) \subset M_{\mu_{\boldsymbol{h}}(y_t,l_t)}(y_t).$$

As in the proof of Lemma 2, put

$$\mathrm{A} = \{\alpha_1 < \alpha_2 < \cdots < \alpha_n : \alpha_i \in J,\ \mu_{\boldsymbol{h}}(y_{\alpha_i}, l_{\alpha_i}) \leqslant m_{\boldsymbol{g}}\}.$$

Assume that A is nonempty. A mere repetition of what has been said in the proof of Lemma 2 allows us to assert that there exist

$$y_0^*, \ldots, y_{n-1}^* \in X$$

such that

$$y_0^* \in W_{\boldsymbol{g}}(g_i) \cap W_{\boldsymbol{h}}(y_{\alpha_1}, l_{\alpha_1})$$

and

$$y_i^* \in W_{\boldsymbol{h}}(y_{\alpha_i}, l_{\alpha_i}) \cap W_{\boldsymbol{h}}(y_{\alpha_{i+1}}, l_{\alpha_{i+1}}) \qquad \text{for } i = 1, \ldots, n-1.$$

Consider the fragment $\boldsymbol{w}_n$ of $(*)$ comprising letters to the right of the letter $-v'_{l_{\alpha_n}}$. Assume that $\boldsymbol{w}_n$ is nonempty. Then the first letter of $\boldsymbol{w}_n$ is cancelled with the letter $-v'_{l_{\alpha_n}}$ of the word $(*)$, the third one with the second, and so on (see (2^*)); the last letter of $\boldsymbol{w}_n$ is $-h_j = -v'_{l_{j\nu}}$. Thus,

$$\boldsymbol{w}_n = v'_{l_{\alpha n}} - h_j,$$

and the word $\boldsymbol{w}_n$ is representable as the sum of pairs of letters having the form $u_{k_s} - u'_{k_s}$ and $v_{l_t} - v'_{l_t}$, where

$$\begin{aligned} u_{k_s}, u'_{k_s} &\in O_{\boldsymbol{g}}(x_s, k_s) \subset M_{\mu_{\boldsymbol{g}}(x_s,k_s)}(x_s), \\ v_{l_t}, v'_{l_t} &\in O_{\boldsymbol{h}}(y_t, l_t) \subset M_{\mu_{\boldsymbol{h}}(y_t,l_t)}(y_t), \end{aligned}$$

and the corresponding numbers $\mu_{\boldsymbol{g}}(x_s, k_s)$ and $\mu_{\boldsymbol{h}}(y_t, l_t)$ are greater than $m_{\boldsymbol{g}}$. For different s, numbers $\mu_{\boldsymbol{g}}(x_s, k_s)$ are different, and for different t, numbers $\mu_{\boldsymbol{h}}(y_t, l_t)$ are different, for they are powers of different prime numbers by $(1^{\mathcal{M}})$ and (1^*). Remark 1 and conditions $(1^{\mathcal{M}})$, $(2^{\mathcal{M}})$, and (5) imply that for some $y_n^* \in X$

$$v'_{l_{\alpha n}}, h_j \in M_{m_{\boldsymbol{g}}}(y_n^*) \subset M_{n_{\boldsymbol{g}}}(y_n^*) \subset M_{n_{\boldsymbol{h}}}(y_n^*).$$

By the construction of A, $\mu_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha n}) \leqslant m_{\boldsymbol{g}}$, hence, $M_{m_{\boldsymbol{g}}}(y_n^*) \subset M_{\mu_{\boldsymbol{h}}(y_{\alpha n},l_{\alpha n})}(y_n^*)$ (see $(2^{\mathcal{M}})$. As
$O_{\boldsymbol{h}}(y_{\alpha_k}, l_{\alpha_k}) =$

$$M_{\mu_{\boldsymbol{h}}(y_{\alpha_k},l_{\alpha_k})}(y_{\alpha k}) \setminus \overline{\bigcup \{M_{\mu_{\boldsymbol{h}}(y_{\alpha_k},l_{\alpha_k})}(y) : y \notin W_{\boldsymbol{h}}(y_{\alpha_k}, l_{\alpha_k})\}},$$

we have $y_n^* \in W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha n})$; this may be shown as in proving (15). On the other hand,

$$h_j \in W_{\boldsymbol{h}}(h_j) = M_{n_{\boldsymbol{h}}}(h_j) \setminus \overline{\bigcup \{M_{n_{\boldsymbol{h}}}(x) : x \notin V_{\boldsymbol{h}}(h_j)\}}$$

implies that

$$y_n^* \in V_{\boldsymbol{h}}(h_j).$$

So,

$$y_n^* \in V_{\boldsymbol{h}}(h_j) \cap W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha n}).$$

If word $\boldsymbol{w}_n$ is empty, $v'_{l_{\alpha n}} = h_j$. Then put

$$y_n^* = v'_{l_{\alpha n}} = h_j.$$

Clearly,

$$y_n^* = h_j \in V_{\boldsymbol{h}}(h_j),$$

and

$$y_n^* = v'_{l_{\alpha n}} \in O_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}) \subset W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}).$$

Precisely as in Lemma 2 we may get

$$W_{\boldsymbol{h}}(y_{\alpha_1}, l_{\alpha_1}) \cup \cdots \cup W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}) \subset O^*$$

for some $O^* \in \zeta_{\boldsymbol{h}}(l)$, where $l = \min\{l_{\alpha_1}, \dots, l_{\alpha_n}\}$,

$$O^* \subset V_{\boldsymbol{h}}(x, l) = M_{m_{\boldsymbol{h}}(x,l)}(x) \backslash \overline{\cup \{M_{m_{\boldsymbol{h}}(x,l)}(y) : y \notin U_{\boldsymbol{h}}(x, l)\}},$$

$$c_{\boldsymbol{h}}(O^*, l) = m_{\boldsymbol{h}}(x, l)$$

for some $x \in X$, and show that $g_i \in U_{\boldsymbol{h}}(x, l)$.

As $y_0^* \in W_{\boldsymbol{h}}(y_{\alpha_1}, l_{\alpha_1}) \subset O^*$ and

$$y_n^* \in W_{\boldsymbol{h}}(y_{\alpha_n}, l_{\alpha_n}) \cap V_{\boldsymbol{h}}(h_j) \subset O^* \cap V_{\boldsymbol{h}}(h_j) \neq \varnothing,$$

conditions $O^* \in \zeta_{\boldsymbol{h}}(l)$ and (3^{ζ}) imply that

$$y_0^* \in U_{\boldsymbol{h}}(h_j) \in \gamma_{\boldsymbol{h}}(l(\boldsymbol{h}) + 1);$$

we also have $h_j \in U_{\boldsymbol{h}}(h_j)$. On the other hand,

$$g_i \in U_{\boldsymbol{h}}(x, l),$$
$$y_0^* \in O^* \subset V_{\boldsymbol{h}}(x, l) \subset U_{\boldsymbol{h}}(x, l),$$

and

$$U_{\boldsymbol{h}}(x, l) \in \gamma_{\boldsymbol{h}}(l + 1).$$

As, by (2), $\gamma_{\boldsymbol{h}}(l + 1) \circ \gamma_{\boldsymbol{h}}(l(\boldsymbol{h}) + 1)$ is a refinement of $\gamma_{\boldsymbol{h}}(\min\{l, l(\boldsymbol{h})\})$, this proves the lemma for $\mathrm{A} \neq \varnothing$.

Now assume that $\mathrm{A} = \varnothing$, i.e., for all $t \in J$

$$\mu_{\boldsymbol{h}}(y_t, l_t) > m_{\boldsymbol{g}}.$$

For different s, $\mu_{\boldsymbol{g}}(x_s, k_s)$ are different, and for different t, $\mu_{\boldsymbol{h}}(y_t, l_t)$ are different, because they are powers of different prime numbers; besides, all $\mu_{\boldsymbol{g}}(x_s, k_s)$ are greater than $m_{\boldsymbol{g}}$ by

$(2^{\mathcal{M}})$. Remark 1 and condition $(1^{\mathcal{M}})$ imply that there exists $y^* \in X$ for which

$$g_i, h_j \in M_{m_{\boldsymbol{g}}}(y^*).$$

We have:

$$g_i \in O_{\boldsymbol{g}}(g_i) = M_{m_{\boldsymbol{g}}}(g_i) \setminus \overline{\bigcup \{M_{m_{\boldsymbol{g}}}(x) : x \notin W_{\boldsymbol{g}}(g_i)\}},$$

therefore,

$$y^* \in W_{\boldsymbol{g}}(g_i) \subset M_{n_{\boldsymbol{g}}}(g_i)$$

(otherwise $M_{m_{\boldsymbol{g}}}(y^*) \cap O_{\boldsymbol{g}}(g_i) = \varnothing$). Since $n_{\boldsymbol{g}} \geqslant n_{\boldsymbol{h}}$ (see (5)) and by $(2^{\mathcal{M}})$, $h_j \in M_{n_{\boldsymbol{h}}}(y^*)$. As

$$h_j \in O_{\boldsymbol{h}}(h_j) = M_{m_{\boldsymbol{h}}}(h_j) \setminus \overline{\bigcup \{M_{m_{\boldsymbol{h}}}(x) : x \notin W_{\boldsymbol{h}}(h_j)\}},$$

we have

$$y^* \in W_{\boldsymbol{h}}(h_j).$$

By $(2^{\mathcal{M}})$ and (5) $M_{n_{\boldsymbol{g}}}(g_i) \subset M_{n_{\boldsymbol{h}}}(g_i)$; hence,

$$y^* \in M_{n_{\boldsymbol{h}}}(g_i) \cap W_{\boldsymbol{h}}(h_j).$$

Assume that

$$g_i \notin V_{\boldsymbol{h}}(h_j).$$

Then, as

$$W_{\boldsymbol{h}}(h_j) = M_{n_{\boldsymbol{h}}}(h_j) \setminus \overline{\bigcup \{M_{n_{\boldsymbol{h}}}(x) : x \notin V_{\boldsymbol{h}}(h_j)\}},$$

we have

$$M_{n_{\boldsymbol{h}}}(g_i) \cap W_{\boldsymbol{h}}(h_j) = \varnothing$$

at variance with $y^* \in M_{n_{\boldsymbol{h}}}(g_i) \cap W_{\boldsymbol{h}}(h_j)$. Therefore,

$$g_i, h_j \in V_{\boldsymbol{h}}(h_j) \subset U_{\boldsymbol{h}}(h_j) \in \gamma_{\boldsymbol{h}}(l(\boldsymbol{h}) + 1) \prec \gamma_{\boldsymbol{h}}(l(\boldsymbol{h})),$$

which proves the lemma. □

Now let us prove the Principal Lemma. Let $\boldsymbol{g}, \boldsymbol{h} \in A(X)$ and

$$\begin{aligned} \boldsymbol{g} &= \varepsilon_1 g_1 + \cdots + \varepsilon_{l(\boldsymbol{g})} g_{l(\boldsymbol{g})}, \\ \boldsymbol{h} &= \delta_1 h_1 + \cdots + \delta_{l(\boldsymbol{h})} h_{l(\boldsymbol{h})}, \end{aligned}$$

where $\varepsilon_i, \delta_i = \pm 1$ and $g_i, h_i \in X$; if $l(\boldsymbol{g}) = 0$ ($l(\boldsymbol{h}) = 0$), we consider $\boldsymbol{g}$ (respectively, $\boldsymbol{h}$) to be the empty word. Let

$$\begin{aligned} \Gamma_{\boldsymbol{g}} &= \{\gamma_{\boldsymbol{g}}(n) : n \in \mathbb{N}^+\} \in \mathcal{G} \quad \text{and} \\ \Gamma_{\boldsymbol{h}} &= \{\gamma_{\boldsymbol{h}}(n) : n \in \mathbb{N}^+\} \in \mathcal{G}. \end{aligned}$$

Suppose that

$$f(\boldsymbol{g}, \Gamma_{\boldsymbol{g}}) = n_{\boldsymbol{g}} \geqslant n_{\boldsymbol{h}} = f(\boldsymbol{h}, \Gamma_{\boldsymbol{h}})$$

and

$$F(\boldsymbol{g}, \Gamma_{\boldsymbol{g}}) = \Theta_{\boldsymbol{g}} = \{\theta_{\boldsymbol{g}}(n) : n \in \mathbb{N}^+\},$$

$$F(\boldsymbol{h}, \Gamma_{\boldsymbol{h}}) = \Theta_{\boldsymbol{h}} = \{\theta_{\boldsymbol{h}}(n) : n \in \mathbb{N}^+\},$$

$$\boldsymbol{g} - \boldsymbol{h} \in U(\Theta_{\boldsymbol{g}}) + U(\Theta_{\boldsymbol{h}}).$$

The last inclusion means that there exist p and q and words $\boldsymbol{u}$ and $\boldsymbol{v}$ such that

$$\begin{aligned} \boldsymbol{u} &= u_1 - u_1' + \cdots + u_p - u_p' \in U(\Theta_{\boldsymbol{g}}), \\ \boldsymbol{v} &= v_1 - v_1' + \cdots + v_q - v_q' \in U(\Theta_{\boldsymbol{h}}), \end{aligned}$$

for all $i \leqslant p$, $j \leqslant q$

$$\begin{aligned} u_i, u_i' \in U_i &\quad \text{for some} \quad U_i \in \theta_{\boldsymbol{g}}(i), \\ v_j, v_j' \in V_j &\quad \text{for some} \quad V_j \in \theta_{\boldsymbol{h}}(j), \end{aligned}$$

and

$$\boldsymbol{g} - \boldsymbol{h} = \boldsymbol{u} + \boldsymbol{v}.$$

Words $\boldsymbol{u}$ and $\boldsymbol{v}$ may be empty.

If the word $\boldsymbol{u} + \boldsymbol{v}$ is empty, the assertion is trivial, because then $\boldsymbol{g} - \boldsymbol{h} = \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{0}$, and clearly, $\boldsymbol{g} - \boldsymbol{h} \in U(\Gamma_{\boldsymbol{h}})$. Suppose that $\boldsymbol{u} + \boldsymbol{v}$ is nonempty.

Consider the word

$$\boldsymbol{w} \equiv u_1 - u_1' + \cdots + u_p - u_p' + v_1 - v_1' + \cdots + v_q - v_q'$$

as a sequence of letters and not as a reduced word from $A(X)$; we use the symbol $\equiv$ to denote the equality of sequences of letters rather than the reduced forms of words. Fix some order of cancellations in this word to produce the reduced form $\widehat{\boldsymbol{w}}$ of the word $\boldsymbol{w} = \boldsymbol{u}+\boldsymbol{v}$. Consider a sequence of letters $\boldsymbol{w}'$ resulting from some number of the cancellations in the fixed order. We will now define a division of letters of $\boldsymbol{w}'$ into connected pairs and chains connecting letters into pairs by induction on the number of cancellations.

Let the number of cancellations be zero, i.e., $\boldsymbol{w}' \equiv \boldsymbol{w}$. The set of all letters in word $\boldsymbol{w}' \equiv \boldsymbol{w}$ is naturally divided into pairs $(u_i, -u_i')$ and $(v_j, -v_j')$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. We will call these pairs connected (in word $\boldsymbol{w}' \equiv \boldsymbol{w}$). We remind that when we speak of a letter of a word we speak of an element of alphabet $X \cup -X$ and its position in the word. For any pair of letters x and $-y$ connected in $\boldsymbol{w}' \equiv \boldsymbol{w}$ put $C_{\boldsymbol{w}'}(x, -y) \equiv x - y$. We will call the word $C_{\boldsymbol{w}'}(x, -y)$ a *chain* connecting letters x and $-y$ in $\boldsymbol{w}'$.

Let k be a nonnegative integer and for the word $\boldsymbol{w}''$ produced from $\boldsymbol{w}$ by k cancellations, the division of letters into connected pairs and connecting chains are defined. Let $\boldsymbol{w}'$ be the result of $k+1$ cancellations, and let the last cancellation involve letters x and $-x$. If the pair $(x, -x)$ is connected in $\boldsymbol{w}''$ then the set of connected pairs in $\boldsymbol{w}'$ comprises all pairs connected in $\boldsymbol{w}''$ except $(x, -x)$, and all chains connecting these pairs are already defined. If the letters x and $-x$ are not connected in $\boldsymbol{w}''$ then there exist distinct letters y and $-z$ in the word $\boldsymbol{w}''$ for which pairs $(y, -x)$ and $(x, -z)$ are connected. Define connected pairs in $\boldsymbol{w}'$ as all connected pairs in $\boldsymbol{w}''$ except $(y, -x)$ and $(x, -z)$, and add one connected pair $(y, -z)$. Chains connecting all pairs except $(y, -z)$ are the same as in the word $\boldsymbol{w}''$, and the chain connecting letters y and $-z$ in $\boldsymbol{w}'$ is

$$C_{\boldsymbol{w}'}(y, -z) \equiv C_{\boldsymbol{w}''}(y, -x) + C_{\boldsymbol{w}''}(x, -z)$$

(recall that here, we are treating words as unreduced sequences of letters).

The induction construction is completed.

Similar definitions of connected pairs of letters and chains connecting these letters were earlier introduced by M. G. Tkačenko for non-Abelian free groups.

Remark 2. Note that if two chains connect different pairs of letters in $\boldsymbol{w}$ then none of the letters of one chain is a letter of the other, and

$$\boldsymbol{w} \equiv \sum C_{\widehat{\boldsymbol{w}}}(a, -b),$$

where the summation is over all pairs $(a, -b)$ connected in $\widehat{\boldsymbol{w}}$, and $\widehat{\boldsymbol{w}}$ is the reduced form of $\boldsymbol{w}$. It is also clear that any chain is the sum of binomials $u_i - u_i'$ and $v_i - v_i'$. Indeed, all chains in $\boldsymbol{w}$ are such binomials themselves, and at every induction step, chains are produced as the sums of chains at the previous induction step. Therefore, every chain in $\widehat{\boldsymbol{w}}$ has the form

$$u_{k_1} - u'_{k_1} + \cdots + u_{k_s} - u'_{k_s} + v_{l_1} - v'_{l_1} + \ldots v_{l_t} - v'_{l_t}.$$

It is also evident that for every connected pair $(a, -b)$ in $\widehat{\boldsymbol{w}}$, the reduced forms of words $a - b$ and $C_{\widehat{\boldsymbol{w}}}(a, -b)$ coincide.

After the realization of all the cancellations fixed in $\boldsymbol{w}$, we obtain the reduced form of the word $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{g} - \boldsymbol{h}$. From this point on, we will again treat words as elements of $A(X)$, i.e., as their reduced forms. The word $\boldsymbol{u} + \boldsymbol{v}$ is representable as the sum of letters εx, where $\varepsilon = \pm 1$ and $x \in X$. Clearly, the number of letters εx with positive ε's in this sum is equal to the number of εx with negative ε's.

As $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{g} - \boldsymbol{h}$, there is an one-to-one correspondence between letters of $\boldsymbol{u} + \boldsymbol{v}$ and those of $\boldsymbol{g} - \boldsymbol{h}$. Of course, this correspondence may be not unique, if the word $\boldsymbol{u} + \boldsymbol{v}$ (and, therefore, $\boldsymbol{g} - \boldsymbol{h}$) has several letters such that are identical as elements of $X \cup -X$. Fix some one-to-one correspondence between letters of $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{g} - \boldsymbol{h}$.

The word $\boldsymbol{g} - \boldsymbol{h}$ is produced from the sequence of letters

$$\varepsilon_1 g_1 + \cdots + \varepsilon_{l(\boldsymbol{g})} g_{l(\boldsymbol{g})} - \delta_1 h_1 + \cdots - \delta_{l(\boldsymbol{h})} h_{l(\boldsymbol{h})} \tag{22}$$

as the result of cancellations. Therefore, there is a correspondence between the letters of $\boldsymbol{g} - \boldsymbol{h}$ and those letters of (22) that do not appear in the cancellations. This correspondence, as well as cancellations themselves, is also not unique; fix some.

In the following we will always mean these fixed correspondences, e.g., we will say that a letter of $\boldsymbol{u}+\boldsymbol{v}$ is $\varepsilon_i g_i$ for some $i \leqslant l(\boldsymbol{g})$, and so on.

Number all letters of word $\boldsymbol{u}+\boldsymbol{v}$ with positive ε's, i.e., letters which belong to X, in the following way. Consider letters of $\boldsymbol{u}-\boldsymbol{v}$ as letters of $\boldsymbol{g}-\boldsymbol{h}$, and reserve the first numbers for letters which have the form $-\delta_i h_i$ or are connected with letters of this form (in $\boldsymbol{u}-\boldsymbol{v}$). Number the remaining letters arbitrarily. For every possible n denote such nth letter as x_n and the letter connected with x_n in $\boldsymbol{u}+\boldsymbol{v}$ as $-y_n$. Thus all letters of the word $\boldsymbol{u}+\boldsymbol{v}$ are divided into pairs of the form $(x_n, -y_n)$. By Remark 2 for every n

$$x_n - y_n = C_{\boldsymbol{u}+\boldsymbol{v}}(x_n, -y_n)$$

(as usual, $=$ means the equality of reduced forms), and

$$\begin{aligned}(23)\quad & C_{\boldsymbol{u}+\boldsymbol{v}}(x_n, -y_n) \\ &= u_{k_{1,n}} - u'_{k_{1,n}} + u_{k_{2,n}} - u'_{k_{2,n}} + \cdots + u_{k_{p(n),n}} - u'_{k_{p(n),n}} \\ &\quad + v_{l_{1,n}} - v'_{l_{1,n}} + v_{l_{2,n}} - v'_{l_{2,n}} + \cdots + v_{l_{q(n),n}} - v'_{l_{q(n),n}},\end{aligned}$$

where $k_{1,n}, \ldots, k_{p(n),n}$ are different positive integers not exceeding p, and $l_{1,n}, \ldots, l_{q(n),n}$ are different positive integers not exceeding q. Remark 2 also implies that for different n, sets of numbers $\{k_{1,n}, \ldots, k_{p(n),n}\}$ (and $\{l_{1,n}, \ldots, l_{q(n),n}\}$) do not intersect, because all letters in different chains are different.

It is easy to see that

$$x_n - y_n = C_{\boldsymbol{u}+\boldsymbol{v}}(x_n, -y_n) = (**),$$

where $(**)$ stands for one of the four words:

(a)

$$\begin{aligned}
& u_{k_{i_0(n)+1,n}} - u'_{k_{i_0(n)+1,n}} + u_{k_{i_0(n)+2,n}} - u'_{k_{i_0(n)+2,n}} \\
& \quad + \cdots + u_{k_{i_1(n)-1,n}} - u'_{k_{i_1(n)-1,n}} + u_{k_{i_1(n),n}} - u'_{k_{i_1(n),n}} \\
+ \ & v_{l_{j_0(n)+1,n}} - v'_{l_{j_0(n)+1,n}} + v_{l_{j_0(n)+2,n}} - v'_{l_{j_0(n)+2,n}} \\
& \quad + \cdots + v_{l_{j_1(n)-1,n}} - v'_{l_{j_1(n)-1,n}} + v_{l_{j_1(n),n}} - v'_{l_{j_1(n),n}} \\
+ \ & u_{k_{i_1(n)+1,n}} - u'_{k_{i_1(n)+1,n}} + u_{k_{i_1(n)+2,n}} - u'_{k_{i_1(n)+2,n}} \\
& \quad + \cdots + u_{k_{i_2(n)-1,n}} - u'_{k_{i_2(n)-1,n}} + u_{k_{i_2(n),n}} - u'_{k_{i_2(n),n}} \\
+ \ & v_{l_{j_1(n)+1,n}} - v'_{l_{j_1(n)+1,n}} + v_{l_{j_1(n)+2,n}} - v'_{l_{j_1(n)+2,n}} \\
& \quad + \cdots + v_{l_{j_2(n)-1,n}} - v'_{l_{j_2(n)-1,n}} + v_{l_{j_2(n),n}} - v'_{l_{j_2(n),n}} \\
& \cdots \\
+ \ & u_{k_{i_{\nu_n-1}(n)+1,n}} - u'_{k_{i_{\nu_n-1}(n)+1,n}} \\
& \quad + \cdots + u_{k_{i_{\nu_n}(n)-1,n}} - u'_{k_{i_{\nu_n}(n)-1,n}} + u_{k_{i_{\nu_n}(n),n}} - u'_{k_{i_{\nu_n}(n),n}} \\
+ \ & v_{l_{j_{\nu_n-1}(n)+1,n}} - v'_{l_{j_{\nu_n-1}(n)+1,n}} \\
& \quad + \cdots + v_{l_{j_{\nu_n}(n)-1,n}} - v'_{l_{j_{\nu_n}(n)-1,n}} + v_{l_{j_{\nu_n}(n),n}} - v'_{l_{j_{\nu_n}(n),n}},
\end{aligned}$$

where

(1a**)

$$i_0(n) = j_0(n) = 0, \quad i_{\nu_n}(n) = p(n), \quad j_{\nu_n}(n) = q(n),$$

and numbers $k_{\alpha,n}$ and $l_{\beta,n}$ are the same as in (23). In particular, all numbers $k_{\alpha,n}$ are different from each other, and numbers $l_{\beta,n}$ are also so;

(b)

$$\begin{aligned}
& v_{l_{j_0(n)+1,n}} - v'_{l_{j_0(n)+1,n}} + v_{l_{j_0(n)+2,n}} - v'_{l_{j_0(n)+2,n}} \\
& \quad + \cdots + v_{l_{j_1(n)-1,n}} - v'_{l_{j_1(n)-1,n}} + v_{l_{j_1(n),n}} - v'_{l_{j_1(n),n}} \\
+ \ & u_{k_{i_1(n)+1,n}} - u'_{k_{i_1(n)+1,n}} + u_{k_{i_1(n)+2,n}} - u'_{k_{i_1(n)+2,n}} \\
& \quad + \cdots + u_{k_{i_2(n)-1,n}} - u'_{k_{i_2(n)-1,n}} + u_{k_{i_2(n),n}} - u'_{k_{i_2(n),n}} \\
+ \ & v_{l_{j_1(n)+1,n}} - v'_{l_{j_1(n)+1,n}} + v_{l_{j_1(n)+2,n}} - v'_{l_{j_1(n)+2,n}} \\
& \quad + \cdots + v_{l_{j_2(n)-1,n}} - v'_{l_{j_2(n)-1,n}} + v_{l_{j_2(n),n}} - v'_{l_{j_2(n),n}} \\
& \cdots \\
+ \ & u_{k_{i_{\nu_n-1}(n)+1,n}} - u'_{k_{i_{\nu_n-1}(n)+1,n}} \\
& \quad + \cdots + u_{k_{i_{\nu_n}(n)-1,n}} - u'_{k_{i_{\nu_n}(n)-1,n}} + u_{k_{i_{\nu_n}(n),n}} - u'_{k_{i_{\nu_n}(n),n}} \\
+ \ & v_{l_{j_{\nu_n-1}(n)+1,n}} - v'_{l_{j_{\nu_n-1}(n)+1,n}} \\
& \quad + \cdots + v_{l_{j_{\nu_n}(n)-1,n}} - v'_{l_{j_{\nu_n}(n)-1,n}} + v_{l_{j_{\nu_n}(n),n}} - v'_{l_{j_{\nu_n}(n),n}},
\end{aligned}$$

where

(1b**)

$$j_0(n) = i_1(n) = 0, \quad i_{\nu_n}(n) = p(n), \quad j_{\nu_n}(n) = q(n),$$

and numbers $k_{\alpha,n}$ and $l_{\beta,n}$ are the same as in (23);

(c)

$$
\begin{aligned}
& u_{k_{i_0(n)+1,n}} - u'_{k_{i_0(n)+1,n}} + u_{k_{i_0(n)+2,n}} - u'_{k_{i_0(n)+2,n}} \\
& \qquad + \cdots + u_{k_{i_1(n)-1,n}} - u'_{k_{i_1(n)-1,n}} + u_{k_{i_1(n),n}} - u'_{k_{i_1(n),n}} \\
+ \quad & v_{l_{j_0(n)+1,n}} - v'_{l_{j_0(n)+1,n}} + v_{l_{j_0(n)+2,n}} - v'_{l_{j_0(n)+2,n}} \\
& \qquad + \cdots + v_{l_{j_1(n)-1,n}} - v'_{l_{j_1(n)-1,n}} + v_{l_{j_1(n),n}} - v'_{l_{j_1(n),n}} \\
+ \quad & u_{k_{i_1(n)+1,n}} - u'_{k_{i_1(n)+1,n}} + u_{k_{i_1(n)+2,n}} - u'_{k_{i_1(n)+2,n}} \\
& \qquad + \cdots + u_{k_{i_2(n)-1,n}} - u'_{k_{i_2(n)-1,n}} + u_{k_{i_2(n),n}} - u'_{k_{i_2(n),n}} \\
+ \quad & v_{l_{j_1(n)+1,n}} - v'_{l_{j_1(n)+1,n}} + v_{l_{j_1(n)+2,n}} - v'_{l_{j_1(n)+2,n}} \\
& \qquad + \cdots + v_{l_{j_2(n)-1,n}} - v'_{l_{j_2(n)-1,n}} + v_{l_{j_2(n),n}} - v'_{l_{j_2(n),n}} \\
& \cdots \\
+ \quad & u_{k_{i_{\nu_n-2}(n)+1,n}} - u'_{k_{i_{\nu_n-2}(n)+1,n}} \\
& \qquad + \cdots + u_{k_{i_{\nu_n-1}(n)-1,n}} - u'_{k_{i_{\nu_n-1}(n)-1,n}} \\
& \qquad + u_{k_{i_{\nu_n-1}(n),n}} - u'_{k_{i_{\nu_n-1}(n),n}} \\
+ \quad & v_{l_{j_{\nu_n-2}(n)+1,n}} - v'_{l_{j_{\nu_n-2}(n)+1,n}} \\
& \qquad + \cdots + v_{l_{j_{\nu_n-1}(n)-1,n}} - v'_{l_{j_{\nu_n-1}(n)-1,n}} \\
& \qquad + v_{l_{j_{\nu_n-1}(n),n}} - v'_{l_{j_{\nu_n-1}(n),n}} \\
+ \quad & u_{k_{i_{\nu_n-1}(n)+1,n}} - u'_{k_{i_{\nu_n-1}(n)+1,n}} \\
& \qquad + \cdots + u_{k_{i_{\nu_n}(n)-1,n}} - u'_{k_{i_{\nu_n}(n)-1,n}} \\
& \qquad + u_{k_{i_{\nu_n}(n),n}} - u'_{k_{i_{\nu_n}(n),n}},
\end{aligned}
$$

where
(1c**)

$$i_0(n) = j_0(n) = 0, \quad i_{\nu_n}(n) = p(n), \quad j_{\nu_n-1}(n) = q(n),$$

and numbers $k_{\alpha,n}$ and $l_{\beta,n}$ are the same as in (23);

(d)

$$
\begin{aligned}
& v_{l_{j_0(n)+1,n}} - v'_{l_{j_0(n)+1,n}} + v_{l_{j_0(n)+2,n}} - v'_{l_{j_0(n)+2,n}} \\
& \qquad + \cdots + v_{l_{j_1(n)-1,n}} - v'_{l_{j_1(n)-1,n}} + v_{l_{j_1(n),n}} - v'_{l_{j_1(n),n}} \\
+ \ & u_{k_{i_1(n)+1,n}} - u'_{k_{i_1(n)+1,n}} + u_{k_{i_1(n)+2,n}} - u'_{k_{i_1(n)+2,n}} \\
& \qquad + \cdots + u_{k_{i_2(n)-1,n}} - u'_{k_{i_2(n)-1,n}} + u_{k_{i_2(n),n}} - u'_{k_{i_2(n),n}} \\
+ \ & v_{l_{j_1(n)+1,n}} - v'_{l_{j_1(n)+1,n}} + v_{l_{j_1(n)+2,n}} - v'_{l_{j_1(n)+2,n}} \\
& \qquad + \cdots + v_{l_{j_2(n)-1,n}} - v'_{l_{j_2(n)-1,n}} + v_{l_{j_2(n),n}} - v'_{l_{j_2(n),n}} \\
& \ldots \\
+ \ & u_{k_{i_{\nu_n-2}(n)+1,n}} - u'_{k_{i_{\nu_n-2}(n)+1,n}} \\
& \qquad + \cdots + u_{k_{i_{\nu_n-1}(n)-1,n}} - u'_{k_{i_{\nu_n-1}(n)-1,n}} \\
& \qquad + u_{k_{i_{\nu_n-1}(n),n}} - u'_{k_{i_{\nu_n-1}(n),n}} \\
+ \ & v_{l_{j_{\nu_n-2}(n)+1,n}} - v'_{l_{j_{\nu_n-2}(n)+1,n}} \\
& \qquad + \cdots + v_{l_{j_{\nu_n-1}(n)-1,n}} - v'_{l_{j_{\nu_n-1}(n)-1,n}} \\
& \qquad + v_{l_{j_{\nu_n-1}(n),n}} - v'_{l_{j_{\nu_n-1}(n),n}} \\
+ \ & u_{k_{i_{\nu_n-1}(n)+1,n}} - u'_{k_{i_{\nu_n-1}(n)+1,n}} \\
& \qquad + \cdots + u_{k_{i_{\nu_n}(n)-1,n}} - u'_{k_{i_{\nu_n}(n)-1,n}} \\
& \qquad + u_{k_{i_{\nu_n}(n),n}} - u'_{k_{i_{\nu_n}(n),n}},
\end{aligned}
$$

where
(1d**)

$$j_0(n) = i_1(n) = 0, \quad i_{\nu_n}(n) = p(n), \quad j_{\nu_n-1}(n) = q(n),$$

and numbers $k_{\alpha,n}$ and $l_{\beta,n}$ are the same as in (23).

In every case

(2**) x_n is equal to the first letter of the word $(**)$, $-y_n$ is equal to the last letter of $(**)$, and every odd letter in $(**)$, except for the first one, is cancelled with the letter nearest on the left.

In case (b) put

$$i_0(n) = -1, \quad k_{0,n} = k_{i_0(n)+1,n} = p + n, \quad \text{and}$$

$$u_{k_{0,n}} = u_{k_{i_0(n)+1,n}} = u'_{k_{0,n}} = u'_{k_{i_0(n)+1,n}} = v_{l_{j_0(n)+1,n}}.$$

Then

$$u_{k_{0,n}} = u'_{k_{0,n}} \in U_{k_{0,n}} \text{ for some } U_{k_{0,n}} \in \theta_{\boldsymbol{g}}(k_{0,n}) = \theta_{\boldsymbol{g}}(p+n),$$

$$x_n - y_n = u_{k_{0,n}} - u'_{k_{0,n}} + C_{\boldsymbol{u}+\boldsymbol{v}}(x_n, -y_n) =$$

$$u_{k_{i_0(n)+1,n}} - u'_{k_{i_0(n)+1,n}} + (**),$$

and condition (2^{**}) with $u_{k_{i_0(n)+1,n}} - u'_{k_{i_0(n)+1,n}} + (**)$ instead of $(**)$ is fulfilled.

In case (c) put

$$j_{\nu_n}(n) = q(n) + 1, \; l_{q(n)+1,n} = l_{j_{\nu_n}(n),n} = q + n, \quad \text{and}$$

$$v_{l_{q(n)+1,n}} = v_{l_{j_{\nu_n}(n),n}} = v'_{l_{q(n)+1,n}} = v'_{l_{j_{\nu_n}(n),n}} = u'_{k_{i_{\nu_n}(n),n}}.$$

Then

$$v_{l_{q(n)+1,n}} = v'_{l_{q(n)+1,n}} \in U_{l_{q(n)+1,n}} \quad \text{for some}$$

$$U_{l_{q(n)+1,n}} \in \theta_{\boldsymbol{h}}(l_{q(n)+1,n}) = \theta_{\boldsymbol{h}}(q+n), x_n - y_n =$$

$$C_{\boldsymbol{u}+\boldsymbol{v}}(x_n, -y_n) + v_{l_{q(n)+1,n}} - v'_{l_{q(n)+1,n}} =$$

$$(**) + v_{l_{q(n)+1,n}} - v'_{l_{q(n)+1,n}},$$

and condition (2^{**}) with $(**) + v_{l_{q(n)+1,n}} - v'_{l_{q(n)+1,n}}$ instead of $(**)$ is fulfilled.

Analogously, in case (d) put

$$i_0(n) = -1, \quad k_{0,n} = k_{i_0(n)+1,n} = p + n,$$

$$j_{\nu_n}(n) = q(n) + 1, \quad l_{q(n)+1,n} = l_{j_{\nu_n}(n),n} = q + n,$$

and

$$u_{k_{0,n}} = u_{k_{i_0(n)+1,n}} = u'_{k_{0,n}} = u'_{k_{i_0(n)+1,n}} = v_{l_{j_0(n)+1,n}},$$

$$v_{l_{q(n)+1,n}} = v_{l_{j_{\nu_n}(n),n}} = v'_{l_{q(n)+1,n}} = v'_{l_{j_{\nu_n}(n),n}} = u'_{k_{i_{\nu_n}(n),n}}.$$

Then

$$u_{k_{0,n}} = u'_{k_{0,n}} \in U_{k_{0,n}} \text{for some}$$

$$U_{k_{0,n}} \in \theta_g(k_{0,n}) = \theta_g(p+n),$$

$$v_{l_{q(n)+1,n}} = v'_{l_{q(n)+1,n}} \in U_{l_{q(n)+1,n}} \text{for some}$$

$$U_{l_{q(n)+1,n}} \in \theta_h(l_{q(n)+1,n}) = \theta_h(q+n),$$

and

$$x_n - y_n = u_{k_{0,n}} - u'_{k_{0,n}} + C_{u+v}(x_n, -y_n) + v_{l_{q(n)+1,n}} - v'_{l_{q(n)+1,n}}$$

$$= u_{k_{i_0(n)+1,n}} - u'_{k_{i_0(n)+1,n}} + (**) + v_{l_{q(n)+1,n}} - v'_{l_{q(n)+1,n}},$$

and condition (2^{**}) with $u_{k_{i_0(n)+1,n}} - u'_{k_{i_0(n)+1,n}} + (**) + v_{l_{q(n)+1,n}} - v'_{l_{q(n)+1,n}}$ instead of $(**)$ is fulfilled.

Remark 3. Note that as $k_{0,n} = p+n$ and $l_{q(n)+1,n} = q+n$, every number of the form $k_{0,n}$ (if defined) is not equal to $k_{i,m}$ for $m \neq n$ and $i \geqslant 0$ or for $m = n$ and $i \geqslant 1$, because $p+n \neq p+m$ for $n \neq m$, and $k_{i,m} \leqslant p$ for $i \geqslant 1$. Analogously, every number of the form $l_{q(n)+1,n}$ (if defined) is not equal to $l_{i,m}$ for $m \neq n$ and $i \leqslant q(n)+1$ or for $m = n$ and $i \leqslant q(n)$. Therefore, we may assume that for every n $x_n - y_n$ is representable in the form $(**)$, case (a), with observing ($1a^{**}$) and (2^{**}), $k_{i_\alpha(n)+\beta,n} \neq k_{i_{\alpha'}(n')+\beta',n'}$ and $l_{j_\alpha(n)+\beta,n} \neq l_{j_{\alpha'}(n')+\beta',n'}$, if either $\alpha \neq \alpha'$, or $\beta \neq \beta'$, or $n \neq n'$, and

$$u_{k_{i_\alpha(n)+\beta,n}}, u'_{k_{i_\alpha(n)+\beta,n}} \in U_{i_\alpha(n)+\beta,n} \text{ for some}$$

$$U_{i_\alpha(n)+\beta,n} \in \theta_g(k_{i_\alpha(n)+\beta,n}),$$

$$v_{l_{j_\alpha(n)+\beta,n}}, v'_{l_{j_\alpha(n)+\beta,n}} \in V_{j_\alpha(n)+\beta,n} \text{ for some}$$

$$V_{j_\alpha(n)+\beta,n} \in \theta_{\boldsymbol{h}}(l_{j_\alpha(n)+\beta,n}).$$

We remind that any x_n and $-y_n$ are the letters of the word $\boldsymbol{g} - \boldsymbol{h}$, as $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{g} - \boldsymbol{h}$, i.e., one of the four possibilities is realized:

(i) x_n is $-\delta_{i(n)}h_{i(n)}$ and $-y_n$ is $-\delta_{j(n)}h_{j(n)}$ for some $i(n), j(n) \leqslant l(\boldsymbol{h})$; as x_n, y_n, $h_{i(n)}$, and $h_{j(n)}$ are the elements of X, clearly, $-\delta_{i(n)} = \delta_{j(n)} = 1$, and $x_n = h_{i(n)}$, $y_n = h_{j(n)}$;
(ii) x_n is $\varepsilon_{i(n)}g_{i(n)} = g_{i(n)}$ and $-y_n$ is $\varepsilon_{j(n)}g_{j(n)} = -g_{j(n)}$ for some $i(n), j(n) \leqslant l(\boldsymbol{g})$;
(iii) x_n is $\varepsilon_{i(n)}g_{i(n)} = g_{i(n)}$ and $-y_n$ is $-\delta_{j(n)}h_{j(n)} = -h_{j(n)}$ for some $i(n) \leqslant l(\boldsymbol{g})$, $j(n) \leqslant l(\boldsymbol{h})$;
(iv) x_n is $-\delta_{j(n)}h_{j(n)} = h_{j(n)}$ and $-y_n$ is $\varepsilon_{i(n)}g_{i(n)} = -g_{i(n)}$ for some $j(n) \leqslant l(\boldsymbol{h})$, $i(n) \leqslant l(\boldsymbol{g})$.

Note that (**), case (a), is the same as (*), and conditions (1a**) and (2**) are the same as (1*) and (2*). Hence by Remark 3 we can use Lemmas 1–3.

By Lemma 1, possibility (i) should be ruled out. By Lemma 2, in case (ii) we have

$$x_n = g_{i(n)}, y_n = g_{j(n)} \in U_{i(n)j(n)}, \tag{24}$$

$$U_{i(n)j(n)} \in \gamma_{\boldsymbol{h}}(l_{j_\alpha(n)+\beta,n}) \quad \text{for some } \alpha \text{ and } \beta$$

By Lemma 3, in cases (iii) and (iv) we have (24) or

$$y_n = g_{i(n)}, x_n = h_{j(n)} \in U_{i(n)j(n)}, \tag{25}$$

$$U_{i(n)j(n)} \in \gamma_{\boldsymbol{h}}(l(\boldsymbol{h})).$$

The number of pairs $(x_n, -y_n)$ for which possibilities (iii) and (iv) may be realized is not greater than $l(\boldsymbol{h})$, and the method of numbering letters x_n (the first numbers are reserved for letters of the form $-\delta_i h_i$ and for the letters connected with them) implies that in cases (iii) and (iv) $n \leqslant l(\boldsymbol{h})$. Therefore, condition (2) allows us to write $U_{i(n)j(n)} \in \gamma_{\boldsymbol{h}}(n)$ instead of

$U_{i(n)j(n)} \in \gamma_{\boldsymbol{h}}(l(\boldsymbol{h}))$. As $\boldsymbol{g} - \boldsymbol{h} = \boldsymbol{u} + \boldsymbol{v}$, all the letters of the reduced word $\boldsymbol{g} - \boldsymbol{h}$ are divided into pairs of the form $(x_n, -y_n)$. Therefore, we have represented word $\boldsymbol{g} - \boldsymbol{h}$ as

$$\sum_{n \in N_1} x_n - y_n + \sum_{n \in N_2} x_n - y_n,$$

where

$$\begin{aligned} N_1 &= \{n : \text{ for } (x_n, -y_n) \ (24) \text{ is realized}\}, \\ N_2 &= \{n : \text{ for } (x_n, -y_n) \ (25) \text{ is realized}\}. \end{aligned}$$

For every $n \in N_1$

$$x_n, y_n \in U_{i(n)j(n)} \in \gamma_{\boldsymbol{h}}(l_{j_\alpha(n)+\beta,n}),$$

and by Remark 3 numbers $l_{j_\alpha(n)+\beta,n}$ are different from each other for different n. It was already shown that for every $n \in N_2$

$$x_n, y_n \in U_{i(n)j(n)} \in \gamma_{\boldsymbol{h}}(n).$$

Thus,

$$\sum_{n \in N_1} x_n - y_n \in U(\Gamma_{\boldsymbol{h}})$$

(see (1)), and

$$\sum_{n \in N_2} x_n - y_n \in U(\Gamma_{\boldsymbol{h}}).$$

Hence,

$$\boldsymbol{g} - \boldsymbol{h} \in 2U(\Gamma_{\boldsymbol{h}}).$$

The Principal Lemma and, therefore, the theorem is proved. □

Corollary. *Every stratifiable T_1 space can be embedded into an Abelian stratifiable T_0 group as a closed subspace.*

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REFERENCES

1. M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR. Ser. Mat., **12** (1948), 279–324 (Russian); English transl. in: Translat. Amer. Math. Soc., **8** no. 1 (1962), 305–364.
2. J. Ceder, *Some generalizations of metric spaces*, Pacific J. Math., **11** (1961), 105–125.
3. C. R. Borges, *On stratifiable spaces*, Pacific J. Math., **17** (1966), 1–16.
4. G. Gruenhage, *Generalized metric spaces*, Handbook of Set-Theoretic Topology, (K. Kunen, J. E. Vaughan, eds), North-Holland, Amsterdam–N.-Y.–Oxford–Tokyo, 1984, 423–501.
5. A. V. Arkhangel'skii, *Classes of topological groups*, Uspekhi Mat. Nauk, **36** no. 3 (1981), 127–146 (Russian); English transl. in: Russian Math. Surveys, **36** no 3 (1981), 151–174.
6. C. J. R. Borges, *Free topological groups*, J. Austral. Math. Soc., Ser. A, **23** no. 3 (1977), 360–365.
7. A. A. Markov, *On free topological groups*, Doklady Akad. Nauk SSSR **31** (1941), 299–301 (Russian).
8. —, *On free topological groups*, Izv. Akad. Nauk SSSR. Ser. Mat., **9** (1945), 3–64 (Russian); English Transl. in: Translat. Amer. Math. Soc., **8** no. 1 (1962), 195–272.
9. R. Engelking, *General Topology*, PWN, Warszaw, 1977.
10. A. V. Arhangel'skii, *Mappings related to topological groups*, Soviet Math. Dokl., **9** (1968), 1011–1015.
11. M. G. Tkačenko, *On topologies of free groups*, Czechoslov. Math. J., **134** no. 4 (1984), 541–551.
12. —, *On completeness of free Abelian toplogical groups*, Soviet Math. Dokl., **27** no. 2 (1983), 341–345.
13. V. G. Pestov, *Neighborhoods of identity in free topological groups*, Vestnik Mosk. Univ. Ser. 1. Mat. Mech., no. 3 (1985), 8–10. (Russian); English transl. in: Moscow Univ. Math. Bull., **40** no. 3 (1985), 8–12.
14. R. W. Heath, R. E. Hodel, *Characterizations of σ-spaces*, Fund. Math., **77** no. 3 (1973), 271–275.

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