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COHESION IN TOPOLOGICAL SPACES

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ABSTRACT. Provoked by Arhangel'skiĭ's notion of absolute dimension, we define the new concept of cohesion and examine its properties. In particular, it is shown that cohesion is closely related to scattered length and cohesion is completely characterised in scattered spaces. However, examples are produced which are not scattered and have cohesion defined. As an appendix, it is shown that absolute dimension does not agree with classical dimension functions on the unit square.

1. INTRODUCTION

In his paper "Cleavability over the Reals" [1], Arhangel'skiĭ defined inductively a "function of the dimensional type" which he called *absolute dimension.* We have amended the base case of the definition and extended its range to the ordinals to give the new concept of *cohesion.* Cohesion can be viewed as asserting that if a space has cohesion n then the boundaries of all open sets have cohesion less than n. This is similar to the definition of the classical inductive dimensions but is in fact a much stronger assertion giving very different consequences. If a sequential space has cohesion defined on it then it is scattered. The relation between cohesion and scattered spaces is examined and cohesion is completely characterised in scattered spaces. Cohesion is also considered in spaces which are crowded (dense in themselves) and regular crowded spaces of each finite cohesion are produced. However, it is shown that it is difficult to do better than this, as there is no regular space of transfinite cohesion. The behaviour of cohesion under various classes of continuous functions is described. We show in the Appendix that absolute dimension is not defined on the unit square and hence does not agree with the classical dimension functions.

Cohesion is based on the notion of **nowhere dense** subsets of a space. A subset Y of a space X is nowhere dense in X if $Int_X \overline{Y}^X = \emptyset$. From this definition, it is not hard to see that if Y is nowhere dense in X then so too is $\overline{Y}^X$. Also, if Z is nowhere dense in some subset Y of X then Z is nowhere dense in X.

A space X is said to be **scattered** if every subset of X has an isolated point. Taking X^d to denote the set of accumulation points of X, we make the following definition:

$$X^{(0)} = X,$$

$$X^{(\alpha+1)} = (X^{(\alpha)})^d,$$

$$X^{(\lambda)} = \bigcap_{\alpha\in\lambda} X^{(\alpha)} \text{ for } \lambda \text{ a limit ordinal.}$$

X is scattered if and only if $X^{(\gamma)} = \emptyset$ for some ordinal γ and then the scattered length of X, denoted $sl(X)$, is taken to be the smallest such γ.

A space X is said to be **crowded** if it has no isolated points. This term is from [4] and the author finds it more natural to use as an adjective than the usual term "dense in itself".

For a continuous surjection $f : X \to Y$, f is said to be **irreducible** if there is no closed subset A of X such that $A \neq X$ and $f|_A : A \to Y$ is surjective. For any $g : X \to Y$ the **small image of A under g** denoted $g^*(A)$ is defined by:

$$g^*(A) = \{y \in Y : g^{-1}(y) \subseteq A\}.$$

Remaining terms and notation can be found in Engelking [5] and all spaces considered are T_1 unless otherwise stated.

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2. Definition of Cohesion and its Basic Properties

For a topological space X, the **cohesion** of X, abbreviated to $cohX$, is defined by transfinite recursion as follows:

$$cohX = -1 \text{ if and only if } X = \emptyset,$$

$$\text{for } \alpha \in Ord, \ \ cohX \leq \alpha \text{ if for every nowhere dense subset } C \subseteq X, \ cohC < \alpha.$$

For a space X and ordinal α, $cohX = \alpha$ if $cohX \leq \alpha$ and for every $\beta < \alpha$ it is not the case that $cohX \leq \beta$.

Remark. Despite the fact that we will prove that there is no regular space of transfinite cohesion, we have given the definition in its full generality. This is for two reasons. First, in proving this fact we wish to use certain lemmas which tell us about the structure of spaces with transfinite cohesion. Secondly, there may yet be some interesting Hausdorff spaces of transfinite cohesion.

Proposition 2.1. *If X is a space such that $cohX \leq \alpha$ for some ordinal α and $Y \subseteq X$ then $cohY \leq \alpha$.*

Proposition 2.2. *A non-empty space X is discrete if and only if $cohX = 0$.*

Proof: If X is discrete then every subset of X is open. This means that the only nowhere dense subset of X is the empty set so by the definition it follows that $cohX = 0$.

If $cohX = 0$ then every nowhere dense subset has a cohesion of -1. So no non-empty subset is nowhere dense. Consider $\{x\}$. This is closed as X is T_1 but is not nowhere dense so contains a non-empty subset open in X. This must be $\{x\}$. Hence every point of X is open and X is discrete. □

Proposition 2.3. *If X is a space such that, for some ordinal α, $cohX = \alpha$ then, for all $\beta < \alpha$, there exists a closed, nowhere dense subset $C_\beta \subseteq X$ such that $cohC_\beta = \beta$.*

Proof: The proof proceeds by transfinite induction.

If $\alpha = -1$ then it is trivial. Assume the proposition has been proven for all spaces X such that $cohX = \beta$ where $\beta < \alpha$.

Consider the case where $\alpha = \gamma + 1$. If every nowhere dense subset of X has cohesion less than γ then by definition, $cohX \leq \gamma$. Since this is not the case it must be that there is a nowhere dense subset A of X such that $cohA = \gamma$. Define $C_\gamma = \overline{A}^X$ so C_γ is nowhere dense in X. Hence $cohC_\gamma \leq \gamma$ and $A \subseteq C_\gamma$ so by Proposition 2.1, $cohC_\gamma \geq cohA = \gamma$. Therefore $cohC_\gamma = \gamma$.

Now suppose $\beta < \alpha$. If $\beta = \gamma$ then C_β is already defined. If $\beta < \gamma$ then by the inductive hypothesis there exists a $C_\beta \subseteq C_\gamma$ closed and nowhere dense in C_γ such that $cohC_\beta = \beta$. But then C_β is also closed and nowhere dense in X and the hypothesis holds for α.

Consider now the case where α is a limit ordinal. For every $\beta < \alpha$, there exist $\gamma < \alpha$ and a nowhere dense subset of X, A_γ, such that $\beta < \gamma$ and $cohA_\gamma = \gamma$ (otherwise $cohX \leq \beta + 1$). As before taking $C = \overline{A_\gamma}^X$, C is closed and nowhere dense in X with $\gamma \leq cohC < \alpha$. Then $cohC > \beta$ and, by the inductive hypothesis, there exists a C_β closed and nowhere dense in C and hence in X such that $cohC_\beta = \beta$.

The proof is complete. □

One other useful property is:

Proposition 2.4. *If $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover for X such that for some $n \in \omega$, $cohU_\lambda \leq n$ for all $\lambda \in \Lambda$ then*

$$cohX \leq n.$$

Proof: Suppose $cohU_\lambda \leq -1$ for all $\lambda \in \Lambda$, then they are all empty but still form a cover of X so X is empty and $cohX \leq -1$.

Assume now that, for any space X and some $n \in \omega$, the proposition holds and consider the case where $cohU_\lambda \leq n + 1$ for all $\lambda \in \Lambda$.

Suppose A is nowhere dense in X. Taking $C = \overline{A}$, C is nowhere dense and closed in X. Thus $C \cap U_\lambda$ is nowhere dense in U_λ as it is closed in U_λ and if $U \subseteq C \cap U_\lambda$ for some U open and non-empty in U_λ then U is open in X and $U \subseteq C$. This contradicts the fact that C is nowhere dense in X.

But then by definition of cohesion,

$$coh(C \cap U_\lambda) \leq n.$$

Taking $V_\lambda = C \cap U_\lambda$, $\{V_\lambda : \lambda \in \Lambda\}$ is an open cover for C such that $cohV_\lambda \leq n$, for all $\lambda \in \Lambda$. So by the inductive hypothesis, $cohC \leq n$ giving $cohA \leq n$ and hence

$$cohX \leq n + 1. \quad \square$$

The next lemma is useful throughout this work. It seems to be well-known but the author was unable to find it proven anywhere so a proof is given here for completeness.

Lemma 2.5. *If X is a scattered space then X^d is nowhere dense in X.*

Proof: Suppose X is a scattered space and that U is a non-empty open subset of X. By definition of scattered spaces, there exists $x \in U$ which is isolated in U. $\{x\}$ is then open in U and hence in X. Therefore $x \in U \setminus X^d$ which means that no non-empty open subset of X is contained in X^d, that is, X^d is nowhere dense in X. $\square$

This next lemma is also very useful.

Lemma 2.6. *If X is scattered and Y is nowhere dense in X then $Y \subseteq X^d$.*

Proof: Suppose $Y \not\subseteq X^d$. Then Y contains an isolated point of X so in particular contains a non-empty open set. Hence Y is not nowhere dense in X. $\square$

Examples. Take $C_1 \subseteq \mathbb{R}$ to be $C_1 = \{0\} \cup \{\frac{1}{n} : n \in \omega \setminus \{0\}\}$. This is clearly non-empty and not discrete. By Lemmas 2.5

and 2.6, the only possible non-empty, nowhere dense subset of C_1 is $\{0\}$ which clearly has a cohesion of 0.

$$\therefore cohC_1 = 1.$$

Taking C_1 to be the base space, for each $n \in \omega$, inductively define scattered, closed subsets of $\mathbb{Q}$, call them C_n, such that $cohC_n = n$ as follows:

$$C_{n+1} = C_n \cup \{\frac{1}{k_1} + \ldots + \frac{1}{k_n} : k_{i+1} \geq 2k_i(k_i - 1) \text{ for } i = 1, \ldots, n-1\}.$$

This gives sequences of points which converge down to every point of C_n. Hence $C_{n+1}^d = C_n$ and since C_n is scattered so too is C_{n+1}. If A is nowhere dense in C_{n+1} then, by , $A \subseteq C_n$ and $cohA \leq n$. However, C_n is nowhere dense in C_{n+1}, by Lemma 2.5, and $cohC_n = n$ so by definition of cohesion

$$cohC_{n+1} = n + 1.$$

Since each of these spaces is a subset of $\mathbb{Q}$ this shows that if $coh\mathbb{Q}$ exists then it is transfinite. However, we in fact have:

Theorem 2.7. *$coh\mathbb{Q}$ is not defined.*

This is actually a consequence of Theorem 3.1 but the following proof is somewhat shorter.

Proof: Suppose for contradiction that $coh\mathbb{Q}$ is defined. $\mathbb{Q}$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ which contains $\{0\} \times \mathbb{Q}$ as a nowhere dense subset. Thus, by definition of cohesion, $coh(\{0\} \times \mathbb{Q}) < coh(\mathbb{Q} \times \mathbb{Q})$. But as $\{0\} \times \mathbb{Q}$ is also homeomorphic to $\mathbb{Q}$ this gives us our required contradiction. □

3. Cohesion and Scattered Spaces

The last theorem of the previous section was shown essentially by finding a nowhere dense subspace of $\mathbb{Q}$ which was homeomorphic to $\mathbb{Q}$. The result then followed directly from the definition of cohesion. This is not in general possible but it is possible to find in certain spaces a subspace which contains a nowhere dense homeomorph of itself. It then follows, as for $\mathbb{Q}$, that such spaces cannot have cohesion defined on them.

The following theorem gives the details of how such subspaces can be constructed in the general setting.

Theorem 3.1. *If X is T_2, sequential and coh$X$ exists then X is scattered.*

Proof: Suppose X is not scattered. So there exists $A \subseteq X$ which has no isolated points. Define $Y = \overline{A}^X$. Y is a closed subset of X so is also T_2 and sequential. Moreover, if $y \in Y$ is an isolated point of Y then there exists $U \subseteq X$ which is open in X such that $U \cap Y = \{y\}$. But if $y \notin A$ then U is an open set about an element of $\overline{A}^X$ which does not intersect A. As this is not possible, it must be that $y \in A$. But then $U \cap A = \{y\}$ so y is an isolated point of A which is also not possible. Thus Y has no isolated points.

This means that for all $y \in Y$, $y \in \overline{Y \setminus \{y\}}^Y$. So $Y \setminus \{y\} \neq \overline{Y \setminus \{y\}}^Y$ or more simply $Y \setminus \{y\}$ is not closed in Y. Since Y is sequential, this implies that there exists a sequence in $Y \setminus \{y\}$ which converges to a point outside of $Y \setminus \{y\}$. There is only one possible point left in Y which this sequence could converge to and this is y. Denote such a sequence by $\{y_n\}_{n=0}^{\infty}$ and since Y is Hausdorff we can assume all elements of the sequence are distinct.

We need to separate the points of such sequences quite some way so we require the following:

Fact. $\forall n \in \omega$, $\exists U_n(y) \subseteq Y$ open in Y such that $y_n \in U_n(y)$, $y \notin U_n(y)$ and $U_n(y) \cap U_m(y) = \emptyset$ whenever $n, m \in \omega$ and $n \neq m$.

This can be proved using only the fact that Y is Hausdorff.

We now show how, for a given $x \in Y$ contained in some open set U, there exist sets $I_n(x, U) \subseteq U$ for each $n \in \omega$ such that $(I_{n+1}(x, U))^d = I_n(x, U)$, $I_n^{(n)}(x, U) = \{x\}$ and $\forall z \in I_{n+1}(x, U) \setminus I_n(x, U)$, $\exists U_z \subseteq X$ which is open in X with $U_z \cap$

$I_{n+1}(x,U) = \{z\}$ and whenever $z \neq z'$, $U_z \cap U_{z'} = \emptyset$. These I_n are the equivalent of the C_n in Section 2.

Take $I_0(x,U) = \{x\}$ and define $U_x = U$. This trivially satisfies the conditions.

Suppose then that for some $n \in \omega$, if $i \leq n$ the set $I_i(x,U)$ and the corresponding U_z's are defined. Consider a $z \in I_n(x,U) \setminus I_{n-1}(x,U)$ (taking $I_{-1}(x,U) = \emptyset$). Take $\{z_k\}$ to be the sequence contained in U_z converging to z whose existence is demonstrated at the beginning of this proof. Define

$$I_{n+1}(x,U) = I_n(x,U) \cup \{z_k : z \in I_n(x,U) \setminus I_{n-1}(x,U) \text{ and } k \in \omega\}$$

$$U_{z_k} = U_z \cap U_k(z) \text{ as given by the Fact.}$$

Suppose $z, z' \in I_n(x,U) \setminus I_{n-1}(x,U)$. If $z \neq z'$ then for all $j,k \in \omega$, $U_{z_j} \cap U_{z'_k} \subseteq U_z \cap U_{z'} = \emptyset$. And if $z = z'$ then for $j,k \in \omega$ with $j \neq k$, $U_{z_j} \cap U_{z'_k} \subseteq U_j(z) \cap U_k(z) = \emptyset$ by their definition. From this, U_{z_k} does not contain any z'_j for $(j \neq k) \vee (z \neq z')$. Moreover, from the Fact $z \notin U_{z_k}$ hence $U_{z_k} \cap I_{n+1} = \{z_k\}$. Thus the U_{z_k} are the open sets required in the definition of $I_{n+1}(x,U)$.

The U_{z_k} also show that if $z \notin I_{n+1}(x,U) \setminus I_n(x,U)$ then z is an isolated point of $I_{n+1}(x,U)$. And if $z \in I_n$ then by its definition there is a sequence in I_{n+1} converging to z. These two statements together give

$$(I_{n+1}(x,U))^d = I_n(x,U)$$

from which it follows by part of the induction hypothesis that

$$(I_{n+1}(x,U))^{(n+1)} = \{x\}.$$

Hence I_{n+1} is scattered.

We now take

$$Z = \bigcup_{n \in \omega} I_n(y_n, U_n(y)).$$

If $z \in Z$ is isolated then, by the definition of the I_n, it cannot be the case that $z \in I_{n-1}(y_n, U_n(y))$ for any $n \in \omega$ and so it must be that z is isolated in some $I_n(y_n, U_n(y))$. In the opposite direction, if z is isolated in $I_n(y_n, U_n(y))$ for some $n \in \omega$ then

$\{z\} = V \cap I_n(y_n, U_n(y))$ for some V open in Y. But then $\{z\} = Z \cap (V \cap U_n(y))$ as $I_n(y_n, U_n(y)) \subseteq U_n(y)$ and the $U_n(y)$ are pairwise disjoint. This means z is isolated in Z. Hence we have

$$Z^d = \bigcup_{n\in\omega} (I_n(y_n, U_n(y)))^d$$

$$Z^d = \bigcup_{n\in\omega} I_n(y_{n+1}, U_{n+1}(y))$$

which is clearly homeomorphic to Z. It is not hard to see that Z is scattered (with $sl(x) = \omega + 1$) giving that Z^d is nowhere dense in Z.

But if X has a definable cohesion then so too does Y and thus Z. By the definition, $cohZ^d < cohZ$ which is impossible since Z^d is homeomorphic to Z. Thus we have a contradiction.

Hence it must be the case that X is scattered. □

In spaces which have a defined cohesion and are scattered, we have two numbers attached to the space, the cohesion and the scattered length. The next two theorems give the relation between them in scattered spaces.

Theorem 3.2. *For X a scattered space and $n \in \omega$,*

$$sl(X) = n \text{ if and only if } cohX = n - 1.$$

Proof: Firstly suppose X is scattered with $sl(X) = 0$. Then $X = X^{(0)} = \emptyset$ and hence $cohX = -1$.

Assume for the purposes of induction that if $sl(X) = n$ then $cohX = n - 1$.

Consider a space X of scattered length $n+1$. X^d is nowhere dense in X and clearly has scattered length n. Thus $cohX^d = n - 1$.

If C is a nowhere dense subset of X then, by Lemma 2.6, $C \subseteq X^d$. By Proposition 2.1, this gives us that for every nowhere dense subset C of X,

$$cohC \leq n - 1.$$

So by the definition of cohesion,

$$cohX \leq n.$$

However X^d is a nowhere dense subset of X of cohesion $n-1$. Hence

$$cohX = n.$$

So by induction,

$$\text{if } sl(X) = n \text{ then } cohX = n - 1.$$

To do the reverse implication, if $cohX = -1$ then $X = \emptyset$ and hence $sl(X) = 0$. Assume now that if $cohX = n-1$ then $sl(X) = n$. If X is a scattered space such that $cohX = n$, then X^d is nowhere dense in X and, since any nowhere dense subset of X is contained in X^d, this gives

$$cohX^d = n - 1.$$

But then by the inductive hypothesis,

$$sl(X^d) = n$$

which clearly implies that

$$sl(X) = n + 1.$$

So by induction,

$$\text{if } cohX = n - 1 \text{ then } sl(X) = n. \quad \square$$

Theorem 3.3. *If X is scattered and $cohX$ is defined then $sl(X)$ is finite.*

Proof: Suppose $sl(X) = \kappa$ and $cohX = \mu$ for some $\kappa, \mu \in Ord$.

Suppose κ is infinite. Define a function $f : \omega \to \mu$ by

$$f(n) = cohX^{(n)} \text{ for } n \in \omega.$$

Since $X^{(n+1)} = (X^{(n)})^d$, $X^{(n+1)}$ is nowhere dense in $X^{(n)}$. Thus

$$cohX^{(n+1)} < cohX^{(n)}.$$

But then $\{f(n) : n \in \omega\}$ forms a strictly decreasing sequence in the ordinal μ which contradicts the well-ordering of μ. Hence κ cannot be infinite and we have

$$sl(X) \text{ is finite. } \square$$

Putting these last three results together we have the following:

Corollary 3.4. *If X is a sequential Hausdorff space and $cohX$ exists then X is scattered and $sl(X) = n$ for some $n \in \omega$. Moreover, $cohX = n - 1$.*

4. Cohesion in Non-scattered Spaces

The theorems of the previous section show how cohesion and scattered spaces are related. However, are there spaces which are crowded but for which cohesion still exists? Moreover, what cohesions can such spaces have? The answer to the first question is in the affirmative. The examples which we use are called **nodec spaces** which were first defined by van Douwen [3] as those spaces for which every NOwhere DEnse subset is Closed. From the definition, it follows that every nowhere dense subspace of a nodec space is discrete. However, this simply means that a nodec space has cohesion of at most one.

Van Douwen showed in [4] that any crowded maximal topology is nodec but such examples may only be Hausdorff. In [3] though, he proved that there is a countable, regular, crowded, T_1 nodec space, Θ. We give here a description of Θ and we would like to thank Ian Stares for his very useful exposition of the construction of Θ which was given in [8].

The Construction of Θ Given a regular crowded space X with topology $\mathcal{T}$, by Zorn's Lemma, we can find a topology on X which is maximal with respect to being crowded and regular and also contains $\mathcal{T}$. Such spaces are called ultraregular spaces and more can be found on these spaces in Bourbaki, p139 [2].

For a countable ultraregular space X, define a subspace Θ_X by:

$$\Theta_X = \{x \in X : \text{there is no nowhere dense subset } A \text{ of } X \text{ such that } x \in \overline{A} \setminus A\}.$$

Θ_X is clearly a nodec space and it can be shown for any ultraregular space X that Θ_X is non-empty. It then follows that Θ_X is dense in X. If this were not the case, as X is countable and regular hence zero-dimensional, we could find a clopen non-empty subset U of X such that $U \cap \Theta_X = \emptyset$. But as a clopen subset of an ultraregular space, U is ultraregular and $\Theta_U = U \cap \Theta_X = \emptyset$ which is a contradiction. Moreover, Θ_X is crowded otherwise it has an isolated point which must also be isolated in $\overline{\Theta}_X = X$. This is impossible as X is assumed to be crowded.

We will use these spaces as building blocks to construct (countable) regular crowded spaces of each finite cohesion.

Theorem 4.1. *If X and Y are topological spaces such that $cohX = n$ for some $n \in \omega$ and Y is a crowded nodec space then $(X \times Y, \mathcal{T})$ is a topological space such that*

$$coh(X \times Y) = n + 1,$$

where $\mathcal{T}$ is the topology determined by the following basis:

Fix $y_0 \in Y$ and then for $(x, y) \in X \times Y$ a basic open neighbourhood is of the form:

(1) *$\{x\} \times U$ when $y \neq y_0$ and where U is open in Y with $y_0 \notin U$.*

(2) *$\bigcup\{\{a\} \times U_a : a \in V\}$ when $y = y_0$ and where V is an open neighbourhood of x in X and, for all $a \in V$, U_a is an open neighbourhood of y_0 in Y.*

Proof: It is not too hard to check that the definition given does indeed define a topology on $X \times Y$.

First of all, we shall show that $coh(X \times Y) \geq n + 1$.

The set $X \times \{y_0\}$ is a subset of $X \times Y$. It is closed since if $(x, y) \notin X \times Y$ then $y \neq y_0$ and $\{x\} \times (Y \setminus \{y_0\})$, is an open neighbourhood of (x, y) which misses $X \times \{y_0\}$. Moreover it is nowhere dense because any open set, say V, about $(x, y_0) \in$

$X \times Y$ contains $\{x\} \times U$ for some open neighbourhood U of y_0. But y_0 is not isolated so for some $y \in Y \setminus \{y_0\}$, $(x, y) \in \{x\} \times U \subseteq V$. Thus V cannot be a subset of $X \times \{y_0\}$.

Clearly $coh(X \times \{y_0\}) = n$ and so by definition of cohesion,

$$coh(X \times Y) \geq n + 1.$$

Secondly, we show that $coh(X \times Y) \leq n + 1$ and then the proof is complete.

Suppose C is nowhere dense $X \times Y$. Since for all $x \in X$, $\{x\} \times (Y \setminus \{y_0\})$ is open in $X \times Y$ then $C_x = C \cap (\{x\} \times (Y \setminus \{y_0\}))$ is nowhere dense in $\{x\} \times (Y \setminus \{y_0\})$ and hence in $\{x\} \times Y$. Clearly $coh(\{x\} \times Y) = 1$ giving us that $cohC_x = 0$, that is, C_x is closed and discrete in $\{x\} \times Y$. (Note also that C_x is open in C). But then there exists an open neighbourhood U_x of y_0 such that $(\{x\} \times U_x) \cap C_x = \emptyset$.

Take $V = \bigcup_{x \in X}(\{x\} \times U_x)$. By definition of $\mathcal{T}$, V is open in $X \times Y$ and by definition of the U_x's, $C \cap V \subseteq X \times \{y_0\}$ so that $coh(C \cap V) \leq n$.

But we now have that $\{C \cap V\} \cup \{C_x : x \in X\}$ is an open cover of C such that each element of the cover has cohesion at most n. So by Proposition 2.4,

$$cohC \leq n.$$

Hence by definition of cohesion,

$$coh(X \times Y) \leq n + 1. \quad \square$$

It is now reasonably easy to show:

Theorem 4.2. *For all $n \in \omega \setminus \{0\}$ there exists a space X_n which is countable, crowded, regular and $cohX_n = n$.*

Proof: For $n = 1$, take X_1 to be a regular nodec space. Assume X_n has been shown to exist. Now apply the previous theorem with $X = X_n$ and Y a regular nodec space. Define X_{n+1} to be this new space.

It is clear to see that X_{n+1} is countable and that $cohX_{n+1} = n+1$ by the previous result. That X_{n+1} is crowded follows since every open neighbourhood of a point $(x, y) \in X_n \times Y$ contains a set of the form $\{x\} \times U$ where U is an open neighbourhood of y in Y. But no $y \in Y$ is isolated so U contains some point other than y and hence the neighbourhood of (x, y) contains some point other than (x, y).

We must show that X_{n+1} is also a T_1-space. Consider $(x, y) \in X_{n+1}$. The set $U = (X_n \setminus \{x\}) \times Y$ is a basic open set as X_n is T_1. Also $V = Y \setminus \{y\}$ is open in Y as Y is T_1.

Case (1): If $y \neq y_0$ of the last theorem then $X_{n+1} \setminus \{(x, y)\} = \bigcup\{\{a\} \times U_a : a \in X\}$ where $U_a = Y$ for $a \neq x$ and $U_x = V$. Hence the point (x, y) is closed.

Case (2): If $y = y_0$ then $\{x\} \times V$ is open in X_{n+1} and then complement of (x, y) is $U \cup V$ which is open and hence (x, y) is closed.

It remains to show that X_{n+1} is regular. Suppose U is an open neighbourhood of (x, y) in X_{n+1}. We need to find an open set $V \subseteq X_{n+1}$ such that $(x, y) \in V \subseteq \overline{V}^{X_{n+1}} \subseteq U$.

Case (1): If $y \neq y_0$ then U contains an open set of the form $\{x\} \times U'$ for some U' open in Y. In this case there exists a $V \subseteq Y$ open such that $y \in V \subseteq \overline{V}^Y \subseteq U'$. It is not too hard to see that $\overline{\{x\} \times V}^{X_n \times Y} = \{x\} \times \overline{V}^Y$ and so $\{x\} \times V$ is our required open set.

Case (2): If $y = y_0$, then U contains an open set of the form $\bigcup\{\{a\} \times U_a : a \in W\}$ where W is an open neighbourhood of x in X_n and each U_a is an open neighbourhood of y_0 in Y. Take G to be an open set in X_n such that $x \in G \subseteq \overline{G}^{X_n} \subseteq W$. Then for all $a \in \overline{G}^{X_n}$ take an H_a open in Y such that $y_0 \in H_a \subseteq \overline{H_a}^Y \subseteq U_a$. Setting $V = \bigcup\{\{a\} \times H_a : a \in G\}$, it is clear to see that V is an open neighbourhood of (x, y) which is contained in the closed set $\bigcup\{\{a\} \times \overline{H_a}^Y : a \in \overline{G}^{X_n}\}$ which is in turn contained in U.

Hence X_{n+1} is regular and so by induction on the natural numbers the theorem is proven. □

Having found spaces of each finite cohesion, can we do better and find one of transfinite cohesion? The answer to this is almost always "No" as this next theorem shows.

Theorem 4.3. *There is no regular space of transfinite cohesion.*

We actually demonstrate that there is no regular space of cohesion ω. This suffices since Proposition 2.3 says that any regular space of transfinite cohesion contains a subset of cohesion ω which is necessarily regular.

The proof proceeds by demonstrating that if a space of cohesion ω exists then it contains a nowhere dense subset also of cohesion ω but that then contradicts the definition of cohesion. To construct this nowhere dense subset we need a couple of technical lemmas.

Lemma 4.4. *If A, $U \subseteq X$ and U is open with $coh(A \setminus U) \leq n$ and $cohU \leq m$ then $coh(A \cup U) \leq n + m + 1$.*

Proof: Induct on m for a given n. Assume $m = -1$ so $U = \emptyset$ and $cohA \leq n$. Hence $coh(A \cup U) \leq n + -1 + 1$ as required.

Thus suppose it has been proven for $m = k$ and assume $m = k + 1$. If C is nowhere dense in $A \cup U$, then $C \cap U$ is nowhere dense in U as U is open in $A \cup U$. Thus $coh(C \cap U) \leq k$. But also $C \setminus U \subseteq A \setminus U$ so that $coh(C \setminus U) \leq n$ by Proposition 2.1.

So taking $C = X$ in the inductive hypothesis, and noting that $C \cap U$ is open in C,

$$cohC = coh((C \setminus U) \cup (C \cap U)) \leq n + k + 1.$$

But this was for an arbitrary nowhere dense subset of $A \cup U$ hence

$$coh(A \cup U) \leq n + k + 2 = n + (k + 1) + 1.$$

By induction the theorem holds for all m. □

Lemma 4.5. *If X is regular and $cohX = \omega$ then for all $n \in \omega$, there exist $A, U \subseteq X$ such that A is nowhere dense in X, $cohA = n$, U is open in X, $A \subseteq U$ and $coh(X \setminus U) = \omega$.*

Proof: By Proposition 2.3, for X as in the statement of the lemma and some $n \in \omega$, there exists $A \subseteq X$ which is closed and nowhere dense in X such that $cohA = n$.

Suppose that for all open sets U in X which contain A, $coh(X \setminus U) < \omega$. Taking $X \setminus U$ to be A in Lemma 4.4, if $cohU < \omega$ then $cohX < \omega$. Hence $cohU = \omega$ for all such U. Define $\mathcal{U}$ to be the collection of all open sets containing A and index this set by Λ.

Claim.

$$A = \bigcap_{\lambda \in \Lambda} \overline{U_\lambda}$$

Certainly $A \subseteq \bigcap_{\lambda\in\Lambda} \overline{U_\lambda}$ so consider $x \notin A$. By regularity, there exists $\lambda \in \Lambda$ such that

$$A \subseteq U_\lambda \subseteq \overline{U_\lambda} \subseteq X \setminus \{x\}.$$

But then $x \notin \overline{U_\lambda}$ and moreover $x \notin \bigcap_{\lambda\in\Lambda} \overline{U_\lambda}$. Hence

$$\bigcap_{\lambda \in \Lambda} \overline{U_\lambda} \subseteq A$$

and we have our claim.

Suppose now that $coh(X \setminus U_\lambda) \leq M$ for all $\lambda \in \Lambda$ and some $M \in \omega$.

The set $\{X \setminus \overline{U_\lambda} : \lambda \in \Lambda\}$ is an open cover for $X \setminus A$ by the claim. Thus by Proposition 2.4,

$$coh(X \setminus A) \leq M.$$

But we now have that $cohA = n$, $coh(X \setminus A) \leq M$ and $X \setminus A$ is open in X so, by Lemma 4.4, $coh((X \setminus A) \cup A) \leq M + n + 1$ or in other words,

$$cohX < \omega$$

which is a contradiction.

Therefore, for the given n, there exists $\lambda \in \Lambda$ such that $coh(X \setminus \overline{U_\lambda}) \geq n + 1$. By Proposition 2.3, take B to be a subset of $X \setminus \overline{U_\lambda}$ which is nowhere dense in $X \setminus \overline{U_\lambda}$ and for which $cohB = n$. This gives that

$$A \subseteq U_\lambda \subseteq \overline{U_\lambda} \subseteq X \setminus B.$$

Define $W = X \setminus \overline{U_\lambda}$ so that $B \subseteq W$, $cohB = n$ and $\omega \geq coh(X \setminus W) = coh\overline{U_\lambda} \geq cohU_\lambda = \omega$. B is also nowhere dense in X and hence B and W are the sets which satisfy the conditions on the A and U in the statement of the lemma. □

This last lemma allows us to find nowhere dense subsets of a space of cohesion ω of each finite cohesion, which are sufficiently well spaced so that their union is still nowhere dense. But then, this nowhere dense subset has cohesion ω and this is the set we require for the contradiction. The details are as follows:

Proof of Theorem 4.3: Suppose X is a regular space of cohesion ω. First of all we construct nowhere dense subsets of X of each finite cohesion in a particularly nice way. By Lemma 4.5, we can find $C_0, U_0 \subseteq X$ where C_0 is nowhere dense in X, $cohC_0 = 0$, U_0 is open in X, $C_0 \subseteq U_0$ and $coh(X \setminus U_0) = \omega$.

We now define inductively $C_k, U_k \subseteq X$ such that:

(1) C_k is nowhere dense in X

(2) $cohC_k = k$

(3) U_k is open with $C_k \subseteq U_k$

(4) $coh(X \setminus U_k) = \omega$

(5) $U_i \subseteq U_{i+1}$ for $i = 0, 1, 2, \dots, k-1$

(6) $C_{i+1} \subseteq X \setminus U_i$ for $i = 0, 1, 2, \dots, k-1$

Assume that, for $i \leq n$, C_i and U_i have been defined satisfying the inductive assumptions. Define $C_{n+1}, V \subseteq X \setminus U_n$ as given by Lemma 4.5, so that C_{n+1} is nowhere dense subset of $X \setminus U_n$, and hence of X, contained in the set V open in $X \setminus U_n$ such that $cohC_{n+1} = n + 1$ and, $coh((X \setminus U_n) \setminus V) = \omega$.

Take V' to be a set open in X such that $V = V' \cap (X \setminus U_n)$. Take $U_{n+1} = V' \cup U_n$. It is easy to see from their definitions that C_{n+1} and U_{n+1} satisfy all the inductive conditions for $k = n+1$ except (4). But note

$$X \setminus U_{n+1} = X \setminus (V' \cup U_n) = (X \setminus U_n) \setminus V' = (X \setminus U_n) \setminus V$$

Hence

$$coh(X \setminus U_{n+1}) = coh((X \setminus U_n) \setminus V) = \omega$$

Thus C_{n+1} and U_{n+1} are sets satisfying all of the inductive conditions for $k = n+1$ and so the induction is complete.

Define $C = \bigcup_{n=0}^{\infty} C_n$. Clearly $cohC \geq cohC_n$ for all $n \in \omega$ and $C \subseteq X$ so

$$cohC = \omega.$$

It remains to show that C is nowhere dense in X and we have our contradiction. Suppose not then there is an open set U of X such that $U \subseteq \overline{C}$. Thus $U \cap C \neq \emptyset$ and therefore, for some $n \in \omega$, $U \cap C_n \neq \emptyset$. Since $C_n \subseteq U_n$, $V = U \cap U_n$ is a non-empty open set in X.

Moreover for all $i \geq n+1$, $C_{i+1} \subseteq X \setminus U_i \subseteq X \setminus U_n$ by (5) and (6) of the inductive assumptions. Thus $U_n \cap C_{i+1} = \emptyset$ for all $i \geq n$. Thus

$$U_n \cap \bigcup_{i=n+1}^{\infty} C_i = \emptyset$$

and therefore,

$$U_n \cap \overline{\bigcup_{i=n+1}^{\infty} C_i} = \emptyset. \tag{1}$$

Now $U \subseteq \overline{C}$ so $V \subseteq \overline{C}$, that is,

$$V \subseteq \overline{C_0 \cup C_1 \cup \ldots \cup C_n} \cup \overline{\bigcup_{i=n+1}^{\infty} C_i}$$

But by (1), this implies

$$V \subseteq \overline{C_0 \cup C_1 \cup \ldots \cup C_n}.$$

This means that the closure of the union of the first n of the C_k contains a non-empty open set and hence the union of the first n of the C_k is not nowhere dense. But this contradicts the fact that a finite union of nowhere dense sets is nowhere dense.

Thus C must be nowhere dense in X and we can conclude that there is no regular space of transfinite cohesion. □

Given this result we may now feel justified in upgrading Lemma 4.4 to give a theorem very much like a dimension sum theorem.

Theorem 4.6. *If A and B are subsets of some space X, at least one of which is closed, such that $cohA \leq n$, $cohB \leq m$ and $A \cup B = X$ then $cohX \leq n + m + 1$.*

Proof: Assume without loss of generality that A is closed. Then take $U = X \setminus A$ so that U is open in X, $U \subseteq B$ giving $cohU \leq m$ and $coh(A \setminus U) \leq n$. Applying Lemma 4.4, we get that $coh(A \cup U) \leq n + m + 1$, that is,

$$cohX \leq n + m + 1. \quad \square$$

5. Two Mapping Theorems

There are some continuous mappings on scattered spaces which preserve "scattered-ness", so, by the results relating cohesion to scattered spaces, it is reasonable to suppose that these mappings also preserve cohesion. However, as the two concepts are not the same, we need a somewhat different approach.

Theorem 5.1. *If $f : X \to Y$ is an open, continuous surjection and $cohX \leq \alpha$ for some $\alpha \in Ord$ then*

$$cohY \leq \alpha.$$

Proof: The proof is by transfinite induction.

If $cohX = -1$ then X is empty and f is surjective so it must be that Y is empty and hence $cohY = -1$.

Thus assume that the theorem holds for all ordinals $\beta < \alpha$ and that $cohX = \alpha$. Consider $C \subseteq Y$ which is nowhere dense in Y. If $f^{-1}(C)$ is not nowhere dense in X then there exist $U \subseteq X$, open in X such that $U \subseteq \overline{f^{-1}(C)}$. But f is continuous so $\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C})$. Hence

$$U \subseteq f^{-1}(\overline{C}),$$

$$\text{and } f(U) \subseteq \overline{C}.$$

But f is open so $f(U)$ is open and non-empty in Y giving $Int_Y\overline{C}^Y \neq \emptyset$, contradicting the fact that C is nowhere dense.

Therefore $f^{-1}(C)$ is nowhere dense in X and $cohf^{-1}(C) < \alpha$. Define $g = f\mid_{f^{-1}(C)}$ so that $g : f^{-1}(C) \to C$ is a continuous surjection. g is also open since if $V \subseteq f^{-1}(C)$ is open in $f^{-1}(C)$ then $V = U \cap f^{-1}(C)$ for some U open in X. But then

$$g(V) = f(U \cap f^{-1}(C)) = f(U) \cap f(f^{-1}(C)) = f(U) \cap C.$$

And $f(U)$ is open in Y so $g(V)$ is open in C.

Now from the inductive hypothesis $cohC < \alpha$. But this is for an arbitrary nowhere dense subset of Y hence

$$cohY \leq \alpha. \quad \square$$

The technique of this last proof can be carried over for perfect maps using the next two lemmas which may be found in [7].

Lemma 5.2. *For $f : X \to Y$ a continuous surjection, f is closed and irreducible iff for every non-empty open subset U of X, $f^*(U)$ is non-empty and open in Y.*

To apply this usefully we need one more tool:

Lemma 5.3. *If $f : X \to Y$ is perfect then there exists $A \subseteq X$ which is closed in X such that $f\mid_A : A \to Y$ is irreducible and perfect.*

We can now prove our second mapping theorem.

Theorem 5.4. *If $f : X \to Y$ is perfect and $cohX \leq \alpha$ for some $\alpha \in Ord$ then*

$$cohY \leq \alpha.$$

Proof: Assume for the purposes of induction that for all $\beta < \alpha$ the theorem is true and that $cohX = \alpha$.

Take $A \subseteq X$ and $g = f\mid_A\colon A \to Y$ as given in Lemma 5.3. If C is a nowhere dense subset of Y so too is $D = \overline{C}^Y$ Suppose $g^{-1}(D)$ is not nowhere dense in A. Then there exists a non-empty open set $U \subseteq A$ such that $U \subseteq g^{-1}(D)(= \overline{g^{-1}(D)}^X$ as D is closed and g is continuous). But then, by Lemma 5.2, $g^*(U)$ is non-empty and open in Y since g is irreducible. Also $g^*(U) \subseteq g(U) \subseteq D$ which contradicts the fact that D is nowhere dense in Y. Therefore $g^{-1}(D)$ is nowhere dense in X and hence for some $\beta < \alpha$

$$cohg^{-1}(D) \leq \beta < \alpha.$$

Define $h = g\mid_{g^{-1}(D)}\colon g^{-1}(D) \to D$. h is clearly a continuous surjection. As $g^{-1}(D)$ is closed it follows that h is perfect. Hence by the induction hypothesis

$$cohD \leq \beta < \alpha.$$

But $C \subseteq D$ so by Proposition 2.1,

$$cohC < \alpha,$$

and C was an arbitrary nowhere dense subset of Y so, by definition of cohesion,

$$cohY \leq \alpha. \quad \square$$

Examples. Let $f : \omega \to \Theta$ be a denumeration of the nodec space, then f is a continuous bijection. However, $coh\omega = 0$ and $coh\Theta = 1$; so, continuous maps in general do not lower cohesion.

Moreover, if $g : \omega \to \mathbb{Q}$ is a denumeration of the rationals, then it is a continuous bijection with domain having cohesion 0 but for which the range does not even have cohesion defined!

6. Some Further Problems

There remain some interesting problems which, up until now, the author has not been able to answer. Having found that there are no regular spaces of transfinite cohesion, would it be possible to come up with an Hausdorff example? Such an example would be difficult to find as it must not be sequential, scattered or regular.

Question 6.1. *Is there any space of transfinite cohesion?*

The following question is due to Arhangel'skiĭ:

Question 6.2. *Is there a crowded, compact Hausdorff space whose cohesion is defined?*[1]

By Theorem 4.3, such a space has finite cohesion. Also, any compact crowded space is uncountable (as compact countable spaces have an isolated point). But in a compact space of cohesion 1, every nowhere dense subset is closed, so compact, and discrete hence finite. An uncountable crowded space whose nowhere dense subsets are countable is called a Lusin space. Kunen [6] has shown that under MA + $\neg$CH there are no Lusin spaces. Hence we have the partial result:

Proposition 6.3. **(MA + $\neg$CH)** *There is no compact crowded space of cohesion 1.*

This however does not exclude the possibility of examples with higher cohesion in ZFC but simply requires that all subspaces of cohesion 1 of such spaces have to be countable.

The author is very grateful to R. W. Pack for drawing his attention to Kunen's result.

With regard to mappings on spaces of a given cohesion, no space with cohesion was found which had a closed image with higher cohesion.

Question 6.4. *Is there a closed (or even, just a quotient) map which raises cohesion?*

[1] I have answered this in the negative in ZFC

Appendix: Absolute Dimension

In [1], the **absolute dimension** of a space X, denoted $adim X$ is defined inductively to be:

$$adim X = -1 \text{ if and only if } X = \emptyset,$$

$$adim X = 0 \text{ if and only if } ind X = 0,$$

$$\text{for } n \in \omega \setminus \{0\}, \quad adim X \leq n \text{ if for every nowhere dense subset } C \text{ of } X, \quad adim C < n.$$

For a space X, $adim X = n$ means that $adim X \leq n$ but for any $k \in \omega$ such that $k < n$ it is not true that $adim X \leq k$. Arhangel'skiĭ went on from this definition to show that a space which is cleavable over the reals has an absolute dimension of at most one. For our purposes it is sufficient to know that every subset of $\mathbb{R}$ is cleavable over the reals. It is straightforward to see that for a space X, if $adim X = n$ for some $n \in \omega$, then, for every $A \subseteq X$, $adim A \leq n$ holds.

The definition of cohesion is based on that of absolute dimension. So, it will come as no surprise that absolute dimension has rather different properties from the usual inductive dimension functions. In particular, absolute dimension does not agree with these functions on compact metric spaces. If I denotes the closed unit interval of $\mathbb{R}$, then we have:

Theorem. *$adim I^2$ is not defined.*

Proof: Suppose $adim I^2$ is defined. We shall construct nowhere dense subsets C_n of I^2 for each $n \in \omega$ such that $adim C_n \geq n$. The definition of $adim$ then gives that $adim I^2 \geq n+1 \ \forall n \in \omega$ which obviously contradicts the fact that $adim I^2$ is defined.

Trivially $C_0 = \{(0,0)\}$ satisfies the case when $n = 0$. Take $C_1 = I \times \{0\}$. C_1 is cleavable over the reals as it is embeddable in the real line and it is not empty or zero-dimensional so $adim C_1 = 1$. Clearly C_1 is closed and contains no open set in I^2 hence C_1 is nowhere dense in I^2.

Define $C_2 = I \times (\{0\} \cup \{\frac{1}{n} : n \in \omega \text{ and } n \geq 2\})$. This gives a sequence of lines converging down to C_1. As a product of two closed subsets of I, C_2 is closed in I^2 and clearly it cannot contain any open subset of I^2 so C_2 is nowhere dense in I^2. Any open set, U, in C_2 about a point $(x,0) \in I \times \{0\}$ contains an open ball of radius ϵ so $\forall n \in \omega$ such that $\frac{1}{n} \leq \epsilon$, $(x, \frac{1}{n}) \in U$. Thus C_1 contains no non-empty open subset of C_2 and is closed in C_2 so C_1 is nowhere dense in C_2. But $adimC_1 = 1$ hence $adimC_2 \geq 2$. ($adimC_2$ exists because of the assumption that $adimI^2$ exists.)

In general given C_k and noting that $\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$, define $C_{k+1} = C_k \cup \{\frac{1}{n_1} + \ldots + \frac{1}{n_k} : n_1 \geq 2, n_{i+1} \geq 2n_i(n_i - 1) \text{ for } i = 1, \ldots, k-1\}$. As before this gives a sequence of lines converging down to each line in C_k. It can be seen that C_{k+1} is closed and nowhere dense in I^2 (as a countable collection of horizontal lines, C_{k+1} cannot contain a non-empty open subset of I^2). As for C_1 in C_2, C_k is nowhere dense in C_{k+1} and hence $adimC_{k+1} \geq k+1$.

Therefore, $\forall n \in \omega\ \exists C_n \subseteq I^2$ such that $adimC_n \geq n$ which are the sets prophesied at the beginning of the proof and we are done. □

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