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MORE ON TOPOLOGICAL COMPLETIONS OF METRIZABLE SPACES

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ABSTRACT. We continue the investigation of determining the completions and completion remainders of various metrizable spaces. We introduce the study of spaces that are the differences of two completely metrizable spaces and of spaces that are rim-complete: possessing bases whose elements have complete boundaries. These notions are used to characterize the completion remainders of $\mathbb{P}$, the irrationals.

1. INTRODUCTION

All spaces under consideration are assumed to be metrizable. A space is called complete provided that it is completely metrizable (= absolute G_δ-set). By analogy with the notion of rim-compactness, we define a space to be rim-complete if it has a basis whose elements have complete boundaries. A space X is called strongly rim-complete provided that if $H \subseteq U \subseteq X$, with H closed and U open, then there is an open set V in X, such that $H \subseteq V \subseteq U$ and $\beta_X(V)$ is complete. Here, β_X denotes the boundary operator in the space X. Our interest in these notions stems from their usefulness in discussing topological completions of metrizable spaces. For example, a separable space X is a completion remainder of $\mathbb{P}$, the irrationals, if, and only if, X is both rim-complete and the difference of two complete spaces. This is shown in Section 2; it provides an answer to a question raised in [FGO]. Section 3 deals with nonseparable spaces. Section 4 provides a sufficient condition for a space to be rim-complete. Section 5 provides characterizations of spaces that are the differences of complete (absolute G_δ-sets) spaces. Section 6 discusses ambiguous classes and provides two examples of interest.

2. Completion remainders of the irrationals

If X and Y are spaces, Z is a complete space, and there exists a homeomorphism $h : X \to Z$ such that $h(X)$ is dense in Z and $Z \backslash h(X)$ is homeomorphic to Y, then Z is called a completion of X, and Y is called a completion remainder of X; in case $h(X)$ can be taken to be open in such a Z then Y is called a closed completion remainder of X. Denote by $\mathbb{Q}$ or $\mathbb{P}$ the space of rationals or irrationals, respectively, with the usual topology. It was shown in [FGO] that the completion remainders of $\mathbb{Q}$ are the nowhere locally compact Polish spaces, and the problem of determining the completion remainders of $\mathbb{P}$ was posed.

Theorem 1. *In order that the separable space X be a completion remainder of $\mathbb{P}$, it is necessary and sufficient that 1) X be the difference of two complete spaces, and 2) X be rim-complete.*

Proof: (Necessity) Assume that Z is a complete space, $Z = A \cup X$, $A \cap X = \emptyset$, A is dense in Z, and $A \simeq \mathbb{P}$. Then $X = Z \backslash A$ and is thus the difference of two complete spaces. Since A is 0-dimensional there is a basis for the topology of Z at points of X whose elements have boundaries missing A. If $\beta(g)$ is such a boundary it is a closed subset of the complete space Z and is, therefore, complete. So, X is rim-complete.

(Sufficiency) Assume that X is separable, rim-complete, and the difference of two complete spaces. We may assume that $X = B \backslash C$, where B and C are complete and $C \subseteq B$.

We first consider the special case in which both C and $B \backslash C$ are dense in B.

We shall use the following elementary lemma.

Lemma 1. *If A is dense in X, U is open in A, V is open in X, and $V \cap A = U$, then $\beta_X(V) \cap A = \beta_A(U)$.*

There is a sequence $\{G_n : n < \omega\}$ of countable open (in B) covers of X such that

1) $\forall n < \omega$, $G_{n+1} \subseteq G_n$, and if $g \in G_{n+1}$ then $Cl_B(g)$ is a subset of some element of G_n,
2) $\forall n < \omega$, $\forall g \in G_n$, diam $g < 1/n$, where the diameter is taken in a (fixed) complete metric on B,
3) $\forall n < \omega$, $\forall g \in G_n$, $\beta_X(g \cap X) = \beta_B(g) \cap X$ is complete.

Let $Y = \bigcap_{n<\omega}(\bigcup G_n)$. Since $X \subseteq Y$ and X is dense in B, Y is dense in B; also, Y is a G_δ-set in a complete space and is complete. We note that X is dense in Y. Each $\bigcup G_n$ contains a dense G_δ-set C_n in C, so Y contains $\bigcap_{n<\omega} C_n$, a dense G_δ-set C' in C, and C' is dense in B, and $C \cap X = \emptyset$, so $Y \backslash X$ contains a dense set C' in Y. That is, both X and $Y \backslash X$ are dense in Y (and in B).

Let

$$\begin{aligned} A &= (Y \backslash X) \backslash (\bigcup_{n<\omega} \bigcup_{g \in G_n} (\beta_B(g))) \\ &= (Y \cap C) \backslash (\bigcup_{n<\omega} \bigcup_{g \in G_n} (\beta_B(g) \cap (Y \cap C))). \end{aligned}$$

Since $\beta_B(g)$ is closed in B, $\beta_B(g) \cap Y \cap C$ is closed in the complete space $Y \cap C$, so $\bigcup_{n<\omega} \bigcup_{g \in G_n} (\beta_B(g) \cap Y \cap C)$ is an F_σ-set in $Y \cap C$, so its complement A in that space is a G_δ-set in $Y \cap C$ and is, therefore, complete.

By the Baire Category Theorem, A is dense in $Y \backslash X$ and, therefore, in B. Since A has a (countable) basis whose elements have empty boundaries, $\dim A = 0$. Since both A and its complement are dense in B, A is nowhere locally compact.

Therefore, $A \simeq \mathbb{P}$. We note that $A \cap X = \emptyset$ and that A is dense in $A \cup X$.

Now,

$$\begin{aligned} A \cup X &= Y\backslash(\bigcup_{n<\omega} \bigcup_{g\in G_n} (\beta_B(g)\backslash\beta_X(g\cap X))) \\ &= Y\backslash \bigcup_{n<\omega} \bigcup_{g\in G_n} (\beta_B(g)\cap Y\backslash\beta_X(g\cap X)\cap Y))). \end{aligned}$$

Since $\beta_X(g\cap X)$ is complete, $\beta_X(g\cap X)\cap Y$ is complete, and $\beta_B(g)\cap Y$ is closed in Y, so $(\beta_B(g)\cap Y)\backslash(\beta_X(g\cap X)\cap Y)$ is an F_σ-set in Y; so its complement in Y is complete.

This completes the sufficiency proof in the special case.

We now consider the general case: $X = B\backslash C$, where B and C are complete, $C \subseteq B$, and X is rim-complete. If C is empty, it is known [FGO] that X is a completion remainder of $\mathbb{P}$, so we assume $C \neq \emptyset$ and $C \neq B$.

Let $S = \{x \in X : X \text{ is locally complete at } x\}$, and let $T = X\backslash S$. Then S is open in X and is complete; so, S is a closed completion remainder of $\mathbb{P}$ [FGO]. The space T is nowhere locally complete. It is straightforward to show that rim-completeness is inherited by closed subsets; consequently; T is rim-complete. Since S is complete, $T = X\backslash S = (B\backslash C)\backslash S = B\backslash(C\cup S)$ is the difference of two complete spaces.

Let $B_1 = Cl_B(T)$; let $C_1 = (C\cup S)\cap B_1$; let $T_1 = B_1\backslash C_1$. Then B_1 and C_1 are complete, and $C_1 \subseteq B_1$. Since T is nowhere locally complete, C_1 is dense in B_1. Since T is dense in B_1, $B_1\backslash C_1$ is dense in B_1. By the special case, it follows that T is a completion remainder of $\mathbb{P}$.

We now regard X as a subset of a face F in the Hilbert cube I^ω. Since S is open in X, we may assume there is an open set U in F such that $S = U\cap X$ and $Z = F\backslash U$ is a compact space with no isolated points. Applying the construction of the special case to Z, we obtain a copy K of $\mathbb{P}$, $K \subseteq Z$, such that K is dense in $K\cup T$, and $K\cup T$ is complete. Since S is complete, $S \subseteq F$, and S is a closed completion remainder of $\mathbb{P}$, [FGO], it follows that there is a copy J of $\mathbb{P}$, $J \subseteq I^\omega\backslash F$, J is dense in $J\cup S$, and $J\cup S$ is complete. We have $(J\cup K)\cap(S\cup T) = \emptyset$, $J\cup K$ is dense in $(J\cup K)\cup X$, and $(J\cup K)\cup X = (J\cup S)\cup(K\cup T)$ is the union of two complete spaces and is thus complete. Clearly, $J \cup K \simeq \mathbb{P}$.

This completes the proof of Theorem 1.

Remarks. The 0-dimensional case of Theorem 1 was presented by the second author at the Auburn Spring Topology Conference, March 1994. It is known [FGO] that $\mathbb{Q} \times I$ is not a completion remainder of $\mathbb{P}$. It is the difference of two complete spaces, so it is not rim-complete. It can easily be embedded as a closed subset in $\mathbb{R}^3$ of a connected and locally connected set, so rim-completeness is not a consequence of these connectedness properties. Of course, a space may be rim-complete without being the difference of two complete spaces, for example, a 0-dimensional subset of $\mathbb{R}$ that is not a Borel set.

Theorem 2. *If X is separable and not compact, then X is a completion remainder of $\mathbb{P}$ if, and only if, $\mathbb{P}$ is a completion remainder of X.*

Proof: Suppose first that $\mathbb{P}$ is a completion remainder of X. Then there is a complete space $Z = A \cup X$, where $A \simeq \mathbb{P}$, $A \cap X = \emptyset$, and X is dense in Z. Then $X = P \backslash A$, so X is the difference of two complete spaces. Let $p \in U \subseteq X$, U open in X. There is an open set W in Z such that $W \cap X = U$. Since A is 0-dimensional, there is an open set V in Z, $p \in V \subseteq W$, and $\beta_Z(V) \cap A = \emptyset$. Then $W \cap X$ is open in X, $W \cap X \subseteq U$, and $\beta_X(W \cap X) = \beta_Z(V)$ is complete. Therefore, X is rim-complete, and by Theorem 1, X is a completion remainder of $\mathbb{P}$.

Next, suppose X is a completion remainder of $\mathbb{P}$. Then X is rim-complete and the difference of two complete spaces. If X is complete, then by Theorem 1 of [FGO], $\mathbb{P}$ is a completion remainder of X. Let $T = \{x \in X : X \text{ is not locally complete at } x\}$. Regard X as a subset of a face F of the Hilbert cube. As in the sufficiency proof of Theorem 1, there is a copy K of $\mathbb{P}$ in F such that $K \cap T = \emptyset$, K is dense in $K \cup T$, and $K \cup T$ is complete. But T is also dense in $K \cup T$, or at least it can be so taken, for T is dense in $Cl_F(T)$, and K can be taken to be a subset of $Cl_F(T)$. Since T is closed in X, $K \cap X = \emptyset$. So, $X \cup K = (X \backslash T) \cup (K \cup T)$ is a complete space, X is dense in it, and the complement of X in it is K, a copy of $\mathbb{P}$. Therefore, $\mathbb{P}$ is a completion remainder of X.

Theorem 3. *If X is separable, then X is rim-complete if, and only if, there is a completion Z of X such that* $\dim(Z \backslash X) \leq 0$.

Proof: If X is itself complete but not compact, then we can add a single point p to X in such a way that X is dense in $X \cup \{p\}$. Suppose X is not complete, and let T be the set of all points of X at which X is not locally complete. Regard X as a dense subset of a compact space Z. Then $Cl_Z(T)$ is compact, and T and $Cl_Z(T) \backslash T$ are dense in $Cl_Z(T)$. We proceed as in the special case of Theorem 1; this time it does not follow that A is complete. It is true, however, that A is 0-dimensional and that $A \cup T$ is complete. Then $X \cup A$ is complete, and X is dense in $X \cup A$.

Remark. The construction can be strengthened to yield that if X is rim-complete, separable and not compact, and if H and K are disjoint closed sets in X, then X has a completion Z such that $Z \backslash X$ is 0-dimensional and $Cl_Z(H) \cap Cl_Z(K) = \emptyset$.

Theorem 4. *If X is separable and rim-complete, then X is strongly rim-complete.*

Proof: Suppose $H \subseteq U \subseteq X$, where H is closed and U is open. Then H and $X \backslash U$ are disjoint closed sets in X. By the remark above, there is a completion Z of X such that $Z \backslash X$ is 0-dimensional and $Cl_Z(H) \cap Cl_Z(X \backslash U) = \emptyset$. There exist open set W_H and W' in Z such that $Cl_Z(H) \subseteq W_H$, $Cl_Z(X \backslash U) \subseteq W'$, and $Cl_Z(W_H) \cap Cl_Z(W') = \emptyset$. By a standard theorem from dimension theory [HW, 16-17], there is an open set V in Z such that $Cl_Z(H) \subseteq V \subseteq W_H$ and $\beta_Z(V) \cap (Z \backslash X) = \emptyset$. Now, $\beta_Z(V)$ is a closed subset of the complete space Z, so it is complete. Let $S = V \cap X$. Then $H \subseteq S \subseteq W$. Also, $\beta_X(S) = \beta_X(V \cap X) = \beta_Z X(V) \cap X = \beta_Z(V)$ and is complete.

Remark. We do not have an example of a rim-complete space that is not strongly rim-complete.

3. Spaces that may not be separable

In this section we establish the hereditary character of the properties under discussion, and we discuss completions of strongly rim-complete spaces.

Theorem 5. *A. If X is rim-complete and M is a G_δ-set in X, then M is rim-complete. B. If X is strongly rim-complete and M is closed in X, then M is strongly rim-complete.*

Proof: (A) Special case 1: M is closed in X. Let $p \in U$, U open in M. There is an open set V in X, $V \cap M = U$. There is an open set W in X, $p \in W \subseteq V$, with $\beta_X(W)$ complete. Then $\beta_M(W \cap M) \subseteq \beta_X(W) \cap M$, and $\beta_X(W) \cap M$ is a closed subset of the complete space $\beta_X(W)$, so it is complete. Now, $\beta_M(W \cap M)$ is closed in M, so it is closed in every subset of M of which it is a subset; i.e., $\beta_M(W) \cap M$ is closed in the complete space $\beta_X(W) \cap M$, so it is complete.

Special case 2: M is a dense G_δ-set in X. Then $M = \bigcap_{n<\omega} U_n$, where U_n is open in X and $U_n \supseteq U_{n+1}$. Let g be open in X, with $\beta_X(g)$ complete. Then $\beta_M(g \cap M) = \beta_X(g) \cap M = \bigcap_{n<\omega}(\beta_X(g) \cap U_n)$. For each n, $\beta_X(g) \cap U_n$ is open in $\beta_X(g)$ and is complete. So, $\bigcap_{n<\omega}(\beta_X(g) \cap U_n)$ is complete.

General case. By special case 1, $Cl_X(M)$ is rim-complete, and M is a dense G_δ-set in $Cl_X(M)$, so by special case 2, M is rim-complete.

(B) The argument is straightforward and is omitted.

Theorem 6. *If X is the difference of two complete spaces and M is a G_δ-subset of X, then M is the difference of two complete spaces.*

We omit the proof, as it is straightforward.

We next address the question of completing spaces that are strongly rim-complete.

Lemma 2. *If X is strongly rim-complete and dense in Z and $\mathcal{U}$ is a σ-discrete open (in Z) cover of X, then there exists a σ-discrete open (in Z) refinement $\mathcal{V}$ of $\mathcal{U}$ which covers X such that if $V \in \mathcal{V}$ then $\beta_Z(V) \cap X$ is complete.*

Proof: For $U \in \mathcal{U}$, let $\{U_n(U) : n < \omega\}$ be a sequence of open sets such that $Cl_Z(U_n(U) \subseteq U_{n+1}(U)$ and $U = \bigcup_{n<\omega} U_n(U)$. Let $H_n(U) = Cl_Z(U_n(U)) \cap X$, $V_n(U) = U_{n+1}(U) \cap X$. There exists $D_n(U)$, open in X, such that $H_n(U) \subseteq D_n(U) \subseteq V_n(U)$ and such that $\beta_X(D_n(U))$ is complete. There exists $W_n(U)$, open in Z, such that $W_n(U) \cap X = D_n(U)$. Then $\beta_Z(W_n(U)) \cap X$ is complete. For each $n < \omega$, $\{W_n(U) : U \in \mathcal{U}$ is σ-discrete, so $\{W_n(U) : n < \omega, U \in \mathcal{U}\}$ is σ-discrete.

Corollary to Lemma 2: If X is strongly rim-complete, then X has a σ-discrete basis whose elements have complete boundaries.

Theorem 7. *If X is strongly rim-complete and the difference of two complete spaces, then X is a completion remainder of a strongly 0-dimensional, complete space.*

Theorem 8. *If X is strongly rim-complete, then there is a completion Z of X such that* $\dim(Z \backslash X) \leq 0$.

Indication of Proofs. The proofs are very much like those of Theorems 1 and 3; Lemma 2 is used to provide a sequence $\{G_n : n < \omega\}$ of σ-discrete open covers of X with the properties 1), 2), 3) listed before. The set A is now strongly 0-dimensional, because it has a σ-discrete basis whose elements have empty boundaries. Instead of using a face F of the Hilbert cube, we use a face F of the ω-power of a suitable hedgehog.

4. Sufficient conditions for a space to be rim-complete

As was noted following Theorem 1, $\mathbb{Q} \times I$ is not rim-complete. In this section we discuss spaces $X = \bigcup_{n<\omega} X_n$, where each X_n is closed, $X_n \cap X_m = \emptyset$ for $n \neq m$, and there is a metric on

X such that diam $X_n \to 0$ as $n \to \infty$. We show that if, in addition, each X_n is rim-complete, then X is rim-complete.

Lemma 3. *If, in a metric space Z, $G = \{X_n : n < \omega\}$ is a collection of mutually exclusive closed sets and diam $X_n \to 0$ as $n \to \infty$, then G is upper semicontinuous.*

Proof: Let U be an open set containing X_0. For each $x \in X_0$, there is an $\alpha_x > 0$ such that $B(x, \alpha_x)$, the open α_x ball centered at x, is contained in U. There is a positive integer n_x such that if $k \geq n_x$ then diam $X_k < \alpha_x/3$. There is a $\delta_x > 0$ such that $\delta_x < \alpha_x/3$ and $B(x, \delta_x)$ misses $\bigcup_{1 \leq i \leq n_x} X_i$. Let $V_x = B(x, \delta_x)$. If $k > 0$ and $X_k \cap V_x \neq \emptyset$, then $k > n_x$, so diam $X_k < \alpha_x/3$, and dist $(x, X_k) < \delta_x < \alpha_x/3$, so $X_k \subseteq B(x, \alpha_x) \subseteq U$. Let $V = \bigcup_{x \in X_0} V_x$. Then $X_0 \subseteq V \subseteq U$, and if $X_j \cap V \neq \emptyset$, then $X_j \subseteq U$. Thus G is upper semicontinuous at X_0, and similarly G is upper semicontinuous at each of its elements.

Lemma 4. *If G is as in Lemma 3, then the decomposition space of $\bigcup G$ obtained by contracting each element of G to a point is normal; hence, homeomorphic to a subset of $\mathbb{Q}$.*

Proof: Normality is given in [K, page 185].

Since the decomposition space is countable and normal, it is metrizable, and every countable metrizable space is homeomorphic to a set of rational numbers.

Theorem 9. *If $X = \bigcup_{n<\omega} X_n$, where $X_n \cap X_m = \emptyset$ for $m \neq n$, diam $X_n \to 0$ as $n \to \infty$, and each X_n is closed in X and rim-complete, then X is rim-complete.*

Proof: Let $x \in U \cap X_0$, where U is open in X. There is an open set W_1 in X such that $x \in W_1$ and $Cl_X(W_1) \subseteq U$. There is an open set V_1' in X_0 such that $x \in V_1' \subseteq W_1 \cap X_0$ and $\beta_{X_0}(V_1')$ is complete. There is an open set V_1 in X such that $V_1 \cap X_0 = V_1'$ and $V_1 \subseteq W_1$. Then $Cl_X(V_1) \subseteq U$ and $\beta_{X_0}(V_1 \cap X_0)$ is complete. Let $X_0' = X_0 \cap V_1$ and let $G = \{X_0'\} \cup \{X_j; 0 < j < \omega\}$. Then G satisfies the conditions of Lemma 3. Let $f : \bigcup G \to Y$ be the quotient mapping from G onto the decomposition space Y, and regard Y as a subset of $\mathbb{Q}$. There is an open set V in X such that $V \cap X_0 = X_0'$, $V \subseteq U$, and V contains every element of G that it intersects. Since $\bigcup G$ is open in X and f is a closed mapping, V could be chosen as $f^{-1}(E)$, where E is a subset of Y without boundary and E contains $f(X_0')$. Clearly, $\beta_Z(V) = \beta_{X_0}(V) = \beta_{X_0}(V \cap X_0) = \beta_{X_0}(V_1 \cap X_0)$.

5. Spaces that are the differences of two absolute G_δ-sets

It seems appropriate in this section to use the terminology 'absolute G_δ-set' instead of the equivalent 'complete space,' because of the constructions which arise.

Theorem 10. *The space X is the difference of two absolute G_δ-sets if, and only if, X is the union of countably many closed subsets, each an absolute G_δ-set.*

Proof: Suppose $X = B \backslash C$, where $C \subseteq B$ and B and C are absolute G_δ-sets. Now $C = \bigcap_{n<\omega} U_n$, where each U_n is open in B. Then $X = B \backslash (\bigcap_{n<\omega} U_n) = \bigcup_{n<\omega}(B \backslash U_n)$. Each $B \backslash U_n$ is closed in the space B and is thus an absolute G_δ-set; also, each $B \backslash U_n$ is closed in every subset of B of which it is a subset.

Next, suppose $X = \bigcup_{n<\omega} G_n$, where each G_n is closed in X and is an absolute G_δ-set. Let Z be a completion of X. Then, for each n, $G_n = \bigcap_{i<\omega} U_{n,i}$, where $U_{n,i}$ is open in Z and $U_{n,i+1} \subseteq U_{n,i}$.

Let $V_{0,i} = U_{0,i}$, and for $1 \leq n < \omega$, let $V_{n,i} = U_{n,i} \backslash Cl_Z(\bigcup_{j<n} G_j)$.

Let $W_i = \bigcup_{n<\omega} V_{n,i}$, and let $B = \bigcap_{i<\omega} W_i$. Then B is an absolute G_δ-set, and $X \subseteq B$. Let $D = \bigcup_{i<\omega} Cl_Z(G_i)$. It follows that $X = B \cap D$. We denote $Cl_Z(G_n)$ by F_n, and we denote the open set $Z \backslash F_n$ by E_n.

Now,

$$\begin{aligned} X &= B \cap D = B \cap (\bigcup_{n<\omega} F_n) \\ &= B \cap (\bigcup_{n<\omega} (Z \backslash E_n)) \\ &= \bigcup_{n<\omega} ((B \cap Z) \backslash (B \cap E_n)) \\ &= (B \cap Z) \backslash \bigcap_{n<\omega} (B \cap E_n), \end{aligned}$$

so X is the difference of the two absolute G_δ-sets $B \cap Z$ and $\bigcap_{n<\omega}(B \cap E_n)$.

Theorem 11. *The space X is the difference of two absolute G_δ-sets if, and only if, $X = \bigcup_{n<\omega} X_n$ and there is a metric ρ on X such that each X_n is complete in the restriction of ρ to X_n.*

Proof: First, let $X = B \backslash C$, where B and C are absolute G_δ-sets and $C \subseteq B$. Let ρ be a complete metric on B. Now, $C = \bigcap_{n<\omega} U_n$, where U_n is open in B. Then $X = B \backslash (\bigcap_{n<\omega} U_n) = \bigcup_{n<\omega}(B \backslash U_n)$. Each $B \backslash U_n$ is closed in X. Now, fix n, and let $\{x_m : m < \omega\}$ be a ρ-Cauchy sequence of points of $B \backslash U_n$. The sequence must converge to a point of B; but $B \backslash U_n$ is closed in B, so it converges to a point of $B \backslash U_n$.

Next, assume $X = \bigcup_{n<\omega} X_n$ and there is a metric ρ on X such that each X_n is complete in the restriction of ρ to X_n. Let $(Z, \overline{\rho})$ be a *metric* completion of (X, ρ). Let $C = Z \backslash X$. Then $X = Z \backslash C$. It now suffices to show that C is an absolute G_δ-set. Each X_n is closed in Z; to see this; let α be a convergent (in Z) sequence of points of X_n. Then α is a $\overline{\rho}$-Cauchy sequence; hence, a ρ-Cauchy sequence; therefore by hypothesis α converges to a point of X_n. We now have $C = Z \backslash (\bigcup_{n<\omega} X_n) = \bigcap_{n<\omega}(Z \backslash X_n)$, so C is an absolute G_δ-set.

Theorem 12. *The space X is the difference of two absolute G_δ-sets if, and only if, X is locally the difference of two absolute G_δ-sets.*

Proof: We first observe that a union of a locally finite collection of complete spaces is complete. Now, suppose X is locally the difference of two absolute G_δ-sets. Since X is paracompact and since being the difference of two complete spaces is inherited by G_δ-subsets, it follows that 1) there exists a locally finite collection $\mathcal{U} = \{U_\alpha : \alpha < \lambda\}$ of open sets covering X such that each U_α is the difference of two absolute G_δ-sets, and 2) there is a closed collection $\mathcal{F} = \{F_\alpha : \alpha < \lambda\}$ such that $F_\alpha \subseteq U_\alpha$, F_α is closed, and $\mathcal{F}$ covers X. Each F_α is the difference of two absolute G_δ-sets, so by Theorem 10, $F_\alpha = \bigcup_{n<\omega} F_{\alpha,n}$, where $F_{\alpha,n}$ is an absolute G_δ-set and $F_{\alpha,n}$ is closed in F_α and therefore in X. For each n, let $H_n = \bigcup_{\alpha<\lambda} F_{\alpha,n}$. Then H_n, as a union of a locally finite collection of closed sets, is closed, and by the initial observation it is an absolute G_δ-set. Also, $X = \bigcup_{n<\omega} H_n$, so by Theorem 10, X is the difference of two absolute G_δ-sets.

6. Ambiguous classes; two examples

Definition. A set A is of ambiguous class 2 if $A \in G_{\delta\sigma} \cap F_{\sigma\delta}$.

We note that if $A = B \backslash C$, for B, $C \in G_\delta$, then $A = B \cap C'$, where C' is an F_σ; so that $A \in G_{\delta\sigma} \cap F_{\sigma\delta}$. The complement of A is a union of an F_σ and a G_δ, so it is also of ambiguous class 2. Example 1 below shows that a set may be of ambiguous class 2, even the union of an absolute G_δ and a σ-compact set, without being the difference of two absolute G_δ-sets. Example 2 provides an instance of a $G_{\delta\sigma}$-set which is not of ambiguous class 2. These examples show that Theorem 10 is the best along those lines that can be expected.

Example 1. In $I \times I$, let $X_1 = \mathbb{Q} \times I$, $X_2 = \mathbb{P} \times \mathbb{P}$, $X_3 = I^2 \backslash (X_1 \cup X_2) = \mathbb{P} \times \mathbb{Q}$, $X = X_1 \cup X_2$. Then X is the union of a G_δ and countably many compact sets, so it is of ambiguous class 2. It is not, however, the difference of two G_δ-sets. We suppose that it is. Then X_3 is a union of an F_σ-set $\bigcup_{n<\omega} F_n$ and a G_δ-set G. For each $r \in \mathbb{Q}$, let $L_r = \mathbb{P} \times \{r\}$.

Since each F_n is first category in L_r (because each compact set in L_r is nowhere dense), then $G_{r,n} = L_r \backslash F_n$ is a dense G_δ-set. The first projection $\text{Proj}_1(G_{r,n})$ is $G'_{r,n}$. We have countably many dense G_δ-sets in I, namely, $G'_{r,n}$, $r \in \mathbb{Q}$, $n < \omega$. Let t be in their intersection. Then $\{t\} \times \mathbb{Q}$ is a closed subset of G, so it is a G_δ-set, which is a contradiction.

Example 2. In I^ω, let $B = \mathbb{Q}^\omega$, $A = I^\omega \backslash B = \bigcup_{n<\omega} A_n$ where $A_n = \prod_{i<\omega} X_i$, $X_i = I$ for $i \neq n$, and $X_n = \mathbb{P}$. Then A_n is a G_δ-set, and A is a $G_{\delta\sigma}$-set, but A is not an $F_{\sigma\delta}$, so A is not ambiguous of class 2. If $A \in F_{\sigma\delta}$ then $B \in G_{\delta\sigma}$. Assume $B = \bigcup_{n<\omega} G_n$, where each G_n is an absolute G_δ-set. We claim that each G_n is nowhere dense in B. If not, then there is a basic open set, for example, $C = U \times \mathbb{Q}^\omega \subseteq Cl_B(G_0)$, where $U = (a,b) \cap \mathbb{Q}$. Then $G_0 \cap C$ is a dense G_δ-set in C. Since each $C_i = \{r_i\} \times \mathbb{Q}$ is a closed nowhere dense set in C for $a < r_i < b$, $\bigcap_{i<\omega}(C \backslash C_i) \cap G_0$ is a dense G_δ-set in G_0; hence, $\bigcap_{i<\omega} (C \backslash C_i) \cap G_0 \neq \emptyset$, because G_0 is complete. But $\bigcap_{i<\omega}(C \backslash C_i) = \emptyset$, a contradiction.

Since G_0 is nowhere dense, there is a basic open set $V_0 = D_0 \times \mathbb{Q}$, where $D_0 = D_{0,0} \times D_{0,1} \times \cdots \times D_{0,n_0}$ is an open set in $\mathbb{Q}^{n_0}$ such that $V_0 \cap G_0 = \emptyset$. In particular, there is $t_0 \in \mathbb{Q}^{n_0}$ such that $t_0 \times \mathbb{Q}^{\omega - n_0} \cap G_0 = \emptyset$. Repeating the argument, we obtain a $t_1 \in \mathbb{Q}^{n_0} \times \cdots \times \mathbb{Q}^{n_0+n_1}$, such that $t_0 \times t_1 \times \mathbb{Q}^{\omega-(n_0+n_1)} \cap G_1 = \emptyset$. Finally, there is a sequence $t = \{t_i : i < \omega\}$, $t \in \mathbb{Q}^\omega \notin \bigcup_{n<\omega} G_n$. Therefore $\mathbb{Q}^\omega \neq \bigcup_{n<\omega} G_n$.

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