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NOT ALL PSEUDO-OPEN MAPS ARE COMPOSITIONS OF CLOSED MAPS AND OPEN MAPS

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ABSTRACT. We give examples of properties preserved by inductively open-compact maps and inductively perfect maps, but which are not preserved by pseudo-open compact maps. It follows that not all pseudo-open compact maps are compositions of inductively open-compact maps and inductively perfect maps.

All maps considered in this paper are continuous and surjective. All spaces are T_1. We write $f^{-1}y$ for the inverse image of a point y.

1. Definitions: Let X, Y be topological spaces, and let $f : X \to Y$. The map f is *pseudo-open* iff $\forall y \in Y \forall$ open $U(f^{-1}y \subset U \Rightarrow y \in \text{int}(f[U]))$.

The class of pseudo-open maps contains all open maps and all closed maps. Moreover, the class of pseudo-open maps is closed under compositions. Therefore, if $f = f_k \circ f_{k-1} \circ \ldots \circ f_0$ for some $k \in \omega$, and if each of the maps f_i is either open or closed, then f is pseudo-open.

Since the introduction of the class of pseudo-open maps in [A1], a number of significant results on this class of maps have been obtained. Let us mention but a few of them: Arhangel'skiĭ ([A1], [A3]) characterized Frechet-Urysohn spaces as pseudo-open images of metrizable spaces. Čoban proved that paracompact images of metrizable spaces under pseudo-open compact maps are metrizable (see [A4]), and Burke [Bu] showed that images of paracompact spaces under pseudo-open compact maps are metacompact. Nevertheless, it appears that there are no known examples of pseudo-open maps which have been shown not to be a composition of closed maps and of open maps. The problem of finding such examples boils down to finding a property of maps that is preserved under compositions, is common to both closed and open maps, but is not shared by pseudo-open maps in general. In this note we give examples of properties that can be used to distinguish pseudo-open maps from compositions of closed maps and open maps, and we construct examples of pseudo-open maps which are not compositions of closed maps and open maps.

2. Definition: Let X, Y be topological spaces, and let $f : X \to Y$. The map f is *compact* iff $f^{-1}y$ is a compact subspace of X for every $y \in Y$.

Note that closed, compact maps are known under the name *perfect maps.*

3. Claim: *Let $f : X \to Y$, $g : Y \to Z$, and $h = g \circ f$. If h is compact, then so are f and g. If each fiber of h is finite or is an infinite convergent sequence, then the same is true for f and g.*

Proof: Let f, g, h be as in the assumptions.

First we show that f is compact. Let $y \in Y$, and let $z = g(y)$. By compactness of h, the set $h^{-1}z$ is a compact subset of X. Moreover, $f^{-1}y$ is a closed subset of the compact set $h^{-1}z$, hence compact.

Now we show that g is compact. Let $z \in Z$. Then $g^{-1}z$ is the image of the compact set $h^{-1}z$ under the continuous function f, hence compact.

This proves the first part of the claim. The proof of the second part is similar.

4. Definition: We call a space X a *CD-space* iff $X = Y \oplus Z$, where Y is an infinite discrete subspace of X and Z is connected and compact.

Note that a Hausdorff space X is a CD-space iff there is a connected compact subspace Z of X such that $X \setminus Z$ is an infinite closed discrete subspace of X.

EXAMPLE 1: Let I be the unit segment with the usual topology, and let Y be an infinite discrete space of cardinality $\leq 2^{\aleph_0}$ (disjoint from I). Put $X = Y \oplus I$. Then X is a CD-space.

We will use Theorem 5 and Corollary 6 (proved below) in discussing Example 1.

5. Theorem: *If X is a CD-space, and f is an open compact mapping or a perfect mapping of X onto Y, then Y is also a CD-space.*

6. Corollary: *Let f be a compact mapping of a CD-space X onto a space Y which is not a CD-space. Then f is not a composition of any finite sequence of mappings each of which is either open or closed.*

Example 1 continued: We fix a one-to-one mapping g of Y into I, and define a mapping f of X onto I as follows: $f(x) = x$ for each $x \in I$, and $f(x) = g(x)$ for each $x \in Y$. Then f is a finite-to-one (and hence compact) mapping of the CD-space X onto the space I which is not a CD-space. By Theorem 5, f can not be represented as a composition of a finite sequence of mappings, each of which is open or closed. Nevertheless, f is pseudo-open, as follows from Observation 8 below.

7. Definition: Let $\mathcal{P}$ be a property of mappings. Let us say that a mapping f of a space X onto a space Y has property $\mathcal{P}$ (*strongly*) *inductively*, or that f is (*strongly*) *inductively* $\mathcal{P}$, if there is a (closed) subspace Z of X such that $f(Z) = Y$, and the restriction of f to Z has property $\mathcal{P}$. If $\mathcal{P}$ is the property of being open with compact fibers, then we say that the map is inductively *open-compact.*

The map f in Example 1 is inductively perfect, even inductively a homeomorphism. To see this, consider the restriction of f to I. The following observation implies that f is pseudo-open:

8. Observation: *Every inductively pseudo-open map is pseudo-open .*

Let us now prove Theorem 5. The proof naturally splits into two parts.

9. Lemma: *Let $X = Y \oplus Z$, where Y is an infinite discrete space and Z is an infinite connected compact space, and let g be a compact open mapping of X onto a space H such that $g[Z]$ is infinite. Then:*

(1) The sets $g[Y]$ and $g[Z]$ are disjoint; and

(2) $H = Y_1 \oplus Z_1$, where $Y_1 = g[Y]$ is an infinite discrete space and $Z_1 = g[Z]$ is an infinite connected compact space.

Proof: Take any point $y \in Y$. The set $\{y\}$ is open in X. Hence the set $\{g(y)\}$ is open in H, that is, the point $g(y)$ is isolated in H. By continuity of $g, g[Z]$ is an infinite connected compact subspace of H. It follows that all points of the set $g[Z]$ are non-isolated in H. Thus the sets $g[Y]$ and $g[Z]$ are disjoint. We have proved (1).

Moreover, $Y_1 = g[Y]$ is an open discrete subspace of H, since the mapping g is open. The subspace $Z_1 = g[Z]$ is also open in H, for the same reason. It follows that $H = Y_1 \oplus Z_1$. The set Y_1 is infinite, since the restriction of g to Y is a finite-to-one mapping. The lemma is proved.

10. Corollary: *The image under an open compact map of a CD-space is a CD-space.*

Proof: The only case not covered by Lemma 9 is the situation where the compact connected part of the space reduces to a single point, i.e., where the whole space is discrete. The proof in this case is trivial.

11. Lemma: *Let X, Y, and Z be as in Lemma 9, and let g be a closed compact mapping of X onto a space H. Then the complement of $g[Z]$ in H is an infinite closed discrete subspace of the space H.*

Proof: Since g is closed and Y is a closed discrete subspace of X, $g[Y]$ is a closed discrete subspace of H. By the continuity of g, $g[Z]$ is a compact subspace of H. It follows that every closed discrete subspace of $g[Z]$ is finite. Therefore, the intersection of $g[Y]$ and $g[Z]$ is a finite set. The restriction of g to Y is a compact mapping, hence it is a finite-to-one mapping. Hence the set K of all points $y \in Y$ such that $g(y) \in g[Z]$ is finite, and the set $Y_1 = g[Y \setminus K]$ is infinite. Clearly, Y_1 is a closed discrete subspace of H, and Y_1 is the complement to $g[Z]$ in H.

12. Corollary: *The image of a CD-space under a perfect map is a CD-space.*

This concludes the proof of Theorem 5.

Example 1 is somewhat unsatisfactory in that the mapping f is both inductively closed and inductively open. Let us call a mapping f (*strongly*) *blended* if it can be represented as the composition of a finite sequence of mappings each of which is either (strongly) inductively open or (strongly) inductively closed. Now we are going to construct an example of a pseudo-open compact map that is not blended.

EXAMPLE 2: Let $(q_n)_{n\in\omega}$ be a one-to-one enumeration of the rationals. Moreover, fix a sequence $(I_n)_{n\in\omega}$ of open intervals in the real line such that for all $n \in \omega$:

(a) $q_n \in I_n$;
(b) $q_k \notin I_n$ for all $k \leq n$;
(c) The length of I_n is at most 1.

For each $n \in \omega$, choose a set $H_n = \{x_y^n : y \in I_n\}$ in such a way that $x_y^n \neq x_z^m$ whenever $\langle n, y\rangle \neq \langle m, z\rangle$. Let $X_0 = \bigcup\{H_n : n \in \omega\}$. Define a metric ϱ on X_0 as follows: For each $n \in \omega$ and $y, z \in I_n$ such that $y \neq z$, let $\varrho(x_y^n, x_z^n) = |y - g_n| + |z - q_n|$. If $y = z$, then set $\varrho(x_y^n, x_z^n) = 0$. Finally, if $n \neq m$, then set $\varrho(x_y^m, x_z^n) = 2$, regardless of whether y, z are different or equal. The bound on the length of the I_n's assures that the triangle inequality holds.

Now let $G = \{y \in \mathrm{R} : \forall n \in \omega \exists m > n(y \in I_m)\}$. By condition (b), all elements of G are irrationals. Choose a set X_1 of size continuum that is disjoint from X_0, and enumerate X_1 in a one-to-one manner by the elements of $G : X_1 = \{x_y^\omega : y \in G\}$.

Let $X = X_0 \cup X_1$, and let τ be the topology on X generated by all open subsets of X_0 in the topology induced by the metric ϱ, together with the family $\mathcal{V} = \{V_y^m : m \in \omega,\ y \in G\}$, where $V_y^m = \{x_y^\omega\} \cup \{x_y^n : n \geq m \ \&\ y \in I_n\}$.

Let M be the set of reals endowed with the topology of the Michael line. Let $f : X \to M$ be defined by : $f(x_y^n) = y$ for all $y \in M$ and $n \leq \omega$.

13. Claim: *The function f defined above is continuous.*

Proof: We show that for each $x \in X$, f is continuous at x. If x is an isolated point, then there is nothing to show. If $x = x_y^\omega$ for some y, then the restriction of f to the neighborhood V_y^m of x is constant, and hence f is continuous at x. The only other points in X are of the form $x_{q_n}^m$ for some n. Let U be a neighborhood of q_n in M. Then, for small enough $\epsilon > 0$, the open ball around $x_{q_n}^n$ of radius ϵ (in the topology on X_0 induced by ϱ) gets mapped into U, and again continuity of f at $x_{q_n}^n$ follows.

14. Claim: *f is pseudo-open.*

Proof: Let $y \in M$, and let U be an open neighborhood of $f^{-1}y$ in X. We want to show that $y \in \mathrm{int}(f[U])$. If y is irrational, then y is isolated in the topology of M and there is nothing to prove. So assume that $y = q_n$ for some n. But then for some

$\epsilon > 0$, the set $\{x_y^n : |q_n - y| < \epsilon\}$ is contained in U, and the image of this set under f is an open neighborhood of q_n.

15. Claim: *f is compact.*

Proof: Let $y \in M$. If $y \notin G$, then the inverse image of y under f is finite. If $y \in G$, then the inverse image of y under f is homeomorphic to a convergent sequence. In both cases $f^{-1}y$ is compact.

16. Claim: *The space X constructed above is regular.*

Proof: Straightforward.

The map f constructed above exhibits three interesting phenomena: it is compact but not tri-quotient, and it does not preserve the properties of having a base of countable order and of being submaximal. Each one of these phenomena implies that f is not blended. Let us now consider these three properties one by one.

17. Definition: [M] A surjective map $f : X \to Y$ is *tri-quotient* if one can assign to each open U in X an open U^* in Y such that :

(i) $U^* \subseteq f[U]$,
(ii) $X^* = Y$,
(iii) $U \subseteq Y$ implies $U^* \subseteq V^*$,
(iv) If $y \in U^*$ and $\mathcal{W}$ is a cover of $f^{-1}y \cap U$ by open subsets of X, then there is a finite $\mathcal{F} \subseteq \mathcal{W}$ such that $y \in (\bigcup \mathcal{F})^*$.

We call $U \mapsto U^*$ a *t-assignment* for f.

18. Lemma: *(a) All open maps and all perfect maps are tri-quotient.*
(b) A map is tri-quotient iff it is inductively tri-quotient.

Proof: Clear. See [M, Theorem 6.5 and Lemma 6.4].

19. Lemma: *The composition of two tri-quotient maps is tri-quotient.*

Proof: See [M, Theorem 7.1].

20. Corollary: *Every strongly blended compact map is tri-quotient.*

Proof: Let f be a compact map and suppose f cam be written as a composition $f_k \circ f_{k-1} \circ \ldots \circ f_0$ such that each of the maps f_i is either inductively open or inductively closed. By Claim 3, each of the maps f_i is compact. By Lemma 18, each of the maps f_i is tri-quotient. By Lemma 19, f is tri-quotient.

21. Lemma: *The map f of Example 2 is not tri-quotient.*

Proof: Suppose to the contrary that f is tri-quotient. Let $U \mapsto U^*$ be a t-assignment for f. Since $q_0 \in X^*$ and since $\{H_0\}$ is an open cover of $f^{-1}q_0 \cap X$, it must be the case that $q_0 \in H_0^*$. Since H_0^* is open, there exists some $n > 0$ such that $q_n \in H_0^*$. But again, $\{\{x^0_{q_n}\}\}$ is an open cover of $f^{-1}q_n \cap H_0$. Thus, $q_n \in \{x^0_{q_n}\}^*$. By Definition 16(i), $\{x^0_{q_n}\}^* = \{q_n\}$, which is impossible, since rationals are not isolated in the topology of the Michael line.

Not being tri-quotient is a property of the map rather than the spaces involved. Let us now consider a property of the space X that is not preserved by f.

22. Definition: Let X be a T_1-space. We say that X *has a base of countable order*, or that X is *BCO* iff there exists a sequence $(\mathcal{B}_n)_{n\in\omega}$ of open bases for X such that for each decreasing sequence $(U_n)_{n\in\omega}$ with $U_n \in \mathcal{B}_n$ for every $n \in \omega$, the collection $\{U_n : n \in \omega\}$ is a base at every $x \in \bigcap_{n\in\omega} U_n$.

The original definition of BCO was given in [A2]; the formulation used here was established in [WoW]. The class of all BCO spaces includes all metrizable spaces, all developable spaces, all T_1 first-countable scattered spaces, in particular, ω_1 with the order topology. The Michael line does not have a base of countable order because paracompact Hausdorff spaces which have a base of countable order are metrizable [A2].

23. Claim: *The space X of Example 2 has a base of countable order.*

Proof: The claim is an immediate consequence of the following more general fact.

24. Lemma: *Suppose X is a first-countable space with the following property: There exists a decomposition $X = X_0 \cup X_1$ into two disjoint subspaces such that:*

(a) *X_0 has a base of countable order;*
(b) *X_1 is a discrete space;*
(c) *X_0 is open in X.*

Then X has a base of countable order.

Proof of the lemma: Let X, X_0, X_1 be as in the assumptions. Let $(\mathcal{B}_n)_{n\in\omega}$ be a sequence of bases that witnesses the BCO of X_0. For each $x \in X_1$, let $\{V_x^m : m \in \omega\}$ be a base at x such that $V_x^m \cap X_1 = \{x\}$ for all $m \in \omega$. Define: $\mathcal{C}_n = \mathcal{B}_n \cup \{V_x^m : m > n \;\&\; x \in X_1\}$. Note that each $\mathcal{C}_n$ is a base in X. Moreover, let $(U_n)_{n\in\omega}$ be a decreasing sequence such that $U_n \in C_n$ for all $n \in \omega$, and let $y \in \bigcap_{n\in\omega} U_n$. In order to show that $\{U_n : n \in \omega\}$ is a base at y, consider two cases:
Case 1: $\exists n \in \omega (U_n \cap X_1) = \emptyset$.

Then $U_n \in \mathcal{B}_n$ for sufficiently large n, and it follows from the choice of the $\mathcal{B}_n$'s that $\{U_n : n \in \omega\}$ is a base at y.
Case 2: Not Case 1.

Since each element of $\mathcal{C}_n$ contains at most one member of X_1, and since the sequence $(U_n)_{n\in\omega}$ was assumed decreasing, there exists exactly one $x \in X$ such that $x \in U_n$ for all $n \in \omega$. But then $U_n = V_x^{m_n}$ for each $n \in \omega$ and some sequence $(m_n)_{n\in\omega}$ with $\lim_{n\to\infty} m_n = \infty$. Thus, $x = y$, and the family $\{U_n : n \in \omega\}$ is a base at y.

25. Corollary: *The property of having a base of countable order is not preserved by pseudo-open compact maps.*

As was shown in [WW], base of countable order is preserved by uniformly monotonically complete open maps (in particular by open-compact maps) with regular domains and T_1 ranges. In [W] it was proved that base of countable order is preserved by perfect maps. Moreover, BCO is hereditary. Hence inductively perfect maps and inductively open-compact maps preserve BCO. It follows from the second part of Claim 3 and the description of the map f of Example 2 that f cannot be blended.

The two reasons for non-blendedness of f given so far rely heavily on the fact that f is compact. Let us conclude this note with an argument that does not rely on compactness of f.

Definition: A space X is said to be *submaximal* if every subset A of X which is dense in X is also open in X. An *I-space* is a space X such that the complement to the set of all isolated points of X is a discrete subspace of X.

The concept of a submaximal space can be found in [Bo]; I-spaces were defined in [AC]. Obviously, every I-space is submaximal, and each subspace of a submaximal space is submaximal.

27. Claim: *The space X of Example 2 is an I-space.*

Proof: The set of nonisolated points of X is $\{x^n_{q_n} : n \in \omega\} \cup \{x^\omega_y : y \in G\}$, which is clearly a discrete subspace of X.

28. Claim: *The Michael line is not submaximal.*

Proof: The set $\{0\} \cup (R\backslash Q)$ is dense, but not open in the Michael line.

It was shown in [AC] that both submaximality and the class of I-spaces are preserved by both inductively open mappings and inductively closed mappings. For the preservation of submaximality by such mappings, one does not even have to assume continuity. Therefore the mapping f of Example 2 is not the composition of any family of inductively open or inductively closed mappings, even discontinuous ones.

29. Remark: Restricting the map f of Example 2 to the subspace $f^{-1}\mathrm{Q}$, one obtains an example of a compact pseudo-open map on a countable metrizable I-space that is not tri-quotient and does not preserve submaximality. Just and Wicke earlier constructed an example of a pseudo-open compact map from a countable subspace of $\mathrm{R} \times \mathrm{Q} \times \mathrm{Q}$ onto $\mathrm{Q} \times \mathrm{Q}$ that is not tri-quotient. In [AC], there is another example of a pseudo-open compact map from a countable metrizable I-space onto a metrizable compact space that is not submaximal.

We conclude this note with two open problems.

Problem 1: Is there a tri-quotient (compact) mapping which is not strongly blended? Which is not blended?

Problem 2: Find topological properties other than submaximality and the I-space property that are inherited by subspaces and are preserved by open mappings and by closed mappings, but are not preserved in general by pseudo-open mappings.

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