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Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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TIGHTNESS IN CHAINS OF HAUSDORFF SPACES

TIM LABERGE AND AVNER LANDVER

ABSTRACT. We investigate unions of chains of topological spaces. A family $\{X_\alpha : \alpha < \kappa\}$ of topological spaces is a κ-chain if $\alpha < \beta < \kappa$ implies that X_α is a subspace of X_β. Any topology on $X = \bigcup_{\alpha<\kappa} X_\alpha$ for which each each X_α is a subspace is called a compatible topology.

We investigate tightness of compatible topologies, compatible topologies that are compact, and chains for which there is only one compatible topology.

0. INTRODUCTION

We say that $\{X_\alpha : \alpha < \kappa\}$ is a *κ-chain* of topological spaces if κ is an infinite regular cardinal and $\alpha < \beta < \kappa$ implies that X_α is a subspace of X_β. Any topology on $X = \bigcup_{\alpha<\kappa} X_\alpha$ for which each X_α is a subspace is called a *compatible topology* on the chain.

Given a κ-chain $\{X_\alpha : \alpha < \kappa\}$, the *fine topology* on $X = \bigcup_{\alpha<\kappa} X_\alpha$ is the set of all $U \subseteq X$ such that for all $\alpha < \kappa$, $U \cap X_\alpha$ is open in X_α. The fine topology is the strongest compatible topology. Except for the next section, we will only be concerned with chains for which the fine topology is Hausdorff.

Recall the definition of the tightness of a point p in a space X:

$$t(p,X) = \sup\{\rho : A \in [X]^\rho,\ p \in \overline{A} \setminus A, \text{ and } \forall B \in [A]^{<\rho}(p \notin \overline{B})\} + \omega.$$

The tightness of the space X is then defined as $t(X) = \sup\{t(p,X) : p \in X\}$. In order to avoid issues of sup = max, we sometimes use the cardinal $t^+(X) = \min\{\lambda : \forall p \in X, t(p, X) < \lambda\}$. I.e., $t^+(X)$ is the smallest cardinal bounding all occurrences of tightness in X.

Given an infinite cardinal κ, we define the space $\kappa \oplus 1$ to be the space with point set $\kappa + 1$ in which points less than κ are isolated and a neighborhood of κ is a set whose complement is bounded.

1. Chains that have no compatible Hausdorff topology

It may happen that the fine topology on a chain of Hausdorff spaces is not Hausdorff, so that there is no compatible Hausdorff topology on the chain. The following example is from a preprint version of [DJW] and is included here with their kind permission.

Theorem 1.1. *There is an ω-chain $X = \bigcup_{n\in\omega} X_n$ of Hausdorff spaces such that the fine topology on X is not Hausdorff.*

Proof: Let $\{Y_n : n \in \omega\}$ be a pairwise-disjoint family of regular non-normal spaces. For each $n \in \omega$, fix a pair $\{H_n, K_n\}$ of disjoint closed subsets of Y_n that are not contained in disjoint open sets. For each $n \in \omega$, let X_n be the space with point set

$$\{p, q\} \cup \bigcup_{k\leq n} Y_k \cup \bigcup_{n<k\leq\omega} (H_k \cup K_k)$$

topologized so that each Y_k $(k \leq n)$ and $H_k \cup K_k$ $(k > n)$ is open and has its original topology. A neighborhood of p is a set of the form

$$B_k(p) = \{p\} \cup \bigcup_{m\geq k} H_m \quad (k \in \omega),$$

and a neighborhood of q is a set of the form

$$B_k(q) = \{q\} \cup \bigcup_{m\geq k} K_m \quad (k \in \omega).$$

If $X = \bigcup_{n\in\omega} X_n$ has the fine topology, then it is clear that X is not Hausdorff. □

The fine topology may not be Hausdorff even if the spaces in the chain are very nice.

Theorem 1.2. *For each infinite regular cardinal κ there is a κ-chain of hereditarily normal spaces $X = \bigcup_{\alpha<\kappa} X_\alpha$ such that the fine topology on X is not Hausdorff.*

Proof: Fix an infinite regular cardinal κ.
Case 1: $\kappa \neq \omega_1$. Let $Z = [(\omega_1 \oplus 1) \times (\kappa \oplus 1)] \setminus \{(\omega_1, \kappa)\}$. Let $\{Y_\alpha : \alpha < \kappa\}$ be a collection of κ-many pairwise disjoint copies of Z.

X is the space with point set $\{p, q\} \cup \bigcup\{Y_\alpha : \alpha < \kappa\}$, topologized so that each Y_α is open and homeomorphic to Z, and so that a neighborhood of p is a set U containing p and a neighborhood of $\omega_1 \times \{\kappa\}$ in each Y_β, for all β greater than some fixed $\alpha < \kappa$. Similarly, a neighborhood of q is a set V that contains q and a neighborhood of $\{\omega_1\} \times \kappa$ in each Y_β, for all β greater than some fixed $\alpha < \kappa$.

Clearly, X is not Hausdorff. For each α and β less than κ, define $Y_{\beta\alpha}$ to be the subspace $Y_\beta \setminus (\{\omega_1\} \times [\alpha, \kappa))$ of Y_β. Set $X_\alpha = \{p, q\} \cup \bigcup\{Y_{\beta\alpha} : \beta < \kappa\}$ as a subspace of X. Then each X_α is hereditarily normal and X has the fine topology from the chain $\bigcup_{\alpha<\kappa} X_\alpha$.
Case 2: $\kappa = \omega_1$. Use a similar construction with $Z = ((\omega \oplus 1) \times (\omega_1 + 1)) \setminus \{(\omega, \omega_1)\}$. □

Despite these negative results, notice that the fine topology on a chain of T_1 spaces is always T_1.

For the remainder of this paper, we will only consider κ-chains for which the fine topology is Hausdorff.

2. Tightness of chains

In this section, we investigate tightness of κ-chains. Cardinal functions on chains were first investigated by Tkachenko [T], and by Hajnal and Juhász [J]. Chains of first countable spaces are discussed in [DJW], where it is shown that $\square(\kappa)$ implies that there is a κ-chain of first countable spaces whose union with the fine topology is not first countable, and that in models obtained by collapsing large cardinals, unions of κ-chains ($\kappa > \omega_1$) of first countable spaces with the fine topology are first countable. Dow has recently shown that PFA implies that if $\kappa > \omega_1$, then unions of κ-chains of first countable spaces with the fine topology are first countable.

The type of question we are most interested in is: Given a chain $X = \bigcup_{\alpha<\kappa} X_\alpha$, can we compute the tightness of X from the tightness of each X_α? More specifically, for which chains is tightness continuous, in the sense that the equation $t(X) = \sup\{t(X_\alpha) : \alpha < \kappa\}$ holds? We can also ask a local version of this question: When is the equation $t(p, X) = \sup\{t(p, X_\alpha) : \alpha < \kappa \text{ and } p \in X_\alpha\}$ true for all $p \in X$?

The following result is from [J].

Theorem 2.1. *Suppose $X = \bigcup_{\alpha<\kappa} X_\alpha$ is compact Hausdorff. If each X_α is countably tight and $\kappa > \omega_1$, then X is countably tight. If $\kappa = \omega_1$, then $t(X) \leq \omega_1$.*

Our first result generalizes a result from [DJW].

Theorem 2.2. *Let $X = \bigcup_{\alpha<\kappa} X_\alpha$ be a κ-chain with a compatible Hausdorff topology τ. If $t^+((X, \tau)) \leq \kappa$, then τ is the fine topology.*

Proof: Suppose, by way of contradiction, that τ is not the fine topology. Then there is a $U \subseteq X$ such that for each $\alpha < \kappa$, $U \cap X_\alpha$ is open in X_α, but U is not open in X.

Then $X \setminus U$ is not closed in X, so there is a $p \in \overline{X \setminus U} \cap U$. By $t^+(X) \leq \kappa$, there is a set $A \in [X \setminus U]^{<\kappa}$ such that $p \in \overline{A}$. Because $|A| < \kappa$ and κ is regular, there is an $\alpha < \kappa$ such that $A \subseteq X_\alpha$. Since X_α is a subspace of X, p is in the X_α-closure of A. But then p is in the X_α-closure of $(X \setminus U) \cap X_\alpha$, so that $(X \setminus U) \cap X_\alpha$ is not closed in X_α, whence $U \cap X_\alpha$ is not open in X_α, a contradiction. $\square$

The following example shows that even if each $t^+(X_\alpha) \leq \kappa$, that there can be a compatible topology on $X = \bigcup_{\alpha<\kappa} X_\alpha$ that makes $t^+(X) > \kappa$.

Example 2.3. Let κ be an infinite successor cardinal. For each $\alpha < \kappa$, set $X_\alpha = [0, \alpha)$ with the order topology. The fine topology on $\kappa = \bigcup_{\alpha<\kappa} X_\alpha$ is just the order topology on κ. However, another compatible topology on κ can be obtained by declaring neighborhoods of 0 to be sets of the form $\{0\} \cup [\alpha, \kappa)$. With this topology, the chain is homeomorphic to $\kappa + 1$ with the order topology, and $t^+(\kappa + 1) = \kappa^+$.

On the other hand, if we put the fine topology on a chain, then the tightness of X is determined by the tightness of the X_α's.

Theorem 2.4. *If $X = \bigcup_{\alpha<\kappa} X_\alpha$ has the fine topology and each $t(X_\alpha) \leq \lambda$, then $t(X) \leq \lambda$.*

Proof: Suppose, by way of contradiction, that each $t(X_\alpha) \leq \lambda$, but there is an $S \subseteq X$ and a $p \in \overline{S}$ such that for all $T \in [S]^{\leq\lambda}$, $p \notin \overline{T}$.

Let $S' = \bigcup\{\overline{T} : T \in [S]^{\leq\lambda}\}$. Notice that for all $T \in [S']^{\leq\lambda}$, $\overline{T} \subseteq S'$: Fix such a T. For each $t \in T$, there is an $A_t \in [S]^{\leq\lambda}$ such that $t \in \overline{A_t}$. Then $\overline{T} \subseteq \overline{\bigcup_{t\in T} A_t}$, and $\bigcup_{t\in T} A_t \in [S]^{\leq\lambda}$.

We can therefore assume that for all $T \in [S]^{\leq\lambda}$, $\overline{T} \subseteq S$, and that S is not closed (because $p \in \overline{S} \setminus S$). Because X has the fine topology, there is an $\alpha < \kappa$ such that $X_\alpha \cap S$ is not closed in X_α. Take an $x \in X_\alpha$ such that x is in the X_α-closure of $X_\alpha \cap S$, but not in S. Because the tightness of X_α is at most λ, there is a $T \in [X_\alpha \cap S]^{\leq\lambda}$ such that x is in the X_α-closure of T. But then $x \in \overline{T}$, whence x is in S, a contradiction. □

We give a similar continuity result for t^+ after the proof of Corollary 2.7. For now, we note that if $\kappa = cf(\lambda) < \lambda$, then $t^+(X)$ may be greater than λ.

Example 2.5. Suppose $\kappa = cf(\lambda) < \lambda$, and $\{\kappa_\alpha : \alpha < \kappa\}$ is an increasing sequence of regular cardinals cofinal in λ, with $\kappa_0 > \kappa$. Set $Y = \kappa \oplus 1$ and for each $\alpha < \kappa$, set $Y_\alpha = \kappa_\alpha \oplus 1$. Let Z be the topological sum of Y and the Y_α's. Let X be the quotient space obtained by identifying the non-isolated point of Y_α with the point α in Y, for each $\alpha < \kappa$.

Let I be the set of isolated points of X. Then $|I| = \lambda$, $\kappa \in \overline{I}$, and for all $J \in [I]^{<\lambda}$, $\kappa \notin \overline{I}$. So $t^+(X) = \lambda^+$.

If we set $X_\alpha = (\kappa \oplus 1) \cup \bigcup_{\beta<\alpha} \kappa_\beta$, then each $t^+(X_\alpha) \leq \kappa_\alpha < \lambda$, and X has the fine topology from the chain $\bigcup_{\alpha<\kappa} X_\alpha$. Notice that each $t(\kappa, X_\alpha) = \kappa$, but $t(\kappa, X) = \lambda$, so that $t^+(X) = \lambda^+$.

At this point it is natural to ask when it is possible to determine the tightness of a point p in $X = \bigcup_{\alpha<\kappa} X_\alpha$ with the fine topology from the tightness of p in each X_α. There is a natural condition on a chain that implies this local version of continuity.

We say that a topology τ on $X = \bigcup_{\alpha<\kappa} X_\alpha$ is *chain-compact* if whenever $S \subseteq X$ and $p \in \overline{S}$, then there is an $\alpha < \kappa$ such that p is in the X_α-closure of $S \cap X_\alpha$. Note that chain-compactness implies that if $p \in X$, then $t(p, X) = \sup\{t(p, X_\alpha) : \alpha < \kappa$ and $p \in X_\alpha\}$, and that if τ is chain-compact, then τ is the fine topology.

The following theorem, which is also implicit in [JS], gives a sufficient condition for chain-compactness.

Theorem 2.6. *If $X = \bigcup_{\alpha<\kappa} X_\alpha$ has the fine topology and for all $\alpha < \kappa$, $t^+(X_\alpha) \leq \kappa$, then X is chain-compact.*

Proof: Suppose $S \subseteq X$. For each $\alpha < \kappa$, let S_α be the X_α-closure of $S \cap X_\alpha$. It suffices to show that $S' = \bigcup_{\alpha<\kappa} S_\alpha$ is closed. By way of contradiction, suppose it is not. Because X has the fine topology, there is an $\alpha < \kappa$ such that $S' \cap X_\alpha$ is not closed. By $t^+(X_\alpha) \leq \kappa$, there is a $p \in X_\alpha \setminus S'$ and a $T \in [S' \cap X_\alpha]^{<\kappa}$ such that p is in the X_α-closure of T.

Now $T \subseteq S'$, so for each $t \in T$, there is a $\beta_t < \kappa$ and an $A_t \in [S \cap X_{\beta_t}]^{<\kappa}$ such that t is in the X_{β_t}-closure of A_t. Because κ is regular, $\beta = \sup\{\beta_t : t \in T\} < \kappa$. But then $\bigcup_{t\in T} A_t \subseteq S \cap X_\beta$, so that p is in the X_β-closure of $S \cap X_\beta$. I.e., $p \in S'$, a contradiction. $\square$

Corollary 2.7. *Suppose* $X = \bigcup_{\alpha<\kappa} X_\alpha$ *has the fine topology and each* $t^+(X_\alpha) \leq \kappa$. *Then* $t^+(X) \leq \kappa$ *and moreover, for each* $x \in X$, $t(x, X) = \sup\{t(x, X_\alpha) : \alpha < \kappa$ *and* $x \in X_\alpha\}$.

Proof: Suppose $x \in \overline{A} \setminus A$. By chain-compactness, there is an $\alpha < \kappa$ such that x is in the X_α-closure of $A \cap X_\alpha$. But $t^+(X_\alpha) \leq \kappa$, so there is a $B \in [A \cap X_\alpha]^{<\kappa}$ such that x is in the X_α-closure of B, so $x \in \overline{B}$.

The 'moreover' is immediate from the fact that X is chain-compact. □

Theorem 2.8. *Let* $X = \bigcup_{\alpha<\kappa} X_\alpha$ *be a* κ*-chain with the fine topology and suppose that each* $t^+(X_\alpha) \leq \lambda$. *If either* $\lambda \leq \kappa$ *or* λ *is regular, then* $t^+(X) \leq \lambda$.

Proof: If $\lambda \leq \kappa$, use Corollary 2.7. If $\lambda = \rho^+$, this follows from Theorem 2.4. If $\lambda > \kappa$ and is a regular limit cardinal, use the proof of Theorem 2.4, replacing $[S]^{\leq\lambda}$ with $[S]^{<\lambda}$ throughout. □

We have a few more examples concerning chain-compactness. The first shows that chain-compactness does not require each $t^+(X_\alpha)$ to be small.

Example 2.9. Let κ be an infinite regular cardinal and set $X = (\kappa^+ + 1) \cup \kappa$, where $\kappa^+ + 1$ has the order topology and κ has the discrete topology. Set $X_\alpha = (\kappa^+ + 1) \cup [0, \alpha)$. Then $X = \bigcup_{\alpha<\kappa} X_\alpha$ is chain-compact even though each $t^+(X_\alpha) = \kappa^{++}$.

The following example shows that the fine topology on a chain may not be chain-compact.

Example 2.10. Define $X = [(\kappa \oplus 1) \times (\omega \oplus 1)] \setminus (\{\kappa\} \times \omega)$. Set $X_\alpha = [(\kappa \oplus 1) \times \{\omega\}] \cup (\alpha \times \omega)$. Notice that X has the fine topology from the chain $\bigcup_{\alpha<\kappa} X_\alpha$. Let $S = \kappa \times \omega$, then $(\kappa, \omega) \in \overline{S}$, but for each $\alpha < \kappa$, (κ, ω) is not in the X_α-closure of $S \cap X_\alpha$, so X is not chain-compact. Note that the tightness of (κ, ω) is κ in X and each X_α.

In the previous example, notice that even though X is not chain-compact, the tightness of points does not increase. As we've already seen in Example 2.5, if the length of the chain is less than the tightness of some X_α, then local continuity of tightness can fail. Here is another such example that does not require a singular cardinal.

Example 2.11. Let $\lambda \geq \kappa^+$ be a regular cardinal and set $X = [(\kappa \oplus 1) \times (\lambda \oplus 1)] \setminus (\{\kappa\} \times \lambda)$. Define (for $\alpha < \kappa$) $X_\alpha = [(\kappa \oplus 1) \times \{\lambda\}] \cup (\alpha \times \lambda)$. X has the fine topology from the chain $\bigcup_{\alpha<\kappa} X_\alpha$. For each $\alpha < \kappa$, $t(X_\alpha) = \lambda$ and $t((\kappa, \lambda), X_\alpha) = \kappa$, but $t((\kappa, \lambda), X) = \lambda$.

We do not know if local continuity of tightness is implied by $t^+(X_\alpha) \leq \kappa^+$ for each $\alpha < \kappa$.

Question 2.12. *If* $X = \bigcup_{\alpha<\kappa} X_\alpha$ *has the fine topology and* $t^+(X_\alpha) \leq \kappa^+$, *is*

$$t(p, X) = \sup\{t(p, X_\alpha) : \alpha < \kappa \text{ and } p \in X_\alpha\}$$

for each $p \in X$ *?*

3. Uniqueness of compatible topologies

We again remind the reader that we are only considering compatible Hausdorff topologies on chains. Because a compact Hausdorff topology on a set is minimal Hausdorff, it is clear that if the fine topology on a chain is compact, then it is the only compatible Hausdorff topology. However, it is certainly possible to have more than one compatible compact topology.

Example 3.1. Let κ be a regular cardinal with the order topology, and for each $\alpha < \kappa$, set $X_\alpha = [0, \alpha)$. Let E be the set of limit ordinals in κ, and for each $\alpha \in E$, define a compatible compact topology τ_α on κ as follows. Points in $\kappa \setminus \{\alpha\}$ have their usual order neighborhoods. A neighborhood of α is a set of the form $(\gamma, \alpha] \cup (\beta, \kappa)$, where $\gamma < \alpha < \beta < \kappa$.

This defines a family of κ-many pairwise non-homeomorphic compact compatible topologies on the chain $\bigcup_{\alpha\in\kappa} X_\alpha$.

Recall that a space Z is *left-separated in type* λ if there is a well-ordering of Z in type λ such that final segments are open. The *hereditary density* of a space X is

$$hd(X) = \sup\{\lambda : \exists Z \subseteq X(Z \text{ is left-separated in type } \lambda\}.$$

We define $hd^+(X)$ to be the least cardinal greater than the lengths of all left-separated subspaces of X.

Theorem 3.2. *Suppose a κ-chain $X = \bigcup_{\alpha<\kappa} X_\alpha$ has the fine topology and $hd^+(X) \leq \kappa$. Then the fine topology is the only compatible topology on the chain.*

Proof: We claim that any compatible topology τ on X satisfies $hd^+((X,\tau)) \leq \kappa$: Otherwise, there is a left-separated sequence of length κ in (X,τ), hence in X with the fine topology, a contradiction.

Because $hd^+(X) \leq \kappa$, $t^+((X,\tau)) \leq \kappa$, so by Theorem 2.2, τ is the fine topology. $\square$

Define $hl(X)$ and $hl^+(X)$ by replacing 'left' with 'right' in the definitions of $hd(X)$ and $hd^+(X)$. Let $s^+(X)$ be the least cardinal greater than the cardinality of all discrete subspaces of X.

Question 3.3. *If $s^+(X) \leq \kappa$ or $hl^+(X) \leq \kappa$, is the fine topology the only compatible topology?*

Question 3.4. *Is it possible to have a κ-chain of Hausdorff spaces with exactly two compatible Hausdorff topologies?*

4. When is the fine topology compact?

In this section, κ will denote an uncountable regular cardinal. Recall that a space X is $<\kappa$-bounded if every subset of X of cardinality less than κ has compact closure. It is not hard to see that if X is $<\kappa$-bounded, then every open cover of X of cardinality less than κ has a finite subcover.

A κ-sequence $\{x_\alpha : \alpha < \kappa\} \subseteq X$ is *free* if for each $\beta < \kappa$,

$$\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geq \beta\}} = \emptyset.$$

Arhangel'skii [A] has shown that for a compact space X, $t(X) = \sup\{\kappa : \exists$ a free κ-sequence in $X\}$.

We are interested in trying to determine when the fine topology on a chain of compact spaces is compact. We will need the following lemma.

Lemma 4.1. *Suppose $X = \bigcup_{\alpha<\kappa} X_\alpha$ is a κ-chain, each X_α is compact, and τ is a compatible topology. Then*

(1) *(X, τ) is $<\kappa$-bounded.*
(2) *If (X, τ) is non-compact, then there is an increasing open cover $\{U_\alpha : \alpha < \kappa\}$ of (X, τ) with no subcover of smaller cardinality such that for each $\alpha < \kappa$, $X_\alpha \subseteq U_\alpha$.*
(3) *If (X, τ) is non-compact, then there is a κ-free sequence in X with the fine topology.*

Proof: (1) This follows immediately from the facts that κ is regular and each X_α is compact.
(2) Fix an open cover $\mathcal{V}$ of (X, τ) that has no finite subcover. Because each X_α is compact and (X, τ) is $<\kappa$-bounded, we can assume that $|\mathcal{V}| = \kappa$ and $\mathcal{V}$ has no subcover of smaller cardinality. For each $\alpha < \kappa$, set $U_\alpha = \bigcup_{\beta<\alpha} V_\beta$. Then $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ is an increasing open cover of X with no subcover of smaller cardinality. By thinning and then re-indexing, we can assume that $X_\alpha \subseteq U_\alpha$ for each $\alpha < \kappa$.
(3) Take $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ as in part (2). Inductively choose points $\{x_\alpha : \alpha < \kappa\}$ and ordinals $\{\gamma(\alpha) : \alpha < \kappa\}$ as follows. Take $x_0 \in X_0$ and set $\gamma(0) = 0$. Given $\{x_\alpha : \alpha < \beta\}$ and $\{\gamma(\alpha) : \alpha < \beta\}$, find a $\gamma(\beta) > \sup\{\gamma(\alpha) : \alpha < \beta\}$ such that $\{x_\alpha : \alpha < \beta\} \subseteq X_{\gamma(\beta)}$, and choose $x_\beta \in X \setminus U_{\gamma(\beta)}$.

It is easy to see that the sequence $\{x_\alpha : \alpha < \kappa\}$ so defined is a free κ-sequence in (X, τ), hence in X with the fine topology. $\square$

Theorem 4.2. *Let $X = \bigcup_{\alpha<\kappa} X_\alpha$ be a κ-chain with each X_α compact. Assume X has the fine topology. Then*

(1) *If there are no free κ-sequences in X, then X is compact.*
(2) *If $\lambda = \sup\{t^+(X_\alpha) : \alpha < \kappa\} < \kappa$, then X is compact*
(3) *If each $t^+(X_\alpha) \leq \kappa$, then X is compact if and only if X has no free κ-sequences.*

Proof: (1) This is immediate from (3) of the preceding lemma. (2) A free κ-sequence in X would give a free λ-sequence in an X_α for some $\alpha < \kappa$, contradicting Arhangel'skii's theorem. (3) [$\Rightarrow$] By Corollary 2.7, $t^+(X) \leq \kappa$, so by Arhangel'skii's theorem there are no free κ-sequences in X, so X is compact. [$\Leftarrow$] Trivial. □

References

[A] Arhangel'skii, A. *The frequency spectrum of a topological space*, Soviet Math. Dokl. **13** (1972), 1185–1189.

[DJW] Dow, A., Juhász, I., and Weiss, W., *Integer-valued functions and increasing unions of first countable spaces*, Israel Jour. Math., **67** (1989), no. 2, 181–192.

[J] Juhász, I., *Cardinal Functions in Topology—Ten Years Later*, Mathematical Centre Amsterdam, 1981.

[JS] Juhász, I. and Szentmiklossy, Z., *Increasing strengthenings of cardinal function inequalities*, Fund. Math. **126** (1986), 209–216.

[T] Tkachenko, M. G., *The behavior of cardinal functions on inverse limit spectra consisting of topological spaces*, Serdica **6** (1980), 42–55.

University of Kansas,
Lawrence, KS 66045

current address: Northern Illinois University
Dekalb, IL 60115
email: laberget@math.niu.edu

University of Kansas,
Lawrence, KS 66045

current address: IBM Israel—Science & Technology, MATAM, Haifa 31905, Israel
email: landver@haifasc3.vnet.ibm.com