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## ON A D-EXTENSION OF A HOMEOMORPHISM

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ABSTRACT. In the paper we study the problem: under what conditions imposed upon sets  $A, B \subset R^2$  does each homeomorphic embedding [4]  $h : A \rightarrow Y$  ( $Y$  is some metric space) possess a Darboux extension uniformly discontinuous on  $B$ .

In many mathematical problems, a rather essential role is played by the possibility of extending a given transformation, defined on a subset of some space, to a function defined on the whole space with the preservation of certain properties. Many authors considered these problems for a function "approaching" Darboux functions (see, for example, [CG], [CL], [GR], [KK], [PR], [RJ], [RGK]).

It seems particularly interesting to consider the following problem: Suppose we have a subset  $A$  of a fixed metric space  $X$ . What additional conditions must be satisfied by these objects in order that any homeomorphic embedding  $h : A \rightarrow Y$  ( $Y$  is any metric space for which such a transformation exists) possess a Darboux extension? A natural demand in this case is the preservation of a large number of points of continuity. In turn in view of this assumption, it becomes material to ask the question whether we shall obtain an essential-Darboux extension (i.e. a Darboux extension  $d$  such that each point from the boundary of the domain of the initial function is a point of discontinuity of  $d$ ) or this extension will not be an essential-Darboux one. For it is worth noting here that the observations of concrete examples show that *there exist metric spaces  $X$*

and  $Y$  as well as a closed subset  $A \subset X$  and a homeomorphism  $h : A \rightarrow Y$  such that there is no Darboux extension of  $h$  being discontinuous exactly on  $Fr(A)$ . On the other hand, there are examples of homeomorphisms defined on closed subsets of a fixed space, for which one cannot find a Darboux extension continuous at each point of the domain of the initial function.

A complement to this problem is the question whether a function being an extension of a homeomorphic embedding can be discontinuous (uniformly discontinuous) exclusively on a preassigned set. All these questions, concentrating upon the problem of Darboux extensions of homeomorphisms, lead to the acceptance of Definition 1. However, before we formulate it, we shall give the basic symbols and terms applied in our paper. In many papers, the notion of a Darboux function was generalized to transformations whose domain (and range) are topological spaces more general than the real line (see, for example, [BB], [CJ], [GK], [JJ], [PR], [PW]). In this article, we study the following version of this notion:

We say that  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are arbitrary topological spaces, is a Darboux transformation if  $f(L)$  is a connected set for each arc  $L \subset X$ .

Throughout the paper we apply the classical symbols and notation (see [ER], [PR]). However, in order to avoid any ambiguities, we shall now present those symbols used in the paper whose meanings are not explained in the main text.

The sets of all real numbers, positive integers, rational numbers and points of the plane are denoted by  $R$ ,  $N$ ,  $Q$  and  $R^2$ , respectively. The natural metric on the plane is denoted by  $\varrho$ , and the closed interval on the plane with end-points  $a$  and  $b$  is denoted by  $[a, b]$ .

For  $f : X \rightarrow Y$ , let  $C_f$  ( $D_f$ ) denote the set of all continuity (discontinuity) points of  $f$ . The combination of the mappings  $\{f_t\}_{t \in T}$  is denoted by the symbol  $\bigvee_{t \in T} f_t$  (for  $T = \{1, 2\}$ , we write  $f_1 \bigvee f_2$ ).

The interior and boundary of a set  $B$  in a subspace  $A$  of  $R^2$  are denoted by  $Int_A(B)$  and  $Fr_A(B)$ , resp. (in the case when  $A = R^2$ , we write  $Int B$  and  $Fr B$ , resp.). We say that  $A$  and  $B$  are strongly disjoint if  $\varrho(A, B) > 0$ . The complement (closure) of  $A$  is denoted by  $A'$ ,  $(\overline{A})$ . For any  $A \subset R^2$  and  $\alpha > 0$ , assume  $A^\alpha = \{x \in R^2 : \varrho(A, x) = \alpha\}$ . Moreover, let  $H_a^b$  be a symbol of a closed half-line with end-point  $a$  such that  $b \in H_a^b$ .

By the symbol  $K(x, r)$  ( $S(x, r)$ ) we denote an open ball (sphere) with center  $x$  and radius  $r$ . In the case  $r = 0$ , we assume that  $K(x, r) = S(x, r) = \{x\}$ .

Let  $L \subset R^2$  be an arc and let  $a, b \in L$ . There exists exactly one arc  $L' \subset L$  with end-points  $a$  and  $b$ . Denote  $L' = L_L(a, b)$ .

Throughout the paper, we assume the continuum hypothesis. Let  $*$  be some property of a fixed space  $X$ . We say that a set  $A$  is maximal with respect to  $*$  if, for any set  $B$  such that  $A \subset B$  and  $B$  possesses the property  $*$ ,  $A = B$ .

Let  $X, Y$  be arbitrary metric spaces. For any  $f : X \rightarrow Y$ ,  $A \subset X$  and  $x_0 \in \overline{A}$ , by  $L_A(f, x_0)$  we denote the cluster set of  $f$  at  $x_0$  with respect to  $A$  (i.e.  $\alpha \in L_A(f, x_0)$  if there exists a sequence  $\{a_n\} \subset A$  such that  $\lim_{n \rightarrow \infty} a_n = x_0$  and  $\lim_{n \rightarrow \infty} f(a_n) = \alpha$ ). We say that a function  $f$  is uniformly discontinuous on  $B$  (u-disc on  $B$ ) if there exists  $\delta > 0$  such that  $L_B(f, b) \setminus K(f(b), \delta) \neq \emptyset$  for each  $b \in B$  (the  $\delta$  described above is called a coefficient of the uniform discontinuity of  $f$  on  $B$ ).

**Definition 1.** *Let  $A, B$  be disjoint subsets of a metric space  $X$ . We say that a set  $A$  possesses the property of a D-extension of a homeomorphism, with the u-disc on  $B$ , if, for every metric space  $Y$  and for each homeomorphism  $h : A \rightarrow h(A) \subset Y$ , there exist two Darboux functions  $d_1^h, d_2^h : X \rightarrow Y$  which are some extensions of  $h$ , such that  $d_1^h|_{B'}$  and  $d_2^h|_{B' \setminus Fr A}$  are continuous and  $B$  is a maximal set of uniform discontinuity for  $d_1^h$  and  $d_2^h$ .*

In the proof of the Theorem, we shall use (although this fact

will not always be noted down) the following:

**Lemma.** ([PR], [PRJ]). *Let  $A \subset R^2$  be a closed convex set with a nonempty interior. Let  $H_p$  denote an arbitrary half-line with end-point  $p$  for  $p \in \text{Int } A$ . Then, if there exists a real number  $\alpha_0 > 0$  such that either  $H_p \cap A^{\alpha_0} \neq \emptyset$  or  $H_p \cap \text{Fr } A \neq \emptyset$ , then  $H_p \cap A^\alpha$  ( $\alpha > 0$ ) and  $H_p \cap \text{Fr } A$  are singletons.*

**Theorem.** *Let  $A$  and  $B$  be convex, non-singleton and strongly disjoint subsets of the plane. Then  $A$  possesses the property of a  $D$ -extension of a homeomorphism, with the  $u$ -disc on  $B$ , if and only if  $A$  and  $B$  are closed.*

*Proof. Necessity.* Let  $Y$  be an arbitrary metric space such that there exists a homeomorphic embedding  $h : A \rightarrow Y$ . Moreover assume the notations from the definition of a  $D$ -extension of a homeomorphism.

We shall now show that  $B$  is a closed set. Suppose to the contrary that there exists an element  $b_0 \in B' \setminus \text{Int } B'$ . Let  $\delta > 0$  be a coefficient of the uniform discontinuity of  $d_2^h$  on  $B$ .

We shall prove

$$(1) \quad L_{B \cup \{b_0\}}(d_2^h, b_0) \setminus K(d_2^h(b_0), \frac{\delta}{4}) \neq \emptyset.$$

Let  $S = S(d_2^h(b_0), \frac{\delta}{4})$ . Moreover, let  $n_0$  be a fixed positive integer such that

$$(2) \quad K(b_0, \frac{1}{n_0}) \cap \text{Fr } A = \emptyset \text{ and } d_2^h(K(b_0, \frac{1}{n_0}) \cap B') \subset K(d_2^h(b_0), \frac{\delta}{4}).$$

Let  $n \geq n_0$ . Consider the ball  $K(b_0, \frac{1}{n})$ . There exists some  $b_n \in B \cap K(b_0, \frac{1}{n})$ . Let us consider the following cases:

1.  $d_2^h(b_n) \notin K(d_2^h(b_0), \frac{\delta}{4})$ . Thus there exists an  $x_n \in K(b_0, \frac{1}{n})$  such that  $d_2^h(x_n) \in S$  and so, according to (2) and to the fact that  $n \geq n_0$ , we infer that  $x_n \in B$ .
2.  $d_2^h(b_n) \in K(d_2^h(b_0), \frac{\delta}{4})$ . Thus

$$(3) \quad K(d_2^h(b_0), \frac{\delta}{4}) \subset K(d_2^h(b_n), \frac{\delta}{2}).$$

Since there exists some  $\beta_n \in L_B(d_2^h, b_n) \setminus K(d_2^h(b_n), \delta)$ , we have that there exists some  $c_n \in K(b_0, \frac{1}{n})$  such that  $d_2^h(c_n) \notin K(d_2^h(b_n), \frac{\delta}{2})$ . From (3) and the Darboux property of  $d_2^h$  it follows that there exists an  $x_n \in K(b_0, \frac{1}{n})$  such that  $d_2^h(x_n) \in S$ . By (2),  $x_n \in B$ .

In this way we describe the sequence  $\{x_n\}_{n=n_0}^\infty \subset B \cup \{b_0\}$  for which  $\lim_{n \rightarrow \infty} x_n = b_0$  and  $d_2^h(x_n) \in S$ . Let  $\{x_{k_n}\}$  be a subsequence of  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} d_2^h(x_{k_n}) = \xi$ . Then  $\xi \in L_{B \cup \{b_0\}}(d_2^h, b_0) \setminus K(d_2^h(b_0), \frac{\delta}{4})$ , which ends the proof of (1).

Condition (1) contradicts the supposition that  $B$  is the maximal set which  $d_2^h$  is uniformly discontinuous.

We can easily verify (using, for example, an extension of type  $d_1^h$ ) that  $A$  is a closed set, too.

*Sufficiency.* Let  $Y$  be a metric space such that there exists a homeomorphic embedding  $h : A \rightarrow Y$ . Let  $\varrho(A, B) = 3 \cdot \gamma > 0$ . Consider the set  $A^* = \{x \in R^2 : \varrho(x, A) \leq \gamma\}$ .

First, we shall define functions  $d_{1,1}^h, d_{1,2}^h : A^* \rightarrow Y$ . Let us consider the following cases:

1°  $Int A \neq \emptyset$ . Let  $x_0 \in Int A$ . If  $x \in A$ , then we put  $d_{1,1}^h(x) = d_{1,2}^h(x) = h(x)$ .

For the simplicity of the notation, we assume  $\gamma_n = \frac{\gamma}{n}$  ( $n = 1, 2, \dots$ ) and  $A_n^* = \{x \in R^2 : \gamma_{n+1} \leq \varrho(x, A) \leq \gamma_n\}$  for  $n = 1, 2, \dots$ . Moreover for any  $z \notin A$ , let  $z^0$  denote a point of the intersection of the half-line  $H_{x_0}^z$  and  $Fr A$ . Let  $\alpha_z$  denote a number such that  $z \in A^{\alpha_z}$ .

Let us define  $t_n : A_n^* \rightarrow Y$  in the following way. Let  $z \in A_n^*$ . If  $n$  is an even number, then we put  $t_n(z) = h(x_z)$  where  $x_z \in [x_0, z^0]$  and  $\varrho(z^0, x_z) = (\gamma_n - \alpha_z) \cdot \frac{\varrho(x_0, z^0)}{\gamma_n - \gamma_{n+1}}$ . If  $n$  is an odd number, then we put  $t_n(z) = h(y_z)$  where  $y_z \in [x_0, z^0]$  and  $\varrho(x_0, y_z) = (\gamma_n - \alpha_z) \cdot \frac{\varrho(x_0, z^0)}{\gamma_n - \gamma_{n+1}}$ .

It is not difficult to verify that  $t_n$  is a continuous function for every  $n$ .

Since the mappings  $t_n$  are compatible ([ER], Proposition 2.1.13 p.100), we may define a continuous function  $t = \bigvee_{n=1}^{\infty} t_n$  mapping  $A^* \setminus A$  into  $Y$ . Let now  $d_{1,2}^h(x) = t(x)$  for  $x \in A^* \setminus A$ . Of course,  $d_{1,2}^h$  is discontinuous at each point of  $Fr A$ .

Let us define  $d_{1,1}^h$  on  $A^* \setminus A$  in an analogous way as  $t_1$ , i.e. for  $z \in A^* \setminus A$ , let  $d_{1,1}^h(z) = h(x_z)$  where  $x_z \in [x_0, z^0]$  and  $\varrho(x_0, x_z) = (\gamma - \alpha_z) \cdot \frac{\varrho(x_0, z^0)}{\gamma}$ . It is easy to see that  $d_{1,1}^h : A^* \rightarrow Y$  is a continuous function.

2°  $Int A = \emptyset$ . In this case,  $A$  is either a nondegenerate closed interval, a half-line or a line. Let  $x_0 \in A$  be a point such that  $x_0 \notin Fr_L(A)$  where  $L$  is a line which contains  $A$ . Let  $L_1, L_2$  be two different closed half-lines included in  $L$  with endpoint  $x_0$ . In the case when  $L_i \cap A$  (for  $i = 1$  or  $i = 2$ ) is an unbounded set, let  $\{k_n^i\}_{n=1}^{\infty}$  be a sequence of points of the set  $L_i \cap A$  such that  $\varrho(x_0, k_n^i) = n$ . In the case when  $L_i \cap A$  (for  $i = 1$  or  $i = 2$ ) is a bounded set, let  $\{k_n^i\} = Fr_L(A) \cap L_i$ . Then we assume  $k_n^i = k_1^i$  ( $n = 1, 2, \dots$ ). Let now  $\{\gamma_n\}_{n=1}^{\infty}$  be some sequence converging to zero monotonically, such that  $\gamma_1 = \gamma$ .

Define a mapping  $\tau' : \bigcup_{n=1}^{\infty} A^{\gamma_n} \rightarrow Y$  by letting

$$\tau'(x) = \begin{cases} h(x_0) & \text{for } x \in A^{\gamma_1}, \\ h(k_n^1) & \text{for } x \in A^{\gamma_{2n+1}} \ (n = 1, 2, \dots), \\ h(k_n^2) & \text{for } x \in A^{\gamma_{2n}} \ (n = 1, 2, \dots). \end{cases}$$

Of course,  $\tau'$  is a continuous function defined on a closed subset of  $A^* \setminus A$ . One can readily see, that there exists a continuous extension  $\tau : A^* \setminus A \xrightarrow{onto} h(A) \subset Y$  of  $\tau'$  such that, for every arc  $L \subset A^*$ , if  $L \cap A \neq \emptyset \neq L \setminus A$ , then  $\tau(L \setminus A) = h(A)$ .

Let now

$$d_{1,2}^h(x) = \begin{cases} h(x) & \text{for } x \in A, \\ \tau(x) & \text{for } x \in A^* \setminus A. \end{cases}$$

Then  $d_{1,2}^h : A^* \rightarrow Y$  and  $C_{d_{1,2}^h} = A^* \setminus Fr(A)$ .

Now, we are going to define a function  $d_{1,1}^h : A^* \rightarrow Y$ . Let  $\Pi'(x) = h(x)$  for  $x \in A$  and  $\Pi'(x) = h(x_0)$  for  $x \in A^\gamma$ . Thus  $\Pi' : A \cup A^\gamma \rightarrow Y$  is a continuous function defined on a closed subset of  $A^*$ . Let  $d_{1,1}^h$  be a continuous extension of  $\Pi'$  over  $A^*$ .

Let us denote  $B^* = \{x \in R^2 : \varrho(x, B) \leq \gamma\}$ . For  $x \in \{x \in R^2 : \varrho(x, A) \geq \gamma \wedge \varrho(x, B) \geq \gamma\}$ , we assume  $d_{2,1}^h(x) = h(x_0)$ .

Now we shall define a mapping  $d_{3,1}^h : B^* \setminus B \rightarrow Y$ . Let  $[x_0, x_1]$  be a nondegenerate interval included in  $A$ . Then, similarly as in the case of the mapping  $d_{1,2}^h$  defined in  $2^\circ$ , we may construct a continuous function  $d_{3,1}^h : B^* \setminus B \rightarrow h([x_0, x_1])$  such that  $d_{3,1}^h(x) = h(x_1)$  for each  $x \in B^{\frac{\gamma}{2n}}$  ( $n = 1, 2, \dots$ ),  $d_{3,1}^h(x) = h(x_0)$  for each  $x \in B^{\frac{\gamma}{2n+1}}$  ( $n = 0, 1, 2, \dots$ ) and, for every arc  $L \subset B^*$ , if  $L \cap B \neq \emptyset \neq L \setminus B$ , then  $d_{3,1}^h(L \setminus B) = h([x_0, x_1])$ .

Remark that the family of sets  $\{A^*, \{x \in R^2 : \varrho(x, A) \geq \gamma \wedge \varrho(x, B) \geq \gamma\}, B^* \setminus B\}$  is a finite family of closed sets in  $R^2 \setminus B$ . Hence ([ER], Proposition 2.1.13, p.100) the mapping  $\tilde{d}_1^h = d_{1,1}^h \nabla d_{2,1}^h \nabla d_{3,1}^h : R^2 \setminus B \rightarrow Y$  is continuous and the mapping  $\tilde{d}_2^h = d_{1,2}^h \nabla d_{2,1}^h \nabla d_{3,1}^h : R^2 \setminus B \rightarrow Y$  is continuous at each point of the set  $R^2 \setminus (B \cup Fr A)$ .

Let us define a mapping  $r : B \rightarrow Y$  in the following way. Let  $b_0 \in B$  be a fixed point. On the family  $\{S(b_0, \alpha) : S(b_0, \alpha) \cap B \neq \emptyset \wedge \alpha \geq 0\}$  we define the following equivalence relation:

$$S(b_0, \alpha_1) * S(b_0, \alpha_2) \text{ if and only if } \alpha_1 - \alpha_2 \in Q.$$

The set of equivalence classes of  $*$  is denoted by  $\mathcal{S}$ . Moreover, let  $f$  map  $\mathcal{S}$  onto  $h([x_0, x_1])$ . For  $x \in B$ , let  $S_x$  be an element of  $\mathcal{S}$  such that there exists some  $\beta$  for which  $x \in S(b_0, \beta) \in S_x$ . Thus we assume  $r(x) = f(S_x)$  ( $x \in B$ ).

Let us consider

$$d_1^h = \tilde{d}_1^h \nabla r : R^2 \rightarrow Y,$$

$$d_2^h = \tilde{d}_2^h \nabla r : R^2 \rightarrow Y,$$

We shall show that the above functions are the required mappings.

Indeed. First, we prove that  $d_1^h, d_2^h$  are Darboux functions. The proof of this fact is limited to the mapping  $d_2^h$  because in the case of  $d_1^h$ , the reasoning is analogous (a bit simpler).

Let now  $L$  be an arbitrary arc included in  $R^2$ . Consider the following cases:

1)  $L \subset A$  or  $L \subset R^2 \setminus (A \cup B)$ . Therefore  $d_2^h(L)$  is connected because  $d_2^h$  is continuous on each of these sets.

2)  $L \subset B$ .

2a)  $L \subset S(b_0, \alpha)$  for some  $\alpha > 0$ . Thus  $d_2^h(L)$  is a singleton.

2b)  $L \setminus S(b_0, \alpha) \neq \emptyset$  for each  $\alpha \geq 0$ . Therefore it is easy to see, that for any  $S \in \mathcal{S}$ , there exists a positive number  $\alpha_S$  such that  $S(b_0, \alpha_S) \subset S$  and  $L \cap S(b_0, \alpha_S) \neq \emptyset$ . Consequently,  $d_2^h(L) = h([x_0, x_1])$ .

3) None of the above-mentioned cases holds. Assume to the contrary that  $d_2^h(L)$  is not connected, and so,  $d_2^h(L) = P \cup Q$  where  $P$  and  $Q$  are nonempty separated sets. Put  $P_1 = L \cap (d_2^h)^{-1}(P)$  and  $Q_1 = L \cap (d_2^h)^{-1}(Q)$ . Thus  $P_1$  and  $Q_1$  are nonempty disjoint sets and  $L = P_1 \cup Q_1$ , so  $P_1$  and  $Q_1$  are not separated sets. Without loss of generality we may assume that there exists  $o \in P_1 \cap \overline{Q_1}$ . Let  $\{o_n\}_{n=1}^\infty$  be a sequence of elements of  $Q_1$  such that  $o = \lim_{n \rightarrow \infty} o_n$ . Of course,  $o \notin \text{Int } A \cup (R^2 \setminus (A \cup B))$ . Let us consider the following cases:

3a)  $o \in \text{Fr } A$ . It is easy to see that, in this case,  $\{o_n\}_{n=1}^\infty$  cannot contain a subsequence whose elements belong to  $A$ . So, without loss of generality we may assume that  $o_n \notin A$  for  $n = 1, 2, \dots$

Let us first consider the case when  $\text{Int } A \neq \emptyset$ . Let  $\Xi_k$  denote an open and convex angle containing the half-line  $H_{x_0}^0$  such that the arc measure of this angle is equal to  $\frac{1}{k}$  ( $k = 1, 2, \dots$ ). Moreover, let  $n_k$  (for any  $k = 1, 2, \dots$ ) be a positive integer such that  $L_L(o_{n_k}, o) \subset \Xi_k$ .

Fix  $k$ . Let  $L_k = L_L(o_{n_k}, o)$ . Let us observe that  $d_{1,2}^h(L_k \setminus \{o\})$  is a connected set included in  $Q$ . Thus there exists an  $n_0$  such

that  $L \cap A^{\gamma_n} \neq \emptyset$  for  $n \geq n_0$ , and so,  $d_{1,2}^h(L_k \setminus \{o\}) \cap h(Fr(A) \cap \Xi_k) \neq \emptyset$ . Consequently, let  $r_k \in Fr(A) \cap \Xi_k$  be an element such that  $h(r_k) \in d_{1,2}^h(L_k \setminus \{o\})$ . Of course,  $h(r_k) \in Q$ .

One can readily see that  $\lim r_k = o$ , and so,  $\lim d_{1,2}^h(r_k) = d_{1,2}^h(o)$ , which contradicts the assumption that  $P$  and  $Q$  are separated sets.

Let us now analyze the situation where  $A$  is a boundary set. Similarly as above we may assume that  $\{o_n\}_{n=1}^\infty \subset A^* \setminus A$  (moreover, without loss of generality, we may assume that  $L_L(o_n, o) \subset A^*$  for  $n = 1, 2, \dots$ ). It is easy to observe that the set  $d_{1,2}^h(L_n \setminus \{o\})$  is connected for  $L_n = L_L(o_n, o)$ , and  $d_{1,2}^h(L_n \setminus \{o\}) = h(A)$ . Since  $d_{1,2}^h(L_n \setminus \{o\}) \subset Q$ , the set  $h(A) \subset Q$ . This means that  $d_{1,2}^h(o) \in Q$ , which is impossible.

3b)  $o \in B$ . Observe that, in this case, we have

$$h([x_0, x_1]) \subset d_2^h(L).$$

Thus either  $h([x_0, x_1]) \subset P$  or  $h([x_0, x_1]) \subset Q$ . Let us consider the above cases:

3b<sub>1</sub>)  $h([x_0, x_1]) \subset Q$ . Therefore  $o \in B'$  (in the opposite case,  $d_2^h(o) \in h([x_0, x_1]) \subset Q$ , and so,  $o \notin P_1$ ), which contradicts the assumption, of 3b).

3b<sub>2</sub>)  $h([x_0, x_1]) \subset P$ . Since  $o \in B$ , there exists an  $n_*$  such that  $o_n \in B^*$  for  $n \geq n_*$ . This means that  $d_2^h(o_n) \in h([x_0, x_1]) \subset P$  for  $n \geq n_*$ , and thus,  $d_2^h(o_n) \notin Q$  for  $n \geq n_*$ , which is impossible.

The contradiction obtained proves that  $d_2^h$  is a Darboux function.

Now, we shall show that  $B$  is the maximal set on which the functions  $d_1^h$  and  $d_2^h$  are uniformly discontinuous. Let  $\delta = \frac{1}{3} \cdot dia h([x_0, x_1]) > 0$ . We shall prove that  $\delta$  is a coefficient of the uniform discontinuity of  $d_1^h$  (the proof in the case of  $d_2^h$  is identical).

Let now  $b \in B$  and denote  $K = K(d_1^h(b), \delta)$ . Thus there exists an element  $\xi \in h([x_0, x_1]) \setminus K$ . Let  $\alpha_b \geq 0$  be a number such that  $b \in S(b_0, \alpha_b)$  and let  $S_\xi \in \mathcal{S}$  be an equivalence class such that  $f(S_\xi) = \xi$ . Let us fix  $S(b_0, \mu) \in S_\xi$  ( $\mu > 0$ ). Moreover, let

$\{p_n\}_{n=1}^{\infty}, \{q_n\}_{n=1}^{\infty}$  be two sequences of rational numbers, such that  $\{\mu + p_n\}_{n=1}^{\infty}$  is an increasing sequence and  $\{\mu + q_n\}_{n=1}^{\infty}$  is a decreasing sequence and

$$\lim_{n \rightarrow \infty} (\mu + p_n) = \lim_{n \rightarrow \infty} (\mu + q_n) = \alpha_b$$

(in the case  $b = b_0$ , we consider, of course only  $\{q_n\}_{n=1}^{\infty}$ ).

Since  $B$  is a convex set and it is not a singleton, then there exists an interval  $I_b \subset B$  with end-point  $b$ . Let  $I_b^n$  denote an interval with end-point  $b$ , included in  $I_b$ , such that  $\text{dia } I_b^n = \frac{1}{n} \text{dia } I_b$ . In virtue of the fact that the non-singleton  $\varrho_{b_0}(I_b^n) = \{\varrho(x, b_0) : x \in I_b^n\}$  (for  $n = 1, 2, \dots$ ) is a connected set containing  $\alpha_b$ , (for  $n = 1, 2, \dots$ ) there exists an element  $b_n \in I_b^n$  such that either  $\varrho(b_n, b_0) = \mu + p_k$  (for some positive integer  $k$ ) or  $\varrho(b_n, b_0) = \mu + q_k$  (for some positive integer  $k$ ). It is easy to see that  $\lim_{n \rightarrow \infty} b_n = b$ . It is obvious that  $d_1^h(b_n) = \xi$  ( $n = 1, 2, \dots$ ), and so,  $\xi \in L_B(d_1^h, b) \setminus K$ .

The fact that  $B$  is the maximal set on which  $d_1^h$  is uniformly discontinuous is obvious.

**Problem.** It seems interesting to ask the question whether the assumption of the convexity of the sets  $A$  and  $B$  can be weakened in an essential way. It could also be interesting to obtain a result analogous to the above theorem, where the domain of the transformations under consideration would be some metric space. Finally, it is worth while to raise the question: can one construct appropriate Borel extensions (or measurable ones of class  $\alpha$  - cf. ([ER], p.83))?

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