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BASES, π-BASES AND QUASI-DEVELOPMENTS

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ABSTRACT. The aim of this paper is to give a characterization of those spaces having a σ-disjoint base or a σ-disjoint π-base by means of some quasi-development-like properties. Moreover we show that a space is almost countably subcompact iff it has a countably complete π-base.

Let $\mathcal{A} = \{A_t : t \in T\}$ be a collection of subsets of a set X, the star of a set $M \subset X$ with respect to $\mathcal{A}$ is the set $St(M, \mathcal{A}) = \bigcup\{A_t : M \cap A_t \neq \emptyset\}$. The star of a one-point set $\{x\}$ is denoted $St(x, \mathcal{A})$. A sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of a space X is called a development for X if for every $x \in X$ and any open neighbourhood U of x there exists a natural number i such that $St(x, \mathcal{U}_i) \subset U$.

A sequence $\{\mathcal{G}_n : n \in N\}$ of families of open sets of a space X is called a :

1) quasi-development for X if for every $x \in X$ and any open neighbourhood U of x there is a natural number i such that $x \in St(x, \mathcal{U}_i) \subset U$ [3]. A space is called quasi-developable if it has a quasi-development;

2) strong quasi-development for X if for every $x \in X$ and any open neighbourhood U of x there exist an open neighbourhood V of x and a natural number i such that $x \in \bigcup\{G : G \in \mathcal{G}_i\}$ and $x \in St(V, \mathcal{G}_i) \subset U$;

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3) quasi-development of order 2 for X if for every $x \in X$ and any open neighbourhood U of x there exists a natural number i such that $x \in \bigcup\{G : G \in \mathcal{G}_i\}$ and $St^2(x, \mathcal{G}_i) \subset U$ (where $St^2(x, \mathcal{G}_i) = St(St(x, \mathcal{G}_i), \mathcal{G}_i)$.

A space X is called screenable [4] if every open cover of X has a σ-disjoint open refinement. X is called hereditarily screenable if every subspace of X is screenable.

Theorem 1. *For every space X the following conditions are equivalent :*

(i) X has a σ-disjoint base.

(ii) X has a strong quasi-development.

(iii) X has a quasi-development of order 2.

(iv) X is quasi-developable and hereditarily screenable.

Proof: (i) $\to$ (ii) If $\mathcal{B} = \bigcup_{n\in N} \mathcal{B}_n$ is a σ-disjoint base for X (where each $\mathcal{B}_n$ is a disjoint family) then $\{\mathcal{B}_n : n \in N\}$ is a strong quasi-development for X.

(ii) $\to$ (iii) Let $\{\mathcal{G}_n : n \in N\}$ be a strong quasi-development for X. For every $m, n \in N$ set $\mathcal{V}_{m,n} = \{A \cap B : A \in \mathcal{G}_m, B \in \mathcal{G}_n\}$. Obviously $\mathcal{V}_{m,n}$ is a refinement of $\mathcal{G}_m$ and $\mathcal{G}_n$. We claim that $\{\mathcal{V}_{m,n} : m, n \in N\}$ is a quasi-development of order 2 for X. Let $x \in X$ and let G be an open neighbourhood of x. Then there exist an open neighbourhood V of x and a natural number n such that $x \in \bigcup\{U : U \in \mathcal{G}_n\}$ and $St(V, \mathcal{G}_n) \subset G$. Moreover there exist an open neighbourhood W of x and a natural number m such that $x \in \bigcup\{U : U \in \mathcal{G}_m\}$ and $St(W, \mathcal{G}_m) \subset V$, so $St(x, \mathcal{G}_m) \subset V$. Obviously $x \in \bigcup\{V : V \in \mathcal{V}_{m,n}\}$; moreover $St^2(x, \mathcal{V}_{m,n}) = St(St(x, \mathcal{V}_{m,n}), \mathcal{V}_{m,n}) \subset St(St(x, \mathcal{G}_m), \mathcal{V}_{m,n}) \subset St(V, \mathcal{V}_{m,n}) \subset St(V, \mathcal{G}_n) \subset G$.

(iii) $\to$ (iv) Let $\{\mathcal{U}_n : n \in N\}$ be a quasi-development of order 2 for X. Obviously $\{\mathcal{U}_n : n \in N\}$ is a quasi-development for X. To see that X is hereditarily screenable it is enough to show that every open subspace of X is screenable. Let Y be an open subspace of X and let $\mathcal{V}$ be an open cover of Y. Let us consider a well-ordering $\preceq$ on Y and let us associate inductively an open subset A_x of X to each $x \in Y$. Given

$x \in Y$, if there is a $y \prec x$ such that $x \in A_y$ then we set $A_x = A_{\overline{x}}$, where $\overline{x} = min\{y \in Y : x \in A_y\}$. If $x \notin A_y$ for each $y \prec x$ then choose as A_x a member of $\mathcal{V}$ containing x. Then $\mathcal{A} = \{A_x : x \in Y\}$ is a subset of $\mathcal{V}$. Now let us show the following property :

$\forall x, y \in Y$ if $y \in A_x$ and $x \in A_y$ then $A_x = A_y$ (*).

Let us suppose that there exist $x, y \in Y$ such that $y \in A_x$ and $x \in A_y$ and $A_x \neq A_y$. Then $x \neq y$ and we may assume $x \prec y$. Since $\{z : z \prec y, y \in A_z\} \neq \emptyset$ we have $A_y = A_{\overline{y}}$ where $\overline{y} = min\{z : z \prec y, y \in A_z\}$ and obviously $\overline{y} \prec x$. Hence $\{z : z \prec x, x \in A_z\} \neq \emptyset$. Therefore $A_x = A_{\overline{x}}$ where $\overline{x} = min\{z : z \prec x, x \in A_z\}$ and $\overline{x} \prec \overline{y} \prec x \prec y$. Since $y \in A_x = A_{\overline{x}}$ we have $\overline{y} \preceq \overline{x}$, a contradiction. Now for every $A \in \mathcal{A}$ and for any natural number n define $M_{A,n} = \{x \in \bigcup\{U : U \in \mathcal{U}_n\} : A_x = A, St^2(x, \mathcal{U}_n) \subset A\}$, and $S_{A,n} = \bigcup\{St(x, \mathcal{U}_n) : x \in M_{A,n}\}$. Let us show that for every fixed n we have :

$$\forall A, B \in \mathcal{A} \qquad S_{A,n} \cap S_{B,n} \neq \emptyset \Rightarrow A = B.$$

If $S_{A,n} \cap S_{B,n} \neq \emptyset$ then there are $x \in M_{A,n}$ and $y \in M_{B,n}$ such that $St(x, \mathcal{U}_n) \cap St(y, \mathcal{U}_n) \neq \emptyset$, i.e. there are $U, V \in \mathcal{U}_n$ such that $x \in U$, $y \in V$ and $U \cap V \neq \emptyset$. So

$$St^2(x, \mathcal{U}_n) = St(St(x, \mathcal{U}_n), \mathcal{U}_n) \supset St(U, \mathcal{U}_n) \supset V \ni y$$

and

$$St^2(y, \mathcal{U}_n) = St(St(y, \mathcal{U}_n), \mathcal{U}_n) \supset St(V, \mathcal{U}_n) \supset U \ni x.$$

Since $St^2(x, \mathcal{U}_n) \subset A_x$ and $St^2(y, \mathcal{U}_n) \subset A_y$ ($x \in M_{A,n}$ and $y \in M_{B,n}$), $y \in A_x$ and $x \in A_y$. Hence by (*) we have $A_x = A_y$, so $A = B$. Therefore the family $\mathcal{S}_n = \{S_{A,n} : A \in \mathcal{A}\}$ is a disjoint family for any natural number n. Moreover, for every $A \in \mathcal{A}$ and for any n we have $S_{A,n} \subset A$. We claim that $\bigcup_n \mathcal{S}_n$ is a σ-disjoint open refinement of $\mathcal{V}$. It remains to show only that $\bigcup\{S : S \in \mathcal{S}_n$ for some $n\} = Y$. If $x \in Y$, then A_x is an open neighbourhood of x and there is a natural number n such that $x \in \bigcup\{U : U \in \mathcal{U}_n\}$ and $St^2(x, \mathcal{U}_n) \subset A_x$. Hence $x \in M_{A_x,n} \subset S_{A_x,n}$ and $x \in \bigcup\{S : S \in \mathcal{S}_n$ for some $n\}$.

(iv) $\to$ (i) was observed by Aull [2]. We give a proof for the sake of completeness. Let $\{\mathcal{G}_n : n \in N\}$ be a quasi-development for X. X is hereditarily screenable so for any $n \in N$ there exists a family $\mathcal{B}_n = \bigcup_k \mathcal{B}_{n,k}$ of open sets of $\bigcup\{G : G \in \mathcal{G}_n\}$ such that $\mathcal{B}_n$ is a refinement of $\mathcal{G}_n$ and every $\mathcal{B}_{n,k}$ is disjoint. Thus $\mathcal{B} = \bigcup_{n,k} \mathcal{B}_{n,k}$ is a σ-disjoint family of open sets of X. Let us show that $\mathcal{B}$ is a base for X. Let $x \in X$ and let U be an open neighbourhood of x. Since $\{\mathcal{G}_n : n \in N\}$ is a quasi-development for X there exists a natural number n such that $x \in St(x, \mathcal{G}_n) \subset U$. Since $x \in \bigcup\{G : G \in \mathcal{G}_n\}$ and $\mathcal{B}_n$ refines $\mathcal{G}_n$ there exist a $B \in \mathcal{B}_n$ and a $G \in \mathcal{G}_n$ such that $x \in B \subset G \subset St(x, \mathcal{G}_n) \subset U$. $\square$

Remark 2. Let $\mathcal{A}$ be an infinite maximal family of infinite subsets of N such that $A \cap B$ is finite whenever $A, B \in \mathcal{A}$, $A \neq B$. Let $\psi(N) = \mathcal{A} \cup N$ and describe a topology on $\psi(N)$ as follows : the points of N are isolated; basic neighborhoods of a point $A \in \mathcal{A}$ are sets of the form $\{A\} \cup (A - F)$ where F is a finite subset of N (this is the space ψ described in [6]). ψ(N) is an example of a developable space which is not screenable (it is not even meta-Lindelöf [5]).

A π-base for a space X is a family $\mathcal{B}$ of non-empty open subsets of X such that for every non-empty open set U of X there exists a $B \in \mathcal{B}$ such that $B \subset U$.

A sequence $\{\mathcal{G}_n : n \in N\}$ of families of open subsets of a space X is called a :

i) π-quasi-development for X if for every non-empty open subset U of X there is an $x \in X$ and a natural number i such that $\emptyset \neq st(x, \mathcal{G}_i) \subset U$;

ii) strong π-quasi-development for X if for every non-empty open subset U of X there is an open set V and a natural number i such that $\emptyset \neq St(V, \mathcal{G}_i) \subset U$. A space is called strongly π-quasi-developable if it has a strong π-quasi-development.

$\beta\omega$ is an example of a strongly π-quasi-developable space (it has a countable π-base) which is not quasi-developable (it is not even first countable).

Theorem 3. *A space X has a σ-disjoint π-base if and only if it is strong π-quasi-developable.*

Proof: If $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is a σ-disjoint π-base for X then $\{\mathcal{B}_n : n \in N\}$ is a strong π-quasi-development for X. Now let $\{\mathcal{G}_n : n \in N\}$ be a strong π-quasi-development for X. For every $n \in N$ let $\mathcal{B}_n$ be a maximal disjoint family of non-empty open subsets of X such that for every $B \in \mathcal{B}_n$ there exists a $G \in \mathcal{G}_n$ with $B \subset G$. We claim that $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is a σ-disjoint π-base for X. Let U be a non-empty open set of X. By hypothesis there are an open set V of X and a natural number i such that $\emptyset \neq St(V, \mathcal{G}_i) \subset U$. Let $G \in \mathcal{G}_i$ be such that $G \cap V \neq \emptyset$; obviously $G \subset U$. Now let us consider the open set $H = G \cap V$. By the maximality of $\mathcal{B}_i$ it follows that there is a $B \in \mathcal{B}_i$ such that $H \cap B \neq \emptyset$. Hence $\emptyset \neq G \cap V \cap B \subset B \cap V$. Let $A \in \mathcal{G}_i$ be such that $B \subset A$. Then $\emptyset \neq B \cap V \subset A \cap V$ and $\emptyset \neq B \subset A \subset St(V, \mathcal{G}_i) \subset U$. □

Remark 4. H.E. White,Jr. showed that a first countable Hausdorff space has a σ-disjoint π-base if and only if it has a dense metrizable subspace [9]. From this result it easily follows that a Hausdorff first countable space is strongly π-quasi-developable iff it has a dense metrizable subspace.

A collection $\mathcal{F}$ of non-empty subsets of a space X is called a regular filterbase if, whenever $F_1, F_2 \in \mathcal{F}$, some $F_3 \in \mathcal{F}$ has $\overline{F_3} \subset F_1 \cap F_2$ [7]. Let $\mathcal{B}$ be a π-base for a space X. A sequence $\{U_n \in \mathcal{B} : n \in N\}$ is called a regular $\mathcal{B}$-sequence if $\overline{U}_{n+1} \subset U_n$ for any n. A space X is called almost countably subcompact if there is a π-base $\mathcal{B}$ for X such that if $\mathcal{F} \subset \mathcal{B}$ is a countable regular filterbase, then $\bigcap \mathcal{F} \neq \emptyset$ [1]. A π-base $\mathcal{B}$ for a space X is said to be countably complete if every regular $\mathcal{B}$-sequence has non-empty intersection.

Proposition 5. *X is almost countably subcompact iff it has a countably complete π-base.*

Proof: The necessity is obvious. Now let $\mathcal{B}$ be a countably complete π-base for a space X. We claim that X is almost countably subcompact (cf. lemma 1.1 in [8]). Let us consider a countable regular filterbase $\mathcal{F} = \{F_n : n \in N\} \subset \mathcal{B}$. Define inductively a sequence $\{G_n : n \in N\} \subset \mathcal{F}$ in this way: $G_1 = F_1$ and $G_{n+1} \in \mathcal{F}$ such that $\overline{G}_{n+1} \subset G_n \cap F_{n+1}$. Then $\{G_n : n \in N\}$ is a regular $\mathcal{B}$-sequence so $\emptyset \neq \bigcap_n G_n \subset \bigcap \mathcal{F}$. $\square$

Remark 6. If X has a σ-disjoint π-base and a countably complete π-base then X has a σ-disjoint countably complete π-base. In fact let $\mathcal{B}$ be a countably complete π-base and $\bigcup_n \mathcal{G}_n$ be a σ-disjoint π-base for X. For every natural number n and every $G \in \mathcal{G}_n$ choose a $B_G \in \mathcal{B}$ such that $B_G \subset G$ and let $\mathcal{A}_n = \{B_G : G \in \mathcal{G}_n\}$. $\bigcup_n \mathcal{A}_n$ is a σ-disjoint family of non-empty open subsets of X. We claim that $\mathcal{A} = \bigcup_n \mathcal{A}_n$ is also a countably complete π-base for X. Let U be a non-empty open subset of X, let $n \in N$ and $G \in \mathcal{G}_n$ be such that $G \subset U$. Then $B_G \in \mathcal{A}$ and $B_G \subset U$. If $\mathcal{F}$ is a regular $\mathcal{A}$-sequence then it is a regular $\mathcal{B}$-sequence, hence $\bigcap \mathcal{F} \neq \emptyset$ and $\mathcal{A}$ is a countably complete π-base.

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