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SIMPLY CONNECTED PLANE CONTINUA HAVE THE FIXED POINT PROPERTY*

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There is an intrinsic relationship between simple connectivity and the fixed point property in the plane. A space is *simply connected* if it is arcwise connected and its fundamental group is trivial. A space S has the *fixed point property* if for every map f of S into S there is a point p of S such that $f(p) = p$.

In first year topology classes, we use simple connectivity to prove the 2-dimensional version of the Brouwer fixed point theorem. We assume the unit disk D admits a fixed point free map f. For each point x of D, we draw a line interval from $f(x)$ through x that stops at a point $r(x)$ on BdD, the boundary of D, thus defining a retract r of D on BdD. The map r induces a surjective homomorphism of the fundamental group $\pi(D)$ of D onto $\pi(BdD)$. Since D is simply connected, $\pi(D) = 1$. But since BdD is a circle, $\pi(BdD)$ is the integers, and we have a contradiction.

According to Brouwer's theorem, every n-cell has the fixed-point property. However, our argument fails if $n = 3$, since the boundary of a 3-cell is a 2-sphere S^2 and $\pi(S^2) = 1$. We use this failure to motivate the introduction of homology and Lefschetz fixed point theory. It follows from Lefschetz's work

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that every homologically acyclic ANR has the fixed point property. Lefschetz's algebraic methods do not extend to all locally connected spaces.

In 1935, Borsuk [Bo2] defined a locally connected homologically acyclic continuum in Euclidean 3-space E^3 without the fixed point property. This example, the Borsuk tornadoes, is 3-dimensional and admits a fixed point free homeomorphism. Noting that its fundamental group is not trivial, Borsuk [Bo2] asked if there exists a simply connected example.

In 1940, Vercenko [V] answered Borsuk's question with a 3-dimensional example in E^4. Kinoshita [K] in 1953 defined a 2-dimensional contractible (therefore homologically acyclic and simply connected) continuum in E^3 without the fixed point property. Bing [Bi1] in 1967 defined a variation of the Borsuk tornadoes that is 2-dimensional and admits a fixed point free homeomorphism. Lysko [L] in 1972 used Kinoshita's example to construct a 3-dimensional contractible continuum without the fixed point property for homeomorphisms. It is not known if there exists a 2-dimensional contractible continuum that admits a fixed point free homeomorphism.

In 1978, Manka [M, 20(a), p 434] asked the following basic question:

Question 1. For an arcwise connected plane continuum to have the fixed point property is it necessary and sufficient that its fundamental group be trivial?

Manka's question is related to the following classical problem:

Question 2. Does every nonseparating plane continuum have the fixed point property?

We are not sure about the origin of Question 2. In a 1930 Fundamenta article, Ayers [A] referred to the analogous question for homeomorphisms as being well known. Ayers [A] showed that every homeomorphism of a locally connected nonseparating plane continuum has a fixed point.

In 1932, Borsuk [Bo1] proved:

Theorem 1. *Every locally connected plane continuum that does not separate the plane has the fixed point property.*

If a locally connected plane continuum M separates the plane, then M contains a simple closed curve C that is not bounded by a disk in M [Mo, Th. 43, p. 193]. It follows that C is a retract of M and there exists a fixed point free map of M into C. Thus for locally connected plane continua, the fixed point property is equivalent to the property of not separating the plane.

An arcwise connected subset S of the plane has a trivial fundamental group if and only if every simple closed curve in S bounds a disk in S. Recall that every locally connected continuum is arcwise connected [Mo, Th. 13, p. 91]. It follows from [Mo, Th. 43, p. 193] that a locally connected plane continuum does not separate the plane if and only if its fundamental group is trivial. Hence Theorem 1 asserts that a locally connected plane continuum has the fixed point property if and only if its fundamental group is trivial, a partial answer to Question 1.

In 1971, the speaker [H1] generalized Theorem 1 by proving:

Theorem 2. *Every arcwise connected nonseparating plane continuum has the fixed point property.*

This was accomplished by showing that the boundary of every arcwise connected nonseparating plane continuum is hereditarily decomposable and applying the Bell-Sieklucki theorem [B1] [S]. Theorem 2 remains true when the arcwise connectivity assumption is replaced by either λ-connectivity [H2] [H5] or weak chainable connectivity [Mi1].

For arcwise connected plane continua, the fixed point property does not imply the nonseparating property. The $\sin \frac{1}{x}$ circle [Bi] is an arcwise connected plane continuum that has the fixed point property and separates the plane. Note that since the $\sin \frac{1}{x}$ circle does not contain a simple closed curve, its fundamental group is trivial.

A space is *uniquely arcwise connected* if it is arcwise connected and does not contain a simple closed curve.

In 1979, the speaker [H3] proved:

Theorem 3. *Every uniquely arcwise connected plane continuum has the fixed point property.*

The unit disk is an example that shows the fixed point property does not imply unique arcwise connectivity in arcwise connected plane continua.

Recently, the speaker [H8] answered Question 1 by showing that every simply connected plane continuum has the fixed point property. Theorems 2 and 3 are corollaries to this result since the fundamental group of every arcwise connected nonseparating plane continuum and every uniquely arcwise connected plane continuum is trivial.

A collection Δ of sets is a *decomposition* of a space if $\cup\Delta$ is the space and the elements of Δ are pairwise disjoint. The following generalization of Theorem 3 is established in [H7].

Theorem 4. *Suppose M is a plane continuum, Δ is a decomposition of M, and each element of Δ is uniquely arcwise connected. Then every map of M that sends each element of Δ into itself has a fixed point.*

We show that Theorem 4 remains true when the elements of Δ are only assumed to be simply connected. Then setting $\Delta = \{M\}$ will establish the fixed point property for every simply connected plane continuum.

The difficulty with this extension can be explained in terms of Bing's dog-chases-rabbit principle [Bi, p. 123]. When the elements are uniquely arcwise connected, there is always a unique arc in the continuum between the dog and the rabbit. This arc serves as a constant guide to the dog during the chase. The Borsuk ray [Bo4] is the appropriate tool.

When the elements are only simply connected, the dog is forced to hunt without the guiding arc. The dog must be able to pursue the rabbit through subsets of the continuum that are open relative to the plane. Thus a ray with a special cut property must be used in place of the Borsuk ray.

In spite of this obstacle, almost all of [H7] applies without modification. We use a lemma of Bell [B2, (2.1)] and the constructions of Sieklucki [S] to establish the existence of rays with the cut property that start at each point of the continuum. Then we adjust the argument of [H7] and use it to establish the following:

Theorem 5. *Suppose M is a plane continuum, Δ is a decomposition of M, and each element of Δ is simply connected. Then every map of M that sends each element of Δ into itself has a fixed point* [H8].

A continuum M is *capped* if every simple closed curve in M bounds a disk in M. Every plane continuum with the fixed point property is capped.

Corollary 1. *Suppose M is a capped plane continuum. Then every map of M that sends each arc component into itself has a fixed point.*

A map f of a space S is a *deformation* if there exists a map h of $S \times [0,1]$ onto S such that $h(p,0) = p$ and $h(p,1) = f(p)$ for each $p \in S$.

Corollary 2. *If M is a capped plane continuum, then every deformation of M has a fixed point.*

Corollary 3. *(A restatement of the title of this lecture.) Every capped arcwise connected plane continuum has the fixed point property.*

Semi-aposyndesis and continuum chainability are natural generalizations of arcwise connectivity. In a plane continuum with only finitely many complementary domains both semi-aposyndesis [H4] and continuum chainability [H6] imply arcwise connectivity. However, the speaker and Oversteegen [HO1] have an example that shows arcwise connectivity in Corollary 3 cannot be replaced by either semi-aposyndesis or continuum chainability. A continuum chainable plane continuum with infinitely many complementary domains may be very pathological. Recently, the speaker and Oversteegen [HO2] defined one that admits a fixed point free homeomorphism and does not contain an arc.

Interest in Question 2 intensified in 1978 when Bellamy [Be] constructed a tree-like continuum without the fixed point property. Bellamy [Be] used this example and an inverse limit technique of Fugate and Mohler [FM] to define a tree-like continuum that admits a fixed point free homeomorphism. It is not known if Bellamy's second example is embeddable in the plane. If this example is planar, then the answer to Question 2 is no. After Bellamy's breakthrough, Oversteegen, Rogers [OR1] [OR2] and Minc [Mi2] [Mi3] constructed other interesting tree-like continua without the fixed point property. The most recent example is Minc's tree-like continuum that admits a periodic point free homeomorphism [Mi3].

Other related basic fixed point problems remain unsolved. We do not know if every plane continuum that is an inverse limit of triods has the fixed point property. Bing suggested that we approach this problem by adding assumptions about the fixed points of the bonding maps. In particular, assume that each bonding map fixes each vertex of the triod. While trying to follow Bing's suggestion, Marsh [Ma] proved a general theorem that includes the case where two legs of the triod are fixed by the bonding maps.

In 1967, Knill [Kn] showed that the cone over a spiral to a disk admits a fixed point free map.

Question 3. Does the cone over a spiral to a triod have the fixed point property?

The following more general question is a revision of another Borsuk problem [Bo3] [K] [Kn]:

Question 4. Must the cone over a tree-like continuum have the fixed point property?

Knill [Kn] defined a 2-dimensional continuum M in E^3 with the fixed point property such that $M \times [0,1]$ admits a fixed point free map. In [Bi], Bing asked:

Question 5. If M is a plane continuum with the fixed point property, does $M \times [0,1]$ have the fixed point property?

As a preliminary to Question 5, one might consider the following:

Question 6. If M is a simply connected plane continuum, does $M \times [0,1]$ have the fixed point property?

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