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ON WEAKLY SUSLINIAN CONTINUA

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Dedicated to Professor Andrew Lelek on his 60th birthday

ABSTRACT. In this paper we investigate two classes of continua which we call Suslinian and weakly Suslinian. While the former was introduced by A. Lelek in 1971, the latter is being introduced by the author of this paper. We will show that in the class of locally connected unicoherent continua, these two among other things, are actually equivalent. We also prove that many classical examples of indecomposable continua, such as the Buckethandle and the solenoids, are weakly Suslinian. However, we will see that the class of hereditarily indecomposable continua and the class of weakly Suslinian continua are two mutually exclusive classes.

0. INTRODUCTION

In 1971, the concept of Suslinian curves was introduced by A. Lelek [6]. Suslinian curves, 1-dimensional in nature, have properties which resemble those of ordered sets introduced by M. J. Suslin in the famed Suslin problem. These properties were intended to complete the well-known classification of curves (see [10], p. 99), and suggested an analogue of a decomposition property possessed by rational curves (see [5], p. 285).

In a different context, W. Sierpinski proved that no continuum can be decomposed into a countable family of nonempty,

*This paper is part of the author's dissertation done under the direction of Professor Andrew Lelek at the University of Houston.

disjoint closed sets [9]. Although every nondegenerate continuum can be decomposed into disjoint singleton sets, it is interesting to know when it can be decomposed into mutually disjoint proper nondegenerate subcontinua. In this paper, certain ideas from [6] and [9] are generalized and modified (see Definition 3). Our main interest here is to introduce and investigate a new class of continua which we call weakly Suslinian continua.

The existence of weakly Suslinian continua which are not Suslinian is demonstrated (see Examples 1 and 2). The construction of weakly Suslinian continua of any positive and even infinite dimensions is discussed at the end of Section 3. And weakly Suslinian continua containing subcontinua which are not weakly Suslinian are also given (see Example 2).

It is a natural question to ask whether Suslinian continua are closed under countable unions; A. Lelek and D. R. Brown have asked the author (in private conversations) whether or not objects such as the Buckethandle and solenoids are weakly Suslinian; H. Cook has also raised the question, "Are nondegenerate hereditarily weakly Suslinian continua 1-dimensional?". These questions are all answered in the affirmative in Theorems 1, 2, and 6.

Using results from the study of hyperspaces, it is shown that weakly Suslinian continua and hereditarily indecomposable continua are two mutually disjoint classes of continua (see Lemma 4 and Proposition 5). Furthermore, confluent and weakly confluent images of Suslinian and weakly Suslinian continua are discussed in Theorems 4, 5 and in Example 2.

In Example 3, a non-weakly Suslinian continuum which is the union of two weakly Suslinian continua is investigated. The space in Example 4 is of the same nature. This gives a negative answer to a question raised by A. Lelek.

The last section is mainly devoted to the study of cyclic elements of locally connected continua. The property of being a Suslinian or a weakly Suslinian continuum is proved to be reducible. And the property of being a Suslinian continuum is, on the other hand, shown to be an extensible property (see definition 2).

1. PRELIMINARIES

Throughout this paper the word *continuum* means a nonempty compact connected metric space, and the word *mapping* means a continuous function. The capital letter X will always denote a continuum with metric d, unless otherwise stated. Dimension here means inductive dimension as defined [4]. If X is a nonempty compact metric space, then $\dim(X) = 0$ if and only if X is hereditarily discontinuous (a metric space is *hereditarily discontinuous* provided that it contains no continuum consisting of more than one point). A space is said to be *nondegenerate* if it consists of more than one point. In our terminology, a *curve* means a 1-dimensional continuum. Therefore the curves are nondegenerate sets.

Definition 1. *A metric space X is said to have property Σ provided that X does not contain uncountably many mutually disjoint nondegenerate continua. A continuum satisfying property Σ is called Suslinian.*

This concept first appeared in [6]. Notice that a nondegenerate Suslinian continuum is 1-dimensional. Thus it is sometimes called a Suslinian curve.

A property P of a continuum X is said to be *hereditary* if each subcontinuum of X also has P. Let X be a continuum and Y a subcontinuum of X. If Z is a subcontinuum of Y, then one can easily show that Z is a subcontinuum of X. Hence we have

Proposition 1. *The property of being a Suslinian continuum is hereditary.*

Let X be a metric space and let $p \in X$. By *the composant in X of p* we will mean the set K given by

$$K = \{x \in X : \exists A \subset X,\ A \text{ is proper, closed and connected, such that } p,\, x \in A\}.$$

A curve X is said to be *rational* provided that each point of X is contained in arbitrarily small neighborhoods (relative to X) whose boundaries are countable.

A continuum X is said to be *unicoherent* provided that $A \cap B$ is connected whenever A and B are subcontinua of X such that $A \cup B = X$. A continuum is said to be *hereditarily unicoherent* provided that each of its subcontinua is unicoherent.

A continuum is said to be a *dendrite* provided that it is locally connected and contains no simple closed curve (a homeomorphic image of the unit circle is called a *simple closed curve*). A *dendroid* is an arcwise connected hereditarily unicoherent continuum. Note that every dendrite is a dendroid and every locally connected dendroid is a dendrite.

Definition 2. *Let X be a locally connected continuum. We call a cyclic element of X:*

(i) *every point which separates X,*
(ii) *every set*

$$E_p = \bigcup \{x : \ ord_{p,x} X \geq 2\}$$

(i.e., the set of points x such that no point cuts X between p and x), provided that p does not separate X.

The concept of cyclic element is due to G. T. Whyburn; see [10].

It can be shown that every point of X belongs to at least one cyclic element. Moreover, a non-cut point can belong to only one cyclic element. Indeed only a countable number of points of X can belong to more than one cyclic element and all these points must be cut points.

Let X be a locally connected continuum. Then there are numerous properties of X which depend only on analogous properties of its cyclic elements. The properties called *extensible* are of that kind. We shall call so every property which is a property of X if it is a property of each cyclic element.

We shall say that a property is *reducible* if it is a property of every cyclic element, provided it is a property of X.

2. THE MAIN RESULTS

Definition 3. *A continuum X is said to be weakly Suslinian provided that X is not the union of more than one pairwise disjoint nondegenerate subcontinua of X.*

Proposition 2. *A Suslinian continuum is weakly Suslinian.*

Proof: This follows immediately from Definitions 1 and 3. □

Proposition 3. *The property of being a weakly Suslinian continuum is a topological property.*

Example 1. We construct a weakly Suslinian continuum which is not Suslinian. Let X be the cone over the Cantor set (see Figure 1), then X is not Suslinian since it contains uncountably many disjoint arcs. To show that X is weakly Suslinian let us realize that if Y is a proper subcontinuum of X containing the vertex of the cone, $X \backslash Y$ can then be written as a disjoint union of half-open and half-closed line segments. By Corollary 1 which will be proved later, no half-open and half-closed interval is a union of disjoint nondegenerate closed subintervals. Therefore X is weakly Suslinian.

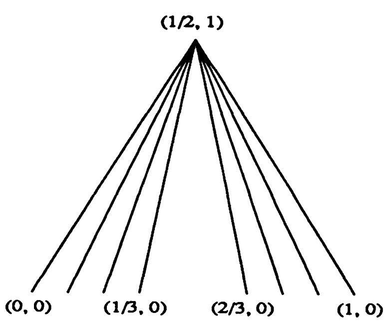

Figure 1: The cone over the Cantor set

Propoition 4. *The bounded closed interval I on the real line is Suslinian and therefore it is weakly Suslinian. Consequently, the same can be said about an arc.*

Corollary 1. *If a and b are real numbers such that $a < b$, then $[a, b)$ is not a union of pairwise disjoint proper nondegenerate compact connected subspaces.*

Proof: Suppose on the contrary $[a, b)$ could be written as such a union, then so could $[a, b) \cup [b, c] = [a, c]$, where c is a real number and $b < c$. Thus by a theorem of Sierpinski [9], this union must be uncountable. Then $[a, c]$ would not be weakly Suslinian. But this is a contradiction of Proposition 4. □

Example 2. This is a weakly Suslinian dendroid X which contains a subdendroid Y such that Y is not weakly Suslinian (see Figure 2). First denote A_i the straight line segment joining $(1, 0)$ and $(0, \frac{-1}{i})$ and then let C be the Cantor set defined on the closed unit interval I. Now set $X = Y \cup Z$, where

$$Y = (C \times I) \cup I \text{ and } Z = \bigcup_{i=1}^{\infty} A_i.$$

Let us first show that Y is not weakly Suslinian. Notice that

$$Y = [\bigcup_{k,m}((\{\tfrac{3k+1}{3^m}, \tfrac{3k+2}{3^m}\} \times I) \cup [\tfrac{3k+1}{3^m}, \tfrac{3k+2}{3^m}])] \cup (\bigcup_{p}\{p\} \times I),$$

where $p \in C$, $\frac{3k+1}{3^m} \neq p \neq \frac{3k+2}{3^m}$, k and m are integers such that $\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \in C$. In other words, Y can be decomposed into countably many disjoint *U's* and uncountably many disjoint *I's*. Y is also called the Cantor comb.

Now if X is not weakly Suslinian, then $X = \bigcup_{j \in J} B_j$, where $\{B_j\}_{j \in J}$ is a collection of mutually disjoint proper nondegenerate subcontinua of X. Assume B_0 contains the point $(1, 0)$, the common endpoint of A_i for all i, then B_0 contains A_i for all i, i.e., $Z \subset B_0$. Consequently, B_0 contains the whole segment $[0, 1]$, which is in the closure of Z. But then $X \backslash B_0$ would be a union of half-open and half-closed intervals. This would, as discussed in Example 1, lead to a contradiction of Corollary 1. Thus X is weakly Suslinian.

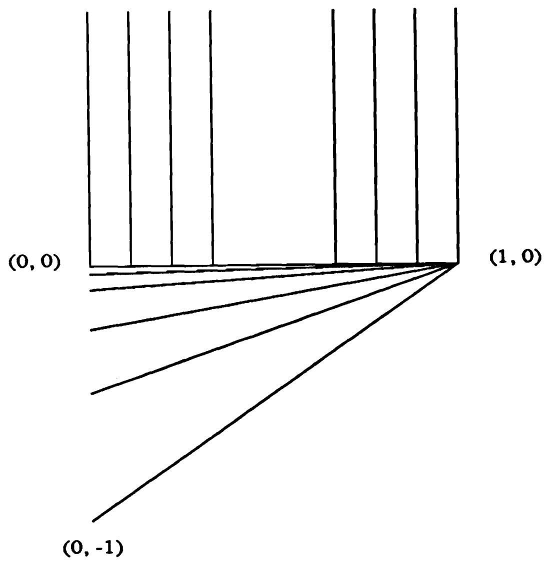

Figure 2: A weakly Suslinian dendroid

Lemma 1. *If a metric space X is a countable union of Suslinian continua, then X has property* Σ.

Proof: Assume the lemma is false. Let X_i be a Suslinian continuum for each $i = 1, 2, \ldots$, then $X = \bigcup_{i=1}^{\infty} X_i$ contains an uncountable collection $\mathcal{C}$ of pairwise disjoint nondegenerate compact connected subspaces of X. Since for compact (metric) spaces the notions of being hereditarily discontinuous and 0-dimensional coincide, there exists $C_0 \in \mathcal{C}$ such that $\dim(C_0 \cap X_i) \leq 0\ \forall i$, for otherwise $\dim(C_j \cap X_k) = 1$ for some positive integer k and uncountably many C_j's. But then we have $C_0 = \bigcup_{i=1}^{\infty}(C_0 \cap X_i)$, thus $\dim(C_0) = 0$, contradicting the fact that C_0 is of dimension 1. $\square$

Theorem 1. *If* $X = \bigcup_{i=1}^{\infty} X_i$ *is a continuum, where* X_i *is a continuum for each* $i = 1, 2, \ldots,$ *then* X *is Suslinian if and only if* X_i *is Suslinian* $\forall i$.

Proof: By Lemma 1, X has property Σ. Therefore X is a Suslinian continuum. Our theorem now follows from Proposition 1. □

The following example shows that, in the plane, the union of two weakly Suslinian continua is not weakly Suslinian.

Example 3. Let us start by constructing the cone over the Cantor set C, denoted M, by joining the point $(\frac{1}{2}, -1)$ with each point of C by a straight line segment. Let I be the closed unit interval $[0, 1]$, then $M \cup (C \times I)$ is homeomorphic to M and thus weakly Suslinian. In a similar fashion we may consider $N \cup (I \times C')$, where N is the cone over a copy of the Cantor set C' on the vertical line segment with endpoints

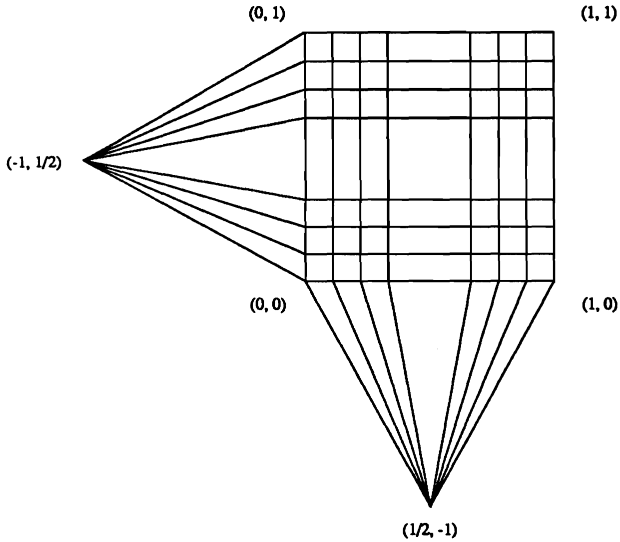

Figure 3: The union of two Cantor fans

(0, 0) and (0, 1), and the vertex of the cone is at $(-1, \frac{1}{2})$. In other words, we may obtain $N \cup (I \times C')$ from $M \cup (C \times I)$ by rotating the latter 90 degrees clockwise in the plane, with $(\frac{1}{2}, \frac{1}{2})$ as the point of rotation. Now we define

$$X = [M \cup (C \times I)] \cup [N \cup (I \times C')].$$

Clearly X is the union of two copies of the cone over the Cantor set. Now let S_i denote the boundary of the square centered at $(\frac{1}{2}, \frac{1}{2})$ and one vertex of S_i is at (i, i). Furthermore, define $A_{(i, j)}$ to be the annulus bounded by S_i and S_j. Then

$$X = [\bigcup_{k, m} (A_{(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})} \cap X)] \cup S_{\frac{1}{3}} \cup (\bigcup_{\alpha} S_\alpha) \cup (S_0 \cup M \cup N),$$

where $\alpha \in C$, $\frac{3k+1}{3^m} \neq \alpha \neq \frac{3k+2}{3^m}$, k and m are non-negative integers.

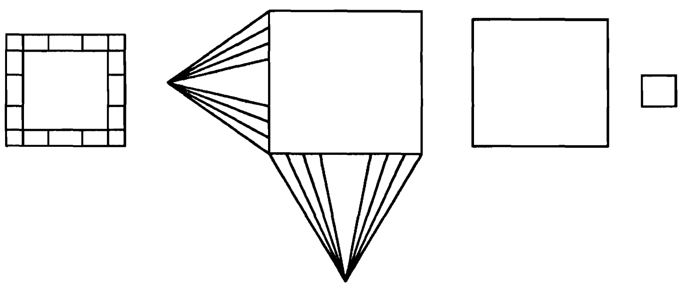

Figure 4: The decomposition of the union of two Cantor fans

It should be clear to the reader that the decomposition of X in this example is somewhat similar to the one in Example 2 (see Figures 3 and 4).

Notice that the continuum in Example 3 is not hereditarily unicoherent. In Example 4 we shall construct a non-weakly Suslinian hereditarily unicoherent continuum in R^3 which is the union of two weakly Suslinian continua.

Lemma 2. *Suppose that X is a continuum, then no composant K of X is a countable union of mutually disjoint proper subcontinua of X.*

Proof: For any $p \in X$, let K be the composant of p in X. By [7, Theorem 3-45], $K = \bigcup_{i=1}^{\infty} K_i$, where $p \in K_i$, and K_i is a proper subcontinuum of X for each i. Assume $K = \bigcup_{i=1}^{\infty} A_i$, $A_i \cap A_j = \emptyset$, for $i \neq j$, A_i is a subcontinuum of X $\forall i$. Then $p \in K_m$ for some m. Thus K_m is a subcontinuum of X, and

$$K_m = \bigcup_{i=1}^{\infty} (K_m \cap A_i).$$

Hence K_m is a countable disjoint union of closed subsets, and we have a contradiction of Sierpinski's theorem [9]. □

Theorem 2. *Suppose X is a continuum such that, if Y is a proper nondegenerate subcontinuum of X, Y is a Suslinian curve. Then X is hereditarily weakly Suslinian.*

Proof: Case I. X is decomposable. By [7, Theorem 3-43], X is the composant of some point $p \in X$. So by [7, Theorem 3-45], X is a countable union of proper subcontinua of X. Hence by Theorem 1 and Proposition 3, X is Suslinian and therefore weakly Suslinian.

Case II. X is indecomposable. By [7, Theorems 3-46 and 3-47], X has uncountably many composants which are disjoint, therefore it suffices to show that no composant K of X is a countable or an uncountable union of mutually disjoint nondegenerate subcontinua of X. Let K be the composant at a point $p \in X$. By Lemma 1, K is not an uncountable union of mutually pairwise disjoint nondegenerate subcontinua of X. By Lemma 2, K is not a countable union of such subspaces either. This ends the proof of Theorem 2. □

As a result of Theorem 2, some well-known continua, including the *Buckethandle* and any solenoid whose proper nondegenerate subcontinua are arcs (see [8], pp. 201-202), are weakly Suslinian.

Lemma 3. *An open ball $B(a, \varepsilon)$ in E^n, $n > 1$, is an uncountable union of pairwise disjoint nondegenerate compact connected subspaces of $B(a, \varepsilon)$.*

Proof: We may write the open ball as

$$B(a, \varepsilon) = \left(\bigcup_{\frac{\varepsilon}{2} < r < \varepsilon} C_r \right) \cup D_{\frac{\varepsilon}{2}},$$

where C_r is the circle of radius r, centered at a, and $D_{\frac{\varepsilon}{2}}$ is the disk of radius $\frac{\varepsilon}{2}$ centered at a. Clearly this decomposition satisfies the requirements. □

Theorem 3. *If X is a continuum contained in E^n, $n > 1$ and $dim(X) = n$, then X is an uncountable union of pairwise disjoint nondegenerate subcontinua of X.*

Proof: By [8, Theorem IV 3], X contains a nonempty subset U which is open in E^n, $n > 1$. Let $B(a, \varepsilon)$ be an open ball contained in U, and let $B(a, \frac{\varepsilon}{2})$ be an open ball inside $B(a, \varepsilon)$. By Lemma 3, $B(a, \frac{\varepsilon}{2})$ is an uncountable union of pairwise disjoint nondegenerate subcontinua of X, therefore it suffices to show that the set A as formulated below is connected. Let us define

$$A = X \backslash B(a, \frac{\varepsilon}{2}) \text{ and } C = B(a, \varepsilon) \backslash B(a, \frac{\varepsilon}{2}).$$

Thus C is connected. Assume that A is not connected, then $A = M \cup N$, where M and N are proper disjoint open subsets of A. Without loss of generality, suppose $C \subset N$, so

$$X = M \cup [N \cup B(a, \frac{\varepsilon}{2})].$$

But then X is not connected, a contradiction. This completes the proof. □

Corollary 2. *If X is a continuum embeddable in E^n, $n > 1$, and $dim(X) = n$, then X is not weakly Suslinian.*

Corollary 3. *If X is a nondegenerate weakly Suslinian continuum in the plane, then dim(X) = 1.*

Definition 4. *A mapping f of a compact space X onto a compact space Y is confluent if, for each subcontinuum C in Y, each component of $f^{-1}(C)$ is mapped onto C by f.*

Definition 5. *A mapping f of a compact space X onto a compact space Y is weakly confluent if, for each subcontinuum C in Y, there is a component K of $f^{-1}(C)$ such that f(K) = C.*

Theorem 4. *Weakly confluent mappings preserve Suslinian continua.*

Proof: Let X be a Suslinian continuum and f be a weakly confluent mapping from X onto a continuum Y. Assume on the contrary Y does not satisfy property Σ, then Y contains an uncountable collection $\{C_i\}_{i \in I}$ of pairwise disjoint nondegenerate proper subcontinua. For each $C_i \in \{C_i\}_{i \in I}$, $f^{-1}(C_i)$ contains a component K_i such that $f(K_i) = C_i$. Thus $\{K_i\}_{i \in I}$ is an uncountable collection of pairwise disjoint nondegenerate proper subcontinua of X. This contradiction completes the proof. □

Theorem 5. *Confluent mappings preserve weakly Suslinian continua.*

Proof: This is because whenever a continuum Y is a confluent image of a continuum X and Y can be decomposed into more than one pairwise disjoint nondegenerate subcontinua of Y, the pre-images of these subcontinua are nondegenerate proper subcontinua of X. □

Weakly confluent mappings, however, do not always preserve weakly Suslinian continua. Let us now review the space X in Example 2. Define

$$r : X \to Y$$

such that r is the identity on Y, and if $(x, y) \in Z$, then

$$r((x, y)) = (x, 0).$$

Thus r is a retraction, hence weakly confluent. But $r(X) = Y$, and Y, as discussed in Example 2, is not weakly Suslinian.

Next we shall investigate the connections between weakly Suslinian continua and hereditarily indecomposable continua. Let us quote Lemma 1.78 from [8].

Lemma 4. *Let X be a hereditarily indecomposable continuum. Let μ be any given fixed Whitney map for $C(X)$. Then, for each $t \in [0, \mu(X)]$, $\mu^{-1}(t)$ is a set-theoretic decomposition of X into subcontinua of X.*

If $0 < t < \mu(X)$, then $\mu^{-1}(t)$ is a set-theoretic decomposition of X into proper nondegenerate subcontinua of X for each t. We thus obtain

Proposition 5. *If X is a nondegenerate hereditarily indecomposable continuum, then X is not weakly Suslinian.*

Theorem 6. *If X is a hereditarily weakly Suslinian continuum, then $\dim(X) \leq 1$.*

Proof: Suppose $\dim(X) > 1$, then, by [1], there exists an essential mapping f from X onto the unit disk D. Let C be a nondegenerate hereditarily indecomposable subcontinuum of D. By a theorem of Mazurkiewicz [7], X contains a subcontinuum L such that $f(L) = C$ (i.e., f is weakly confluent). Following a theorem of H. Cook [2], f restricted to L is confluent. Since by Proposition 5, C is not weakly Suslinian, by Theorem 5, neither is L. Thus X is not hereditarily weakly Suslinian. □

Theorem 7. *Suppose X is a unicoherent locally connected nondegenerate continuum, then the following conditions are equivalent:*

(i) *X is hereditarily unicoherent;*
(ii) *X is a dendroid;*
(iii) *X is a dendrite;*
(iv) *X is hereditarily locally connected;*
(v) *X is rational;*
(vi) *X is Suslinian;*
(vii) *X is hereditarily weakly Suslinian;*
(viii) *$dim(X) = 1$.*

Proof: Since a locally connected continuum is arcwise connected, we have $(i) \Rightarrow (ii)$. And (ii) implies (iii) because every locally connected dendroid is a dendrite. Since every subcontinuum of a dendrite is a dendrite itself, (iii) implies (iv). By [10, p. 94], $(iv) \Rightarrow (v)$. And by [6], (v) implies (vi). Now by Propositions 1 and 2 we immediately have $(vi) \Rightarrow (vii)$. By Theorem 6, it is clear that $(vii) \Rightarrow (viii)$. Finally, by [5, p. 442, Corollary 8], we easily see that $(viii)$ implies (i). □

It is interesting to note that the examples of weakly Suslinian continua we have seen are all 1-dimensional. However, if we replace the vertical line segment joining $(0, -1)$ and $(0, 1)$ of the $\sin[1/x]$-continuum by a disk such that the ray now converges onto the whole disk, using an argument similar to the one in Example 2, we can easily show that the resulting continuum is weakly Suslinian and it is 2-dimensional. Similarly, if such a ray in the Euclidean $(n + 1)$-space converges onto an n-cell, an n-dimensional weakly Suslinian continuum is obtained. An infinite dimensional weakly Suslinian continuum can be constructed if we replace the n-cell by the *Hilbert cube* (i.e., a space which is homeomorphic to the countable product of the closed unit interval I).

3. MORE RESULTS ON LOCALLY CONNECTED CONTINUA

In 1927, G. T. Whyburn introduced the concept of cyclic elements of a locally connected continuum X. Our results in this section have close connection with this concept (see [5] and [10] for more information on this).

Lemma 5. *Suppose X is a continuum and $\mathcal{C} = \{C_i\}_{i=0}^{\infty}$ is a countable collection of subcontinua of X such that $C_0 \cap C_i \neq \emptyset$ for all $i = 1, 2, \ldots$, and $\lim_{i\to\infty} diam(C_i) = 0$. Then $Y = \bigcup_{i=0}^{\infty} C_i$ is a subcontinuum of X.*

Proof: It suffices to show that Y is closed. Let p be a limit point of Y. Then there exists a sequence of points $\{p_j\}_{j=1}^{\infty}$ converging to p. Each p_j belongs to some element C_{i_j} in a subcollection $\mathcal{D}$ of $\mathcal{C}$. If $\mathcal{D}$ is a finite collection, when j is sufficiently large, C_{i_j} will be a constant, consequently $\{p_j\}_{j=1}^{\infty}$ converges to p, so p is in C_{i_j}. However, if $\mathcal{D}$ is an infinite collection, for each p_j we may pick a q_j such that $q_j \in C_0 \cap C_{i_j}$. Hence both $\{p_j\}_{j=1}^{\infty}$ and $\{q_j\}_{j=1}^{\infty}$ converge to p which now lies in C_0. □

Theorem 8. *The property of being a weakly Suslinian continuum is reducible.*

Proof: We will prove the contrapositive of this theorem. Suppose a locally connected continuum X has cyclic elements C_n, $n = 1, 2, \ldots$, then $\lim_{n\to\infty}$ diam$(C_n) = 0$ (see [5], pp. 313-315). If C_1 is a nondegenerate cyclic element which is not weakly Suslinian, then $C_1 = \bigcup_{i\in I} B_i$, where each B_i is a nondegenerate subcontinuum of C_1 and $B_i \cap B_j = \emptyset$ for all i, j, $i \neq j$. Set $A = C_1 \cap (\bigcup_{m=2}^{\infty} C_m)$, then A is at most countable (see [5], p. 315, Theorem 8). For each $p \in A$, p is in B_j for some j. Let K_p be the component containing p in $(X\backslash C_1) \cup \{p\}$, then $K_p \cup B_j$ is a subcontinuum of X. For each i, set

$$Y_i = \bigcup_{p\in B_i\cap A} (B_i \cup K_p).$$

So by Lemma 5, Y_i is a subcontinuum of X for each i. Thus $X = \bigcup_{i \in I} Y_i$. Consequently, X is not weakly Suslinian. □

As an analogue to Theorem 8, we now observe

Theorem 9. *The property of being a Suslinian continuum is reducible.*

Proof: By [5, p. 315, Theorem 7], every cyclic element of a locally connected continuum X is a locally connected subcontinuum of X. Now our theorem follows from Proposition 1. □

In an attempt to settle a question which asks whether the property of being a weakly Suslinian continuum is extensible or not, we came across the following problem.

Problem. *If a locally connected continuum X can be written as an uncountable disjoint union of proper subcontinua only finitely many of which are degenerate, is X weakly Suslinian?*

Our next example, however, gives an affirmative answer to an analogue of this problem when X is not locally connected.

Comment: A simple modification of our next example also gives an affirmative answer to the problem (E. D. Tymchatyn, 09/22/94).

Example 4. The Cantorian Swastika X is shown in Figure 5. X here is clearly the union of four copies of the Cantor comb. Using a similar argument as in Example 2, $X \setminus \{(0, 0)\}$ is a union of pairwise disjoint nondegenerate subcontinua of X. Now we want to show that X is weakly Suslinian. If X has a non-weakly Suslinian decomposition $\mathcal{D}$, let $A \in \mathcal{D}$ such that A contains the origin $(0, 0)$. Without loss of generality, A contains the closed interval $[0, 1/2]$. By Corollary 1, A contains all vertical line segments with endpoints $(c, 0)$ and $(c, 1)$, where c is a point in the Cantor set on I and $0 < c <$

1/2. Hence A contains the vertical line segment with endpoints (0, 0) and (0, 1). In the same fashion A must contain the horizontal line segment with endpoints (0, 0) and (−1, 0). Consequently, we have $A = X$. Thus X is weakly Suslinian.

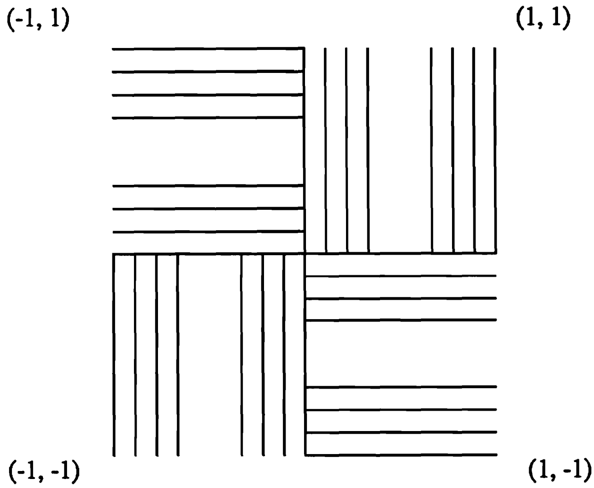

Figure 5: The Cantorian Swastika

Example 4 can be modified to show that there exists a hereditarily unicoherent non-weakly Suslinian continuum which is the union of two weakly Suslinian continua. Let $Z = X \cup B$, where X is the Cantorian Swastika and B is an arc such that the intersection of X and B is the origin (0, 0). It is obvious that Z has all the properties required.

Theorem 10. *The property of being a Suslinian continuum is extensible.*

Proof: Let X be a locally connected continuum such that each cyclic element of X is Suslinian. We classify the subcontinua K of X by the following three types.

Type I. K is a nondegenerate cyclic element.

Type II. K is a singleton point to which some nondegenerate cyclic elements converge.

Type III. K is a maximal subcontinuum of X in the sense that K meets any nondegenerate cyclic element of X at no more than one point.

There are at most countably many subcontinua of these three types (see [5], pp. 312-315). By assumption the subcontinua of type I and type II are all Suslinian. Let Y be a subcontinuum of type III, then Y is a retract of X, and therefore Y is locally connected. Notice that Y contains no simple closed curve, so Y is a dendrite. By Theorem 7, Y is Suslinian. By now we have already shown that X is a countable union of Suslinian continua, hence by Theorem 1, X is Suslinian. □

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