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A NEW CONTINUOUS CELLULAR DECOMPOSITION OF THE DISK INTO NON-DEGENERATE ELEMENTS

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1. Introduction

The main result of this paper is a new construction of a continuous decomposition of a two dimensional disk so that each point of the decomposition space is a non-degenerate non-separating continuum. The decomposition space is homeomorphic to the disk. Even though R. D. Anderson in [1] described a similar non-trivial continuous decomposition of the disk, our construction is of interest because in [4] it is extended to create a non-trivial continuous decomposition of the Sierpiński curve[1] which is homeomorphic to the Sierpiński curve. In [1] R. D. Anderson first describes a construction which results in a non-trivial continuous decomposition of the plane homeomorphic to the plane[2]. He then indicates how the construction can be modified in order to obtain a non-trivial decomposition of the disk homeomorphic to the disk; however, the details are sketchy. The paper goes on to show that there is a continuous decomposition of a 1-dimensional planar curve homeomorphic to the plane. This is done by closing holes in the planar curve. Thus it is of interest if closing holes can be avoided or selectively avoided when decomposing the Sierpiński curve. Our construction described here is based on that of W. Lewis and

[1]By the Sierpiński curve we mean Sierpiński's plane universal curve.

[2]He also mentions in this paper that the construction can be modified so that each member of the decomposition is a pseudo-arc.

J. J. Walsh described in [2] where they show in detail how the plane can be continuously decomposed into pseudo-arcs. Using their result it is simple to create a non-trivial continuous decomposition of the disk with some degenerate members; for example, see [3]. In order to force all members of the decomposition to be non-degenerate we actually modify the details of the Lewis and Walsh construction to obtain a new construction.

Our strategy in describing our decomposition will be to define a sequence $\{P_n\}_{n=1}^{\infty}$ of partitions of the unit square $D = [0,1] \times [0,1]$ into cells with non-overlapping interiors so that the conditions of Proposition 3.1 of [2] are satisfied. For completeness this proposition is stated as Lemma 1 below. Before stating the lemma we introduce the following notation. If P is a collection of sets, then P^* denotes the union of members of P. If p is a set, then $\mathrm{st}^1(p, P) = \{p' \in P : p' \cap p \neq \emptyset\}$ and inductively $\mathrm{st}^i(p, P) = \mathrm{st}^1(\mathrm{st}^{i-1}(p, P)^*, P)$. We abbreviate $\mathrm{st}^1(p, P)$ by $\mathrm{st}(p, P)$.

Lemma 1. **[Lewis and Walsh]** *Let X be a compactum and $\{P_n\}_{n=1}^{\infty}$ be a sequence satisfying:*

(1) *For each n, P_n is a finite collection of non-empty closed subsets of X with $P_n^* = X$, with the elements of P_n having pairwise disjoint interiors, and with $\mathrm{Cl}(\mathrm{Int}(p_n)) = p_n$ for each $p_n \in P_n$.*
(2) *For each $p_{n-1} \in P_{n-1}$, $\mathrm{st}^4(p_{n-1}, P_n)^* \subset \mathrm{st}(p_{n-1}, P_{n-1})^*$.*
(3) *There is a positive number L such that for each pair $p_n, p'_n \in P_n$ with $p_n \cap p'_n \neq \emptyset$, $p_n \subset \mathrm{N}_{L/2^n}(p'_n)$.*
(4) *There is a positive number K such that for each $p_n \in P_n$, there is a $p_{n-1} \in P_{n-1}$ with $p_n \cap p_{n-1} \neq \emptyset$ and $p_{n-1} \subset \mathrm{N}_{K/2^n}(p_n)$.*

Let G be defined by $g \in G$ if $g = \cap_{n=1}^{\infty} \mathrm{st}(p_n, P_n)^*$ where $\cap_{n=1}^{\infty} p_n \neq \emptyset$; then G is a continuous decomposition of X.

The sequence, $\{P_n\}_{n=1}^{\infty}$, will be defined inductively. Assuming we have already constructed $\{P_i\}_{i=1}^{n-1}$, we start stage n of the induction given $\widehat{R}_n$, a division of D into either congruent vertical or congruent horizontal strips.

Definition 2. *A vertical (respectively horizontal) division of D is the collection $R = \{[(i-1)a, ia] \times [0,1] : i \in \{1, ..., 1/a\}\}$ (respectively $\{[0,1] \times [(i-1)a, ia] : i \in \{1, ..., 1/a\}\}$) where $(1/a) \in \{1, 2, 3, ...\}$. The mesh of R denoted by* $\operatorname{mesh}(R)$ *is a. Each member of R is called a vertical strip (respectively horizontal strip).*

Given the vertical (respectively horizontal) division $\widehat{R}$ of D, a division R of D is a refinement of $\widehat{R}$ if for every strip $X \in R$ there is a strip $Y \in \widehat{R}$ so that $X \subset Y$. See Figure 1.

We first give an overview of our construction. There are four positive rational numbers a_n, a'_n, b_n, and c_n and a positive integer k_n which constrain the construction at stage n. Given a vertical (resp. horizontal) division, $\widehat{R}_n$, we construct a refinement, R_n, with mesh a_n. We then partition D into a collection of cells Q_n with non-overlapping interiors by partitioning each strip into cells. To facilitate our discussion we give an informal description of a typical cell $q_n \in Q_n$. See Figure 2. Note that a_n defines the width of q_n. The cell has a height of at least c_n but less than $b_n + c_n$. The thickness; i.e., vertical transverse thickness, of the cell is limited by b_n. The integer k_n defines the number of identical pieces each of width a_n/k_n which make up q_n. By top boundary of q_n we mean the set $\{(x,y) \in q_n : \forall (x,y') \in q_n \ y' \leq y\}$. By left boundary of q_n we mean the left most vertical line segment in q_n. Bottom and right boundaries are similarly defined. Once we have the collection Q_n defined, a homeomorphism $h_n : D \to D$ is defined so that $\{h_n^{-1}(q_n) : q_n \in Q_n\}$ is a collection of identical rectangles with non-over lapping interiors whose union is D. See Figure 3. The set P_n is defined to be $\{h_1 \circ \cdots \circ h_{n-1}(q_n) : q_n \in Q_n\}$. To continue on to stage $n+1$ we use $\{h_n^{-1}(q_n) : q_n \in Q_n\}$ to define $\widehat{R}_{n+1}$, a horizontal (resp. vertical) division of D.

Thus our construction is very similar to the Lewis and Walsh construction. It is different in the exact way in which the cells

Q_n are formed and in the way the function h_n is defined. For example, in Lewis and Walsh when vertical strips are being partitioned the top boundary of a cell is simply the vertical displacement of the bottom boundary by the constant b_n. In our construction, cells within a strip of a vertical division, R_n, do not have congruent top and bottom boundaries. (It is true, however, that because of the restrictions on the cell thickness the top boundary of a cell will lie completely below the displacement of the bottom boundary by b_n.) In addition special attention must be paid in defining h_n within the strips of $\widehat{R}_{n+1}$ that are along the boundary of D. We use the positive real number $a'_n < \mathrm{mesh}(\widehat{R}_{n+1})$ to control h_n. These changes will complicate somewhat the proof that the construction results in a continuous decomposition of D. Notice that like the Lewis and Walsh construction our construction alternates between working with vertical and horizontal divisions. Arbitrarily, we let $\widehat{R}_n$ be a vertical division when n is odd and $\widehat{R}_n$ be a horizontal division when n is even.

2. The Construction

We now describe in more detail our construction at stage n where $n > 1$ is odd. (When n is even, the construction is similar and can be visualized by rotating all figures by ninety degrees.) To start, we create R_n, the refinement of $\widehat{R}_n$, by choosing a_n so that it divides $\mathrm{mesh}(\widehat{R}_n)$.

2.1. Creation of the Cells. To create Q_n, we will define two disjoint polygonal arcs M_n^0 and M_n^1; the first running along the bottom of D, the second running along the top of D. See Figure 4. These polygonal arcs will determine the general shape of the cells of Q_n. We use these two polygonal arcs to help define a sequence of polygonal arcs $\{L_n^j\}_{j=0}^{m_n}$ so that $L_n^0 = M_n^0$ and $L_n^{m_n} = M_n^1$. We use the notation $L_n^j(x)$ to denote y when $(x, y) \in L_n^j$. In order to control the shape of the elements of Q_n we will define $\{L_n^j\}_{j=0}^{m_n}$ so that the following conditions are satisfied:

(i) $b_n \geq \max\{|L_n^j(x) - L_n^{j-1}(x)| : x \in [0,1]\}$ for every $j \in \{1, ..., m_n\}$. Thus b_n will control the thickness of cells.

(ii) $c_n \leq \max\{|L_n^j(x) - L_n^j(x')| : x, x' \in [0,1]\}$ for every $j \in \{0, 1, ..., m_n\}$. Thus c_n will control the height of cells.

(iii) For each $j \in \{1, ..., m_n\}$ and for every $x \in [0, 1 - a_n]$ $L_n^j(x) = L_n^j(x + a_n)$, and the sub-polygonal arc $\{L_n^j : x \in [0, a_n]\}$ can be divided into k_n congruent pieces so that given any two pieces, one is the horizontal displacement of the other. This condition will insure that each cell is made up of k_n identical pieces.

To define M_n^0 and M_n^1 we define a set of points:

$$A_n = \bigcup_{i=1}^{k_n/a_n} \left\{ \left(\frac{a_n}{k_n}(i-1), 0 \right), \left(\frac{a_n}{k_n} \frac{(2i-1)}{2}, c_n \right) \right\} \cup \{(1,0)\}.$$

Define ℓ_n to be the polygonal arc obtained by joining the points of A_n in ascending order by abscissa. Let $M_n^0 = \ell_n + b_n$ and $M_n^1 = 1 - (\ell_n + b_n)$. Now we can define a sequence of polygonal arcs $\{L_n^j\}_{j=0}^{m_n}$ satisfying (i)-(iii) if $\frac{5}{2}c_n + 2b_n < 1$. The fact that we can do this follows from Lemma 3.

Lemma 3. *Let $c > b > 0$. Let M^0 be the polygonal arc connecting the points $A = \{(0,0), (1/2, c), (1,0)\}$ in order and let M^1 be the polygonal arc connecting the points $\{(0,r), (1/2, r - c), (1,r)\}$ in order. If $(5/2)c < r$ and $\widehat{m} \in \{1, 2, 3, ...\}$, then there exists an integer $m > \widehat{m}$ such that $1/(m+2) < b$ and a sequence of polygonal arcs $\{L^j\}_{j=0}^m$ with $L^0 = M^0$ and $L^m = M^1$ such that*

(i) $b \geq \max\{|L^j(x) - L^{j-1}(x)| : x \in [0,1]\}$ *for every* $j \in \{1, ..., m\}$;

(ii) $c \leq \max\{|L^j(x) - L^j(x')| : x, x' \in [0,1]\}$ *for every* $j \in \{0, 1, ..., m\}$.

Proof: See Figure 5. Let $N > \max\{4, \hat{m}/6\}$ be a natural number so that

$$b' = \frac{c}{2N} < \frac{b}{2} \text{ and so that } \epsilon = \frac{2r - 5c}{12N} < \frac{b}{2}.$$

Thus $b' + \epsilon < b$. Let $m = 6N$.

Let $L^0 = M^0$.

For $j = 1, \ldots, N$ let A^j be the set of points

$$\Big\{\Big(0, j\epsilon + L^0(0)\Big), \Big(0.25, j(b' + \epsilon) + L^0(0.25)\Big), \Big(0.5, j\epsilon + L^0(0.5)\Big), \Big(0.75, j(b' + \epsilon) + L^0(0.75)\Big), \Big(1, j\epsilon + L^0(1)\Big)\Big\}.$$

and define L^j to be the polygonal arc connecting the points of A^j in order.

For $j = 1, \ldots, 2N$ let A^{j+N} be the set of points

$$\Big\{\Big(0, j(b' + \epsilon) + L^N(0)\Big), \Big(0.25, j(b' + \epsilon) + L^N(0.25)\Big), \Big(0.5, j\epsilon + L^N(0.5)\Big), \Big(0.75, j(b'+\epsilon)+L^N(0.75)\Big), bigl(1, j(b'+\epsilon)+L^N(1)\Big)\Big\}.$$

and define L^{j+N} to be the polygonal arc connecting the points of A^{j+N} in order.

For $j = 1, \ldots, 2N$ let A^{j+3N} be the set of points

$$\Big\{\Big(0, j(b' + \epsilon) + L^{3N}(0)\Big), \Big(0.25, j\epsilon + L^{3N}(0.25)\Big), \Big(0.5, j\epsilon + L^{3N}(0.5)\Big), \Big(0.75, j\epsilon + L^{3N}(0.75)\Big), \Big(1, j(b' + \epsilon) + L^{3N}(1)\Big)\Big\}.$$

and define L^{j+3N} to be the polygonal arc connecting the points of A^{j+3N} in order.

For $j = 1, \ldots, N$ let A^{j+5N} be the set of points

$$\Big\{\Big(0, j(b' + \epsilon) + L^{5N}(0)\Big), \Big(0.25, j\epsilon + L^{5N}(0.25)\Big), \Big(0.5, j(b' + \epsilon) + L^{5N}(0.5)\Big), \Big(0.75, j\epsilon + L^{5N}(0.75)\Big), \Big(1, j(b' + \epsilon) + L^{5N}(1)\Big)\Big\}$$

and define L^{j+5N} to be the polygonal arc connecting the points of A^{j+5N} in order.

Now $L^m = M^1$ since

$$\begin{aligned}
L^m(0) &= N\epsilon + 5N(b' + \epsilon) = r, \\
L^m(0.25) &= c/2 + 3N(b' + \epsilon) + 3N\epsilon = r - c/2, \\
L^m(0.5) &= c + 5N\epsilon + N(b' + \epsilon) = r - c, \\
L^m(0.75) &= c/2 + 3N(b' + \epsilon) + 3N\epsilon = r - c/2, \\
L^m(1) &= N\epsilon + 5N(b' + \epsilon) = r.
\end{aligned}$$

The fact that $\{L^j\}_{j=0}^m$ meets the constraints (i) and (ii) above follows directly from the construction.

We let $d = 1/(m+2)$. Note that

$$d < \frac{N\epsilon + 5N(b' + \epsilon) + 2b_n}{6N+2} < \epsilon + \frac{5}{6}b' + \frac{1}{3N}b_n < b_n$$

when $N > 4$. □

Letting $r = 1 - 2b_n$ we see that we can apply the above lemma when $b = b_n$ and $c = c_n$ are small enough in order to obtain $m_n = m$ and $\{L_n^j\}_{j=0}^{m_n}$. Let $L_n^{-1}(x) = 0$ and $L_n^{m_n+1}(x) = 1$ for all $x \in [0,1]$. See Figure 6. For $i \in \{1, ..., 1/a_n\}$ and $j \in \{0, ..., (m_n + 1)\}$ define the cells of Q_n as follows:

$$q_{i,j} = \{(x,y) \in D : (i-1)a_n \leq x \leq ia_n \text{ and } L_n^{j-1}(x) \leq y \leq L_n^j(x)\}.$$

Thus

$$Q_n = \{q_{i,j} : i \in \{1, ..., \frac{1}{a_n}\} \text{ and } j \in \{0, ..., (m_n + 1)\}\}.$$

We let $d_n = 1/(m_n + 2)$ and so $d_n < b_n$. Also note that the values of d_n and m_n are independent of k_n and a_n.

2.2. Definition of h_n. We now define a homeomorphism $h_n : D \to D$ so that h_n^{-1} straightens the polygonal arcs $\{L^j\}_{j=-1}^{m_n+1}$ and so that the distance between the straightened polygonal arcs is d_n. We will define h_n so it will map vertical lines onto themselves in a piecewise linear fashion. For the cells which lie along either the top or bottom boundaries of D a great deal of stretching can occur. For example, a vertical segment lying in the center of a symmetrical piece of a cell has length $c_n + b_n$ whereas its pre-image under h_n has length d_n. We define

h_n carefully so we can control exactly where this stretching occurs. We force this stretching to occur between $y = a'_n/16$ and $y = a'_n/8$. Let e_n denote $a'_n/16$ and assume that $a'_n < b_n$. First we define $f_n : D \to \mathbb{R}$.

$$f_n(x,y) = \begin{cases} y & \text{if } 0 \leq y < e_n; \\ 2e_n + \ell_n(x)\left(\frac{y-e_n}{e_n}\right) & \text{if } e_n \leq y < 2e_n; \\ 2e_n + \ell_n(x) + (b_n - 2e_n)\left(\frac{y-2e_n}{d_n-2e_n}\right) & \text{if } 2e_n \leq y < d_n; \\ L_n^{j-1}(x) + (L_n^j(x) - L_n^{j-1}(x))\left(\frac{y-jd_n}{d_n}\right) & \text{if } \exists j \in \{1, \dots, m_n\} jd_n < y < (j+1)d_n; \\ 1 - 2e_n - \ell_n(x) - (b_n - 2e_n)\left(\frac{1-2e_n-y}{d_n-2e_n}\right) & \text{if } (m_n+1)d_n \leq y < 1-2e_n; \\ 1 - 2e_n - \ell_n(x)\left(\frac{1-e_n-y}{e_n}\right) & \text{if } 1-2e_n \leq y < 1-e_n; \\ y & \text{if } 1-e_n \leq y \leq 1. \end{cases}$$

Define $h_n(x,y) = (x, f_n(x,y))$. See Figure 7. Thus h_n^{-1} transforms cells in Q_n into rectangles with non-overlapping interiors each with dimension $a_n \times d_n$; that is,

$$h_n^{-1}(Q_n) = \{[(i-1)a_n, ia_n] \times [jd_n, (j+1)d_n] : i \in \{1, \dots, \tfrac{1}{a_n}\} \text{ and } j \in \{0, \dots, (m_n+1)\}\}.$$

2.3. Creation of P_n and preparation of stage $(n+1)$. We set

$$P_n = H_n(Q_n) = \{h_1 \circ \cdots \circ h_{n-1}(q) : q \in Q_n\}.$$

To continue the construction we define

$$\widehat{R}_{n+1} = \{[0,1] \times [jd_n, (j+1)d_n] : j \in \{0, \dots, (m_n+1)\}\}.$$

Thus $\widehat{R}_{n+1}$ is a horizontal division of D and the construction can continue to stage $(n+1)$. Note that $\text{mesh}(\widehat{R}_{n+1}) = d_n = 1/(m_n+2)$. We define H_{n+1} to be $H_n \circ h_n = h_1 \circ \cdots \circ h_n$. Figure 8 shows the relationship among the various functions and collections defined. Note that $H_{n+1}^{-1}(P_n)$ is the collection of rectangles

$$\{[(i-1)a_n, ia_n] \times [jd_n, (j+1)d_n] : i \in \{1, ..., \tfrac{1}{a_n}\} \text{ and } j \in \{0, ..., (m_n+1)\}\}.$$

3. Application of Construction

We will show that we can apply the above construction to create a sequence $\{P_n\}_{n=1}^{\infty}$ which satisfies Conditions (1)-(4) of Lemma 1. From the above description it can be seen that the exact details of the construction at stage n are controlled by a_n, a'_n, b_n, c_n, and k_n. We now show that at each stage n we can choose a_n, a'_n, b_n, c_n, and k_n so that $\{P_n\}_{i=1}^{n}$ satisfies the Conditions (1)-(4) of Lemma 1 and so $\{P_n\}_{i=1}^{\infty}$ induces the desired continuous decomposition of D.

At stage 1, let $a_1 = 1/128$ so

$$R_1 = \{[(i-1)\frac{1}{128}, i\frac{1}{128}] \times [0,1] : i \in \{1, ..., 128\}\}.$$

Let $c_1 = 1/4$ and $b_1 = 1/2048$. Since $(5/2)c_1 + 2b_1 = 5/8 + 1/1024 < 1$ we can use Lemma 3 to compute m_1 and d_1. Let $a'_1 = d_1/4$. Let $k_1 \geq 8$ so that $4|k_1$ and $k_1 > a_1/a'_1$. Construct Q_1 as above. Let $H_1 = \text{Id}_D$. Thus $P_1 = H_1(Q_1) = Q_1$. We define h_1 as above and $H_2 = h_1$. Finally set $\widehat{R}_2 = \{[0,1] \times [jd_1, (j+1)d_1] : j \in \{0, ..., (m_1+1)\}\}$. At the end of stage 1, Condition (1) holds immediately. Conditions (2) and (4) hold vacuously. For Condition (3) let $L = 1/64$. The fact that we have $a_1 + b_1 < 1/64$ guarantees that Condition (3) is also satisfied.

Now assume we are at the beginning of stage n of our construction having just created $\widehat{R}_n$. Thus the collections R_i, Q_i

and P_i and the homeomorphism h_i have been defined as described previously for $i \in \{1, ..., (n-1)\}$, along with the function H_i for $i \in \{1, ..., n\}$. Let a_i, a'_i, b_i, c_i, and k_i be as described above for each stage i for $i \in \{1, ..., (n-1)\}$. As above we use following notation. Let $m_i + 2$ be the number of strips in the division $\widehat{R}_{i+1}$ and let $\text{mesh}(\widehat{R}_{i+1}) = d_i = 1/(m_i + 2)$ for $i \in \{1, ..., (n-1)\}$. In addition let $\delta_i > 0$ so that $|x - x'| < \delta_i \Rightarrow |H_i(x) - H_i(x')| < 1/2^{i+7}$ for $i \in \{1, ..., n\}$.

At stage $i = n$ we proceed as follows:

Let $a_i = a'_{i-1}$ and let $c_i = a_{i-1}/2$.
Let $b_i > 0$ so that
1a) $b_i < \delta_i/3$;
1b) $b_i < a_i/(2k_{i-1})$.
Define m_i and d_i using Lemma 2.
Let $a'_i = d_i/4$.
Let k_i be an integer so that
2a) $k_i \geq 8$;
2b) $4|k_i$;
2c) $k_i > a_i/a'_i$.

Now construct Q_i as described above in 2.1 and let $P_i = H_i(Q_i)$. Define h_i as described above in 2.2 and let $H_{i+1} = h_1 \circ \cdots \circ h_i$; and define $\widehat{R}_{i+1}$. We assume that choices of these parameters in previous stages $i \in \{1, ..., (n-1)\}$ were also chosen in the above manner. We will now show that $\{P_i\}_{i=1}^{n}$ satisfies Conditions (1)-(4) if $\{P_i\}_{i=1}^{n-1}$ does. We assume without loss of generality that n is odd.

3.1. Conditions (1) and (2). That Condition (1) is satisfied is immediate. That Condition (2) holds is shown as in [2].

3.2. Condition (3). We will show that for each pair $p_n, p'_n \in P_n$ with $p_n \cap p'_n \neq \emptyset$ then $p_n \subset \mathrm{N}_{L/2^n}(p'_n)$ where $L = 1/64$. Let $p_n, p'_n \in P_n$ with $p_n \cap p'_n \neq \emptyset$. Let $q_n = H_n^{-1}(p_n)$ and $q'_n = H_n^{-1}(p'_n)$. Thus both q_n and q'_n are in Q_n. We consider two cases:

Case 1: (Assume neither q_n nor q'_n intersects a vertical edge of D). See Figure 9 showing this case. Now consider $h_{n-1}(q_n)$

and $h_{n-1}(q'_n)$. A worst case possibility would be for q_n and q'_n to be in different strips of $\widehat{R}_n$. In this case

$$h_{n-1}(q_n) \subset \mathrm{N}_{2a_{n-1}/k_{n-1}+b_{n-1}}(h_{n-1}(q'_n)).$$

This follows from the fact that by (1b) we have $b_n < a_n/2k_{n-1} < a_{n-1}/2k_{n-1}$ and the fact that $c_n = a_{n-1}/2 > a_{n-1}/k_{n-1}$. Since $a_{n-1}/k_{n-1} < a'_{n-1} = a_n$ by (2c) we have that

$$h_{n-1}(q_n) \subset \mathrm{N}_{2a_n+b_{n-1}}(h_{n-1}(q'_n)).$$

But $a_n = a'_{n-1} = d_{n-1}/4 < b_{n-1}$; so by (1a) $2a_n + b_{n-1} < \delta_{n-1}$. Thus

$$h_{n-1}(q_n) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q'_n)).$$

But by choice of δ_{n-1} we have that

$$\begin{aligned} p_n &\subset H_{n-1} \circ h_{n-1}(q_n) \\ &\subset \mathrm{N}_{1/2^{n+6}}(H_{n-1} \circ h_{n-1}(q'_n)) \\ &= \mathrm{N}_{1/2^{n+6}}(p'_n) \\ &\subset \mathrm{N}_{L/2^n}(p'_n) \end{aligned}$$

where $L = 1/64$.

Case 2: (Assume q_n intersects a vertical edge of D). See Figure 10. It will suffice to show both $h_{n-1}(q_n) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q'_n))$ and $h_{n-1}(q'_n) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n))$. Notice that both q_n and q'_n must be in the same strip of $\widehat{R}_n$ since q_n is assumed to intersect a vertical edge of of R_n and $a_n = d_{n-1}/4$. Now $e_{n-1} = a'_{n-1}/16 = a_n/16$ since $a_n = a'_{n-1}$. Since q_n consists of more than two pieces by (2a) at least one piece, q_n^p, must be at a distance greater that $a_n/8$ from the left or right edge of D. Thus

$$h_{n-1}(q'_n) \subset \mathrm{N}_{2a_n+b_{n-1}}(h_{n-1}(q_n^p)) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n)).$$

Since $a_{n-1}/k_{n-1} < a_n$ we also have that

$$h_{n-1}(q_n) \subset \mathrm{N}_{2a_n+b_{n-1}}(h_{n-1}(q'_n)).$$

Thus

$$h_{n-1}(q_n) \subset \mathrm{N}_{3b_{n-1}}(h_{n-1}(q'_n)) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q'_n)).$$

3.3. Condition (4). Let $p_n \in P_n$. We will show that there is a $p_{n-1} \in P_{n-1}$ with $p_n \cap p_{n-1} \neq \emptyset$ so that $p_{n-1} \subset \mathrm{N}_{K/2^n}(p_n)$ where $K = 1/16$. To accomplish this we will prove two lemmas. In order to state the lemmas we must introduce the following notation and terminology. For any $q_n \in Q_n$ let q_n^p be any piece of q_n. Note that the width of q_n^p is a_n/k_n. We say that $q_n \in Q_n$ *crosses* q_{n-1}^p if q_n intersects both top and bottom boundaries of $h_{n-1}^{-1}(q_{n-1}^p)$. Note that $h_{n-1}^{-1}(q_{n-1}^p)$ is a rectangle of width d_{n-1} and height a_{n-1}/k_{n-1}.

We now state and prove the following lemmas. Note that we state them for the case when n is odd; however, entirely analogous lemmas exist in the case when n is even.

Lemma 4. *If $q_n \in Q_n$ and q_n^p is further than $a_n/8$ from a vertical edge of D, then we have that $h_{n-1}(q_n) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n^p))$.*

Proof: Let $q_n \in Q_n$. We will consider two cases:

Case 1: (q_n does not intersect a vertical edge of D). See Figure 11 which illustrates this case. As can be seen

$$h_{n-1}(q_n) \subset \mathrm{N}_{b_{n-1}}(h_{n-1}(q_n^p)) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n^p)).$$

Case 2: (q_n intersects a vertical edge of D). See Figure 12 which illustrates this case. Now q_n^p lies at a distance of more than $a_n/8$ from the left and right boundaries of D. Thus $h_{n-1}(q_n) \subset \mathrm{N}_{b_{n-1}+a_n}(h_{n-1}(q_n^p))$. Therefore

$$h_{n-1}(q_n) \subset \mathrm{N}_{b_{n-1}+a_n}(h_{n-1}(q_n^p)) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n^p))$$

and the lemma holds. □

Lemma 5. *If $q_n \in Q_n$ crosses q_{n-1}^p where $q_{n-1} \in Q_{n-1}$, then $q_{n-1}^p \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n))$.*

Proof: Let $q_n \in Q_n$, and $q_{n-1} \in Q_{n-1}$ with q_{n-1}^p so that q_n crosses q_{n-1}^p; that is, q_n contains points on both the top and bottom boundaries of the rectangle $h_{n-1}^{-1}(q_{n-1}^p)$. We will look at two cases:

Case 1: (q_n does not intersect a vertical edge of D). See Figure 13 which shows this case and that

$$q_{n-1}^p \subset \mathrm{N}_{b_{n-1}}(h_{n-1}(q_n)) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n)).$$

Case 2: (q_n intersects a vertical edge of D). See Figure 14 which illustrate this case. Let q_n^p be a piece of q_n which is at a distance of more than $a_n/8$ from the left or right edge of D. Recall that $a_n = a'_{n-1} < d_{n-1} < b_{n-1} < \delta_{n-1}/3$. Thus

$$q_{n-1}^p \subset \mathrm{N}_{a_n+b_{n-1}}(h_{n-1}(q_n^p)) \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n))$$

and the lemma holds. □

Now we can verify that Condition (4) holds. Let $q_n = H_n^{-1}(p_n)$. There are two cases:

Case 1: If q_n intersects the bottom or top edges of D then since $c_n = a_{n-1}/2 > 2(a_{n-1}/k_{n-1}) + a_{n-1}/8$ there is a q_{n-1}^p further than $a_{n-1}/8$ from a horizontal edge of D which q_n crosses.

Case 2: If q_n does not intersect the bottom or top edges of D then since height of q_n is greater than $c_n = a_{n-1}/2$ and since at least half of this height must lie in a strip of R_{n-1} and since the height of any rectangle $h_{n-1}^{-1}(q_{n-1}^p)$ is $a_{n-1}/k_{n-1} < a_{n-1}/2$, there must exist a q_{n-1}^p further than $a_{n-1}/8$ from a horizontal edge of D which q_n crosses.

In any case we will be able to apply Lemma 4. (Actually the n even version of Lemma 4). Let $p_{n-1} = H_{n-1}(q_{n-1})$. Note that $p_{n-1} \cap p_n \neq \emptyset$. Now by Lemma 5 we have that $q_{n-1}^p \subset \mathrm{N}_{\delta_{n-1}}(h_{n-1}(q_n))$ and

$$H_{n-1}(q_{n-1}^p) \subset \mathrm{N}_{1/2^{n+6}}(H_{n-1} \circ h_{n-1}(q_n)) = \mathrm{N}_{1/2^{n+6}}(p_n).$$

But by Lemma 4 we have that

$$h_{n-2}(q_{n-1}) \subset \mathrm{N}_{\delta_{n-2}}(h_{n-2}(q_{n-1}^p))).$$

But p_{n-1} is $H_{n-1}(q_{n-1})$ and

$$\begin{aligned} H_{n-1}(q_{n-1}) &= H_{n-2} \circ h_{n-2}(q_{n-1}) \\ &\subset \mathrm{N}_{1/2^{n+5}}(H_{n-2} \circ h_{n-2}(q_{n-1}^p)) \\ &= \mathrm{N}_{1/2^{n+5}}(H_{n-1}(q_{n-1}^p)). \end{aligned}$$

Therefore $p_{n-1} \subset \mathrm{N}_{1/2^{n+4}}(p_n)$ and $p_{n-1} \subset \mathrm{N}_{K/2^n}(p_n)$ where $K = 1/16$.

Theorem 6. *There is a continuous decomposition of a disk into non-degenerate non-separating continua so that the decomposition space is homeomorphic to a two dimensional disk.*

Proof: Consider the unit disk D and apply the construction as described above to create a decomposition G of D. By Lemma 1 the resulting decomposition is continuous. Since none of the members of P_n for any n separates D, none of the members of G separates D. We can extend G to obtain G' a decomposition of the plane by adding to G the points not in D. Now G' is upper semi-continuous, and so by Moore's theorem is homeomorphic to the plane. Thus G is planar. Also $\mathrm{diam}(g) \geq 1/16$ for all $g \in G$; i.e., each $g \in G$ is non-degenerate. This follows from the facts that $L = 1/64$, $K = 1/16$, and $\mathrm{diam}(p_1) > 1/4$. See Claim C in appendix of [2]. Finally note that each member of G intersects the boundary of D at no more than one point. Thus the natural projection $\pi : D \to G$ maps $\mathrm{Bd}(D)$ homeomorphically onto the $\mathrm{Bd}(G)$. Therefore G is bounded by a simple closed curve and is a disk.

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FIGURES

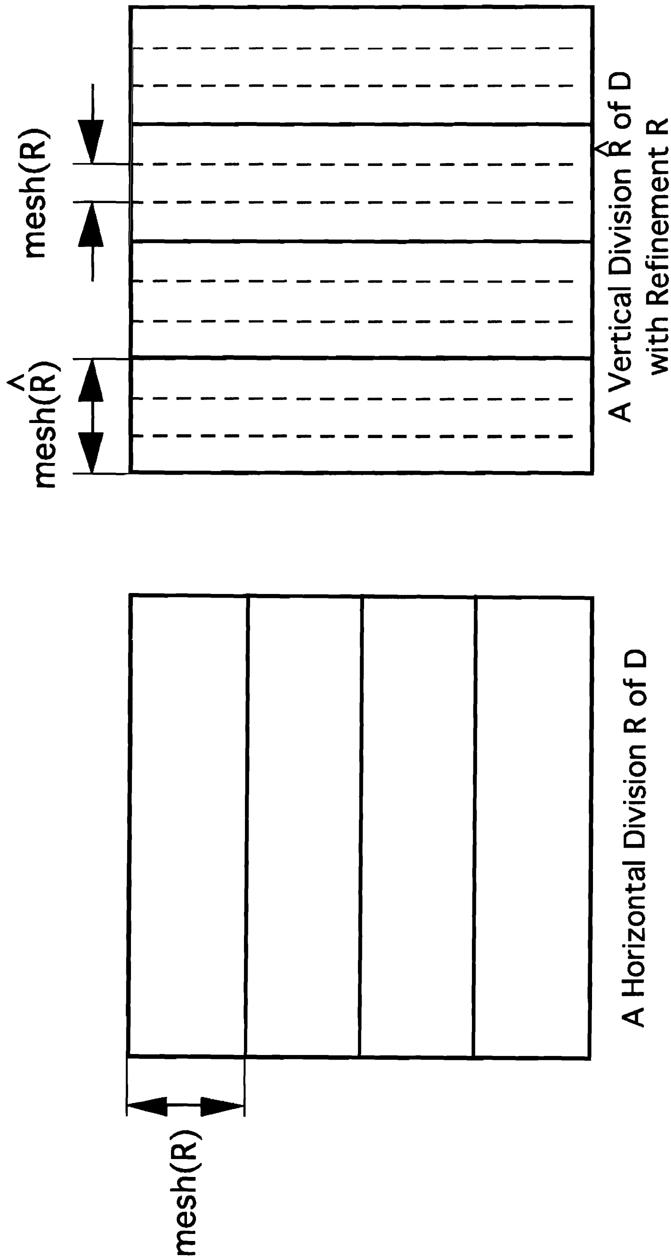

FIGURE 1. Horizontal and vertical divisions.

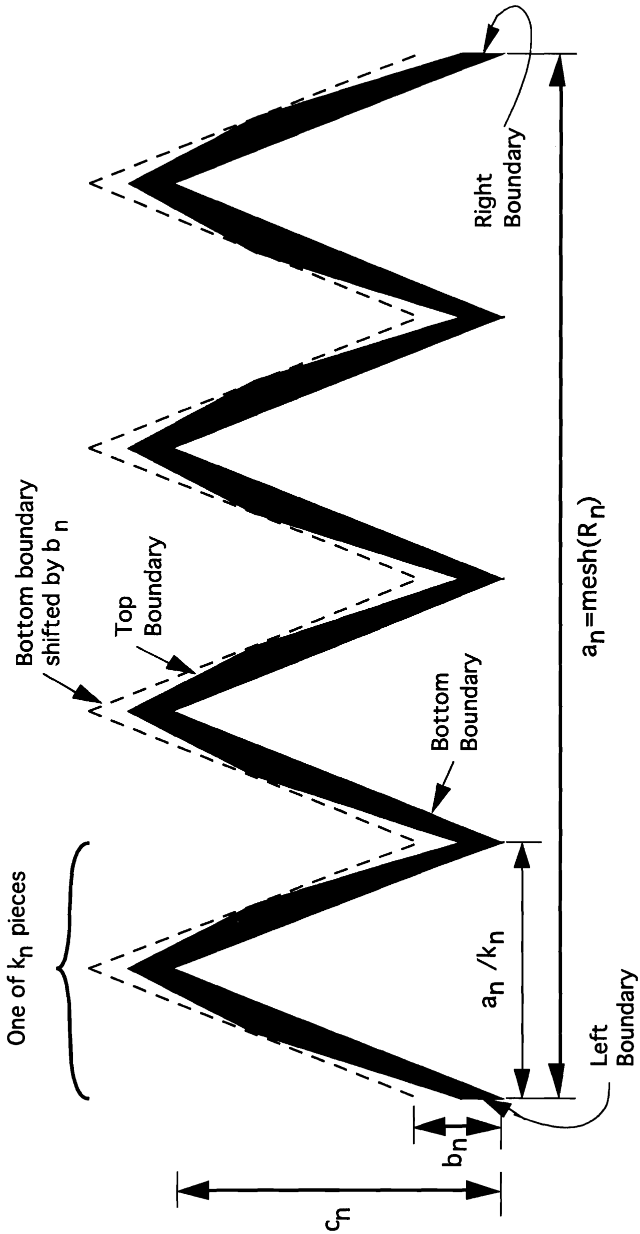

FIGURE 2. A typical cell in Q_n.

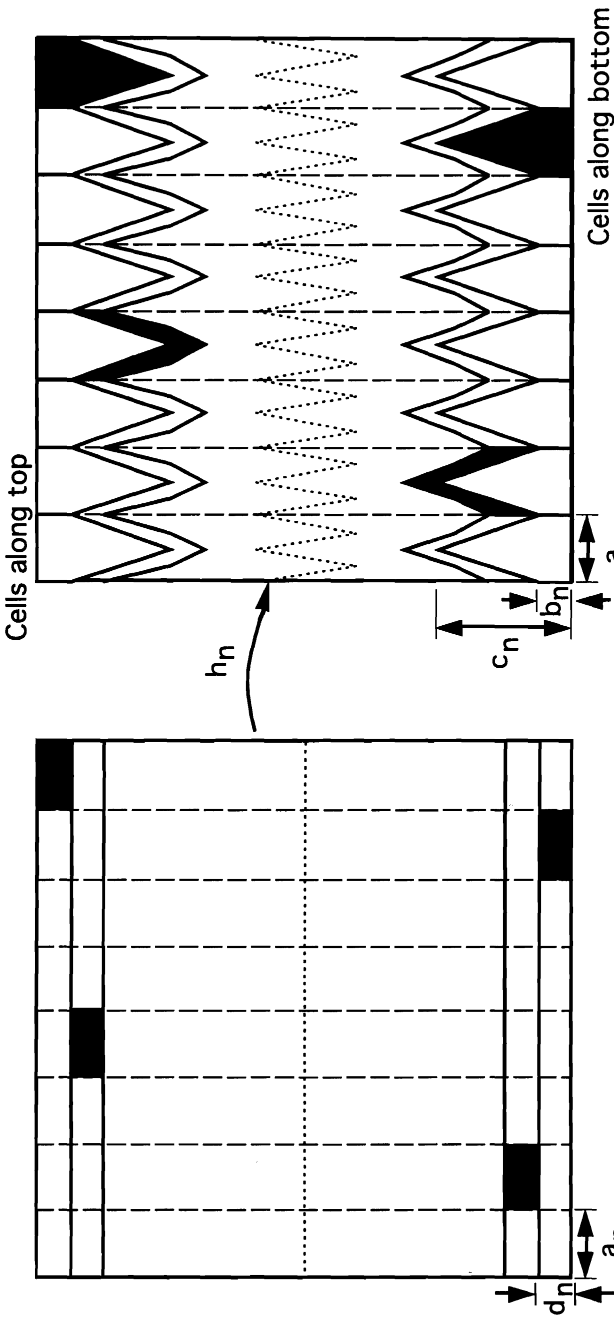

FIGURE 3. Cells of Q_n are pre-images of rectangles.

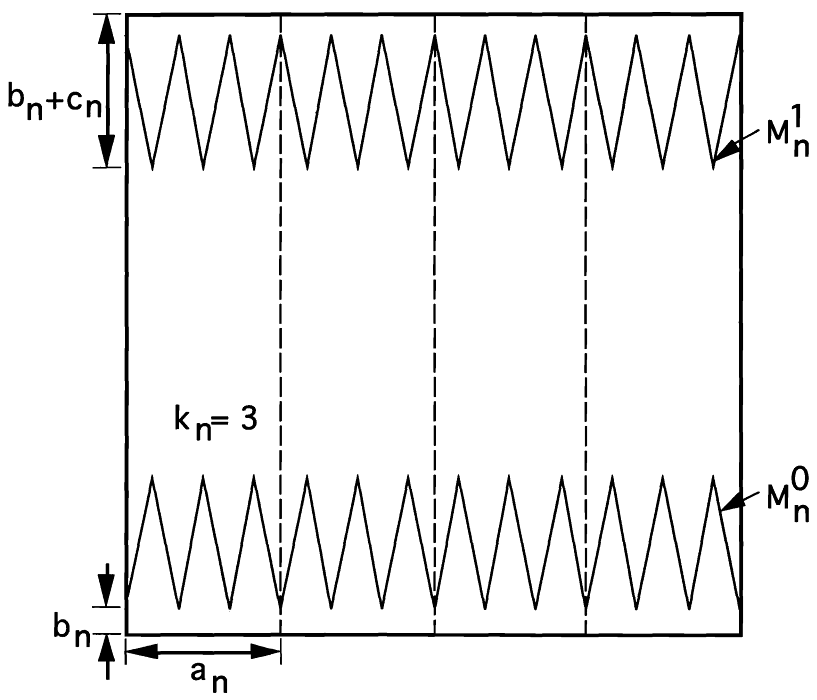

FIGURE 4. Defining M_n^0 and M_n^1 along top and bottom of disk.

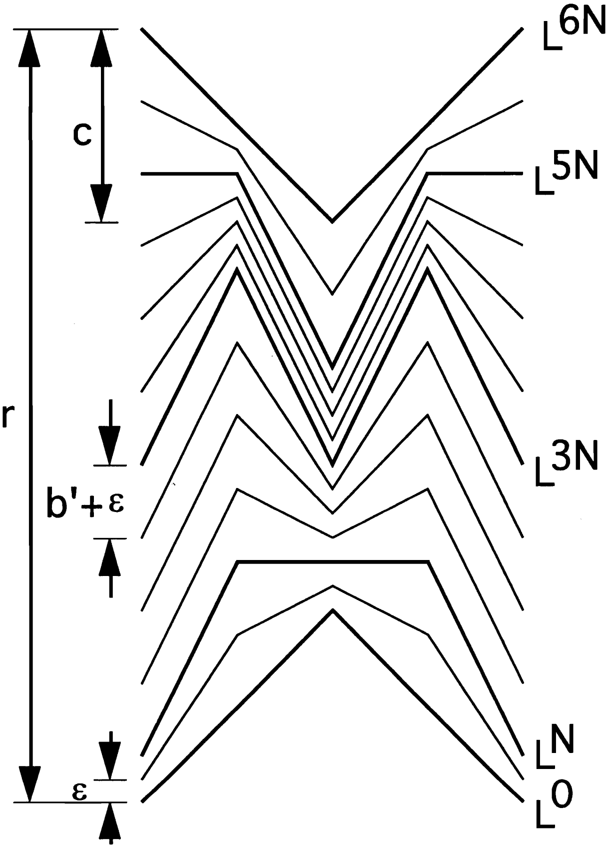

FIGURE 5. Sequence of polygonal arcs $\{L^j\}_{j=1}^{6N}$ in Lemma 3.

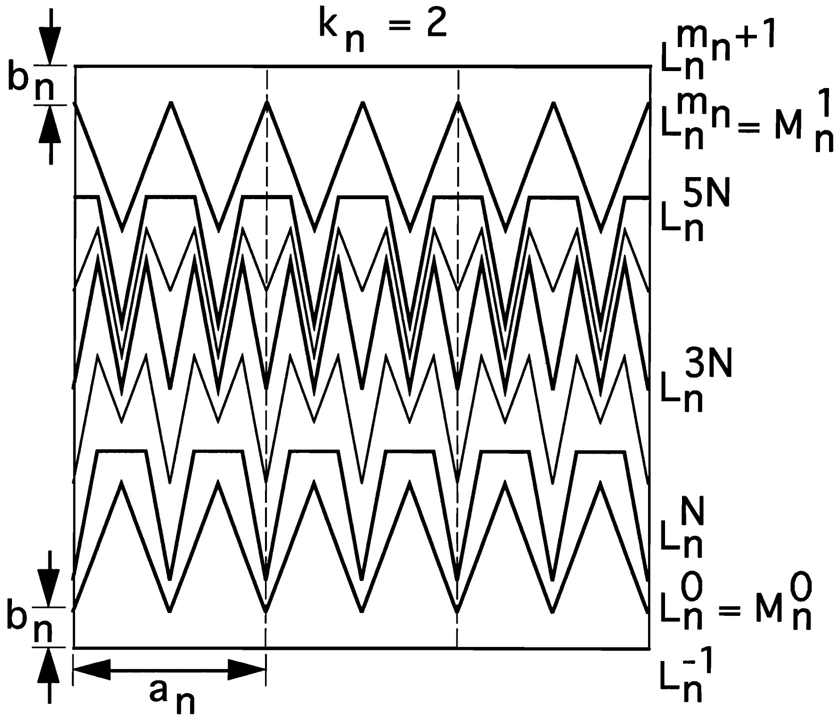

FIGURE 6. The Sequence of polygonal arcs $\{L_n^j\}_{j=-1}^{m_n+1}$.

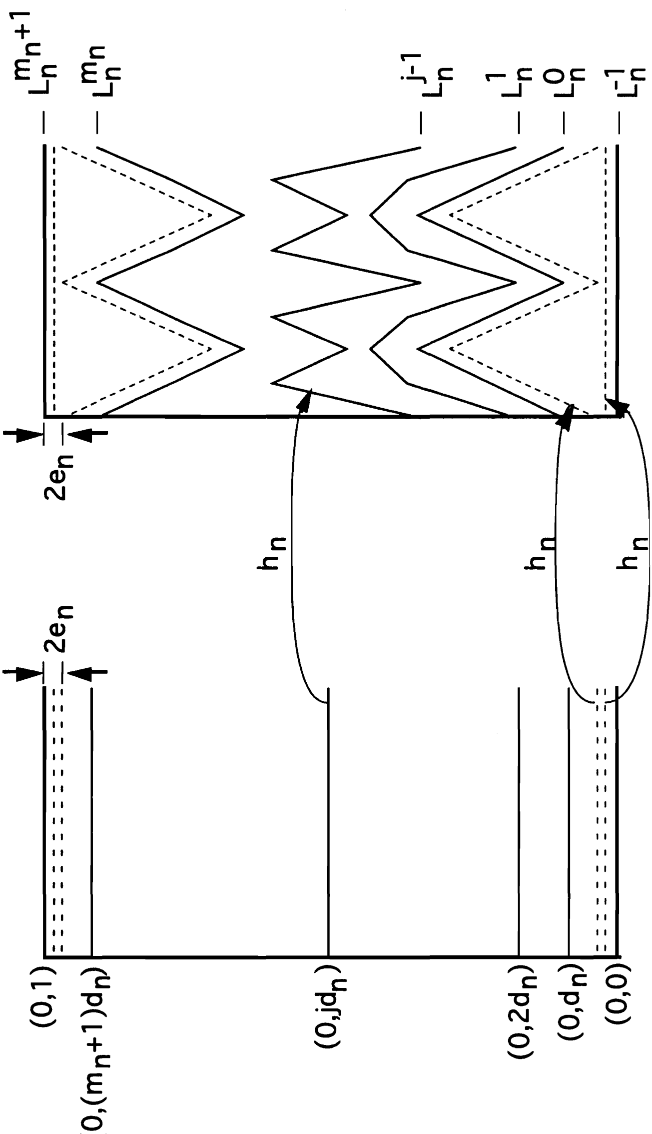

FIGURE 7. Excessive stretching by h_n is confined to between the dotted arcs.

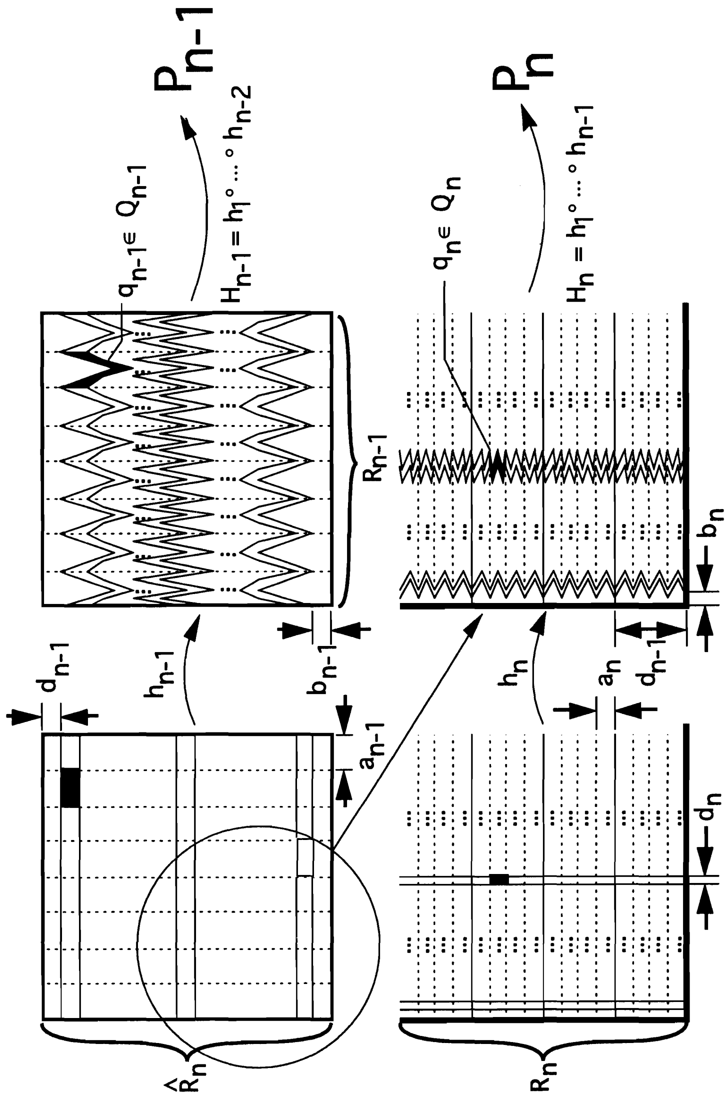

FIGURE 8. The relative relationships among the collections and functions.

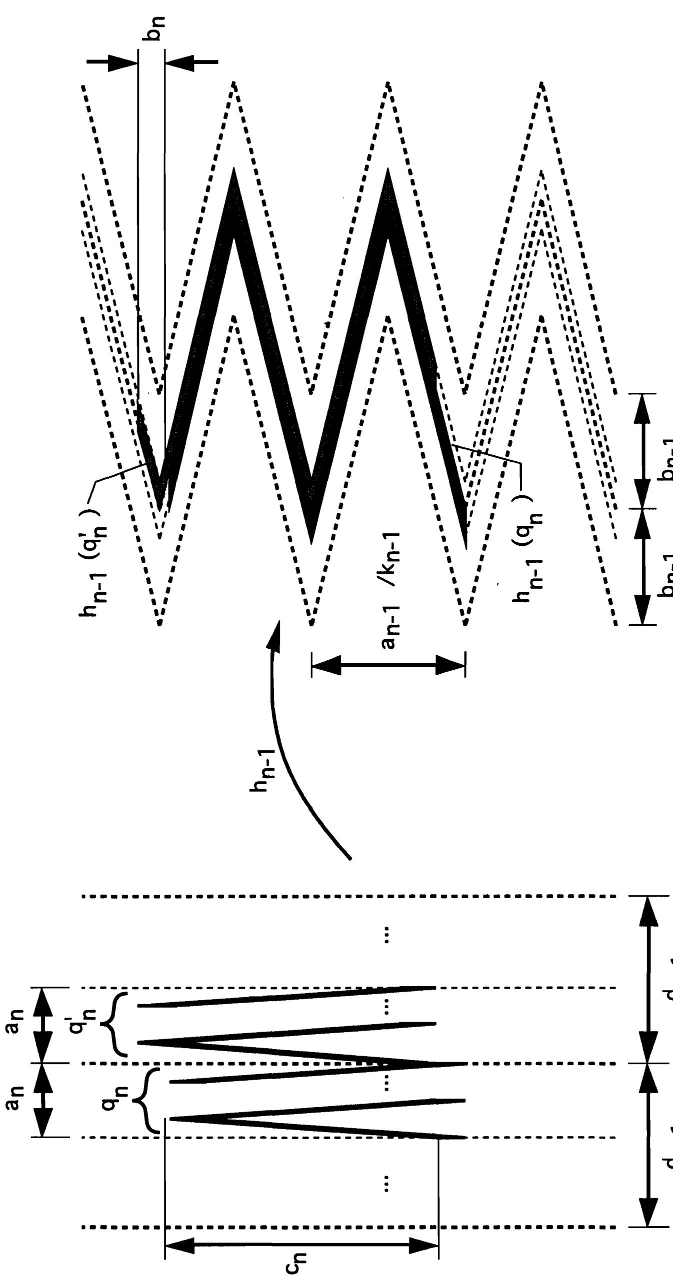

FIGURE 9. Case 1 for Condition 3.

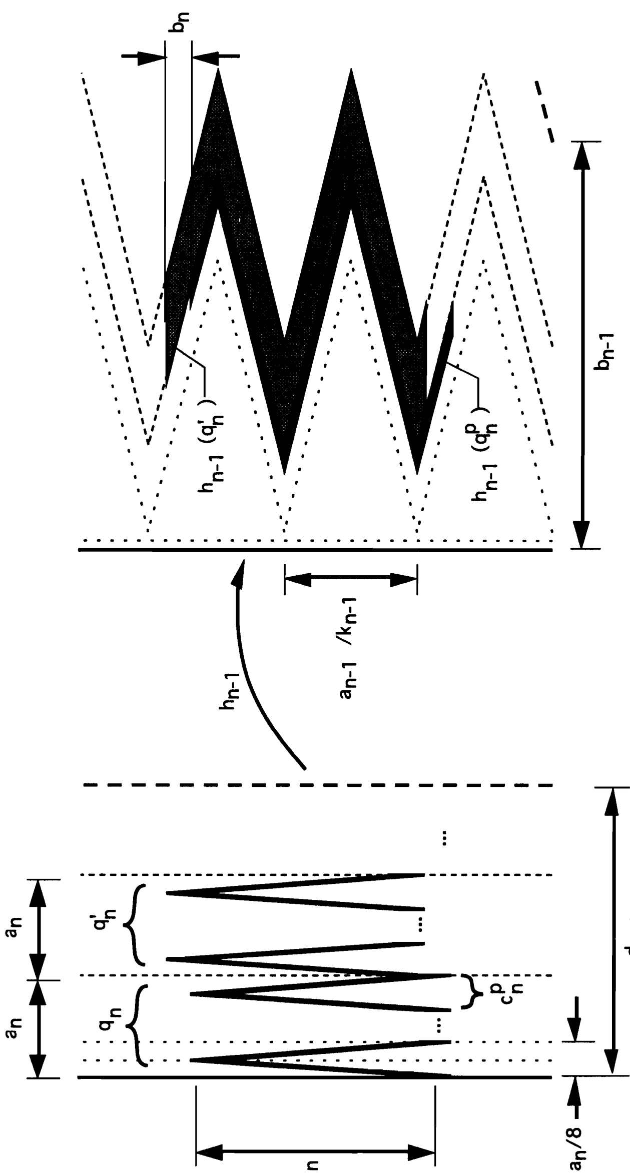

FIGURE 10. Case 2 for Condition 3.

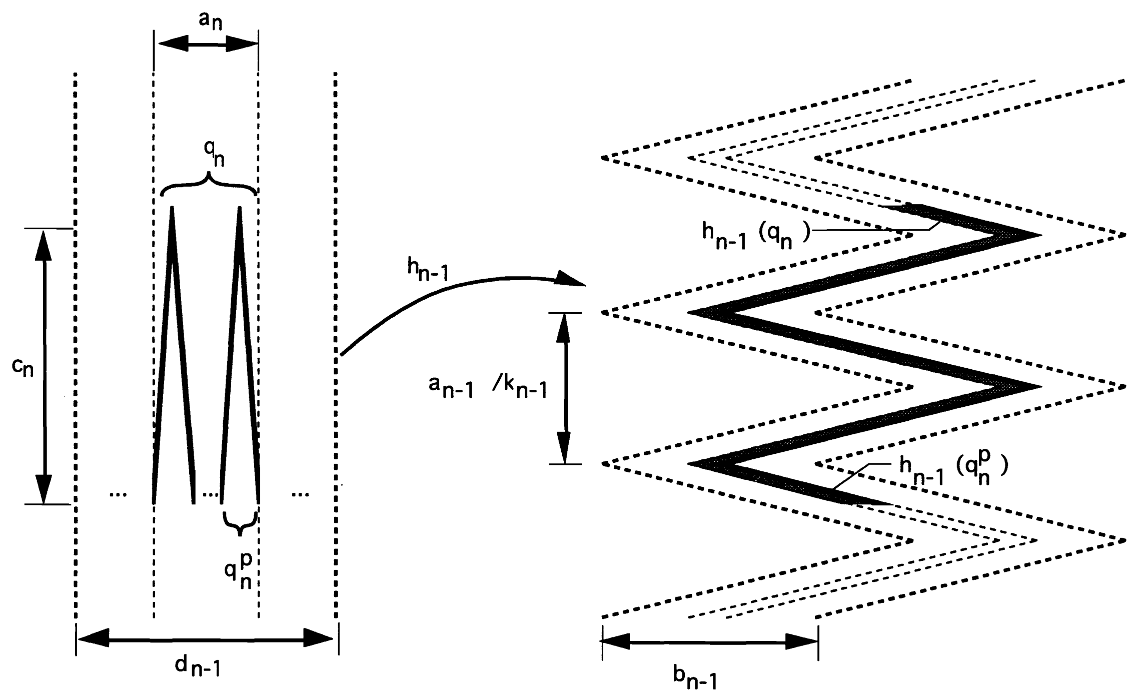

FIGURE 11. Case 1 for Lemma 4.

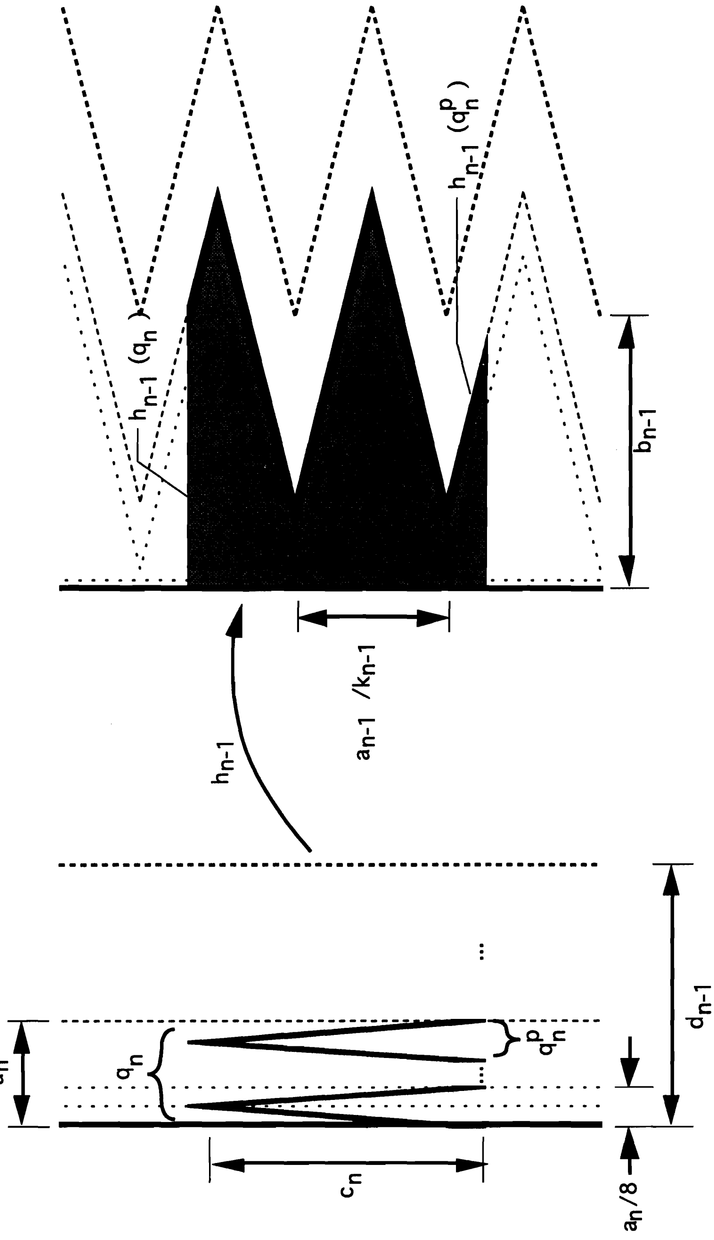

FIGURE 12. Case 2 for Lemma 4.

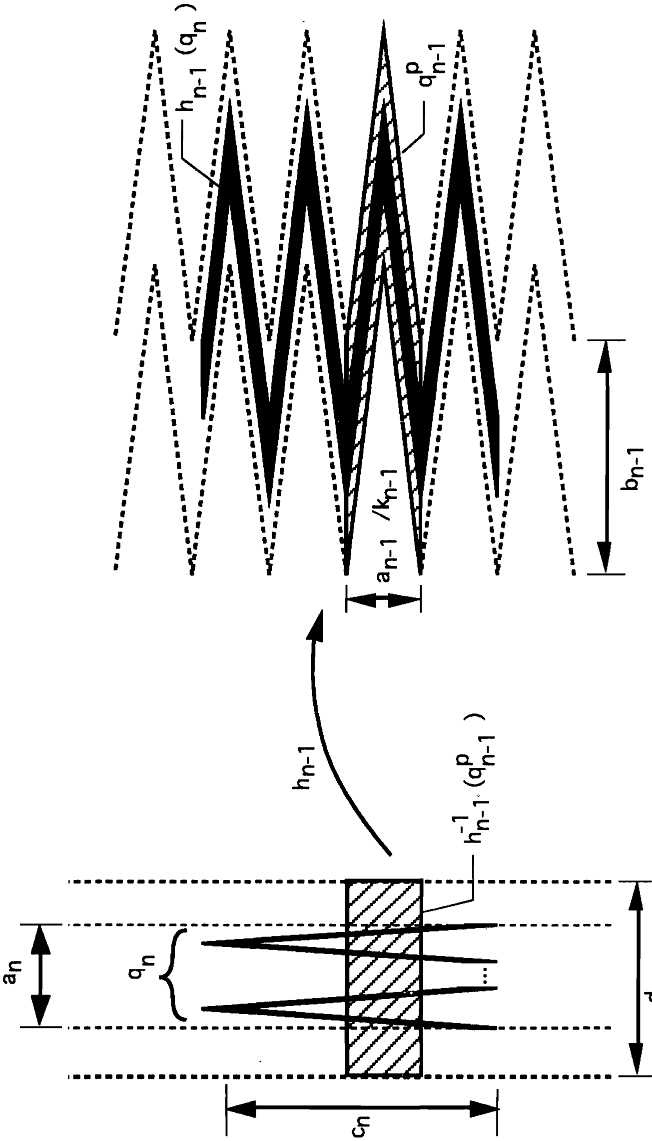

FIGURE 13. Case 1 for Lemma 5.

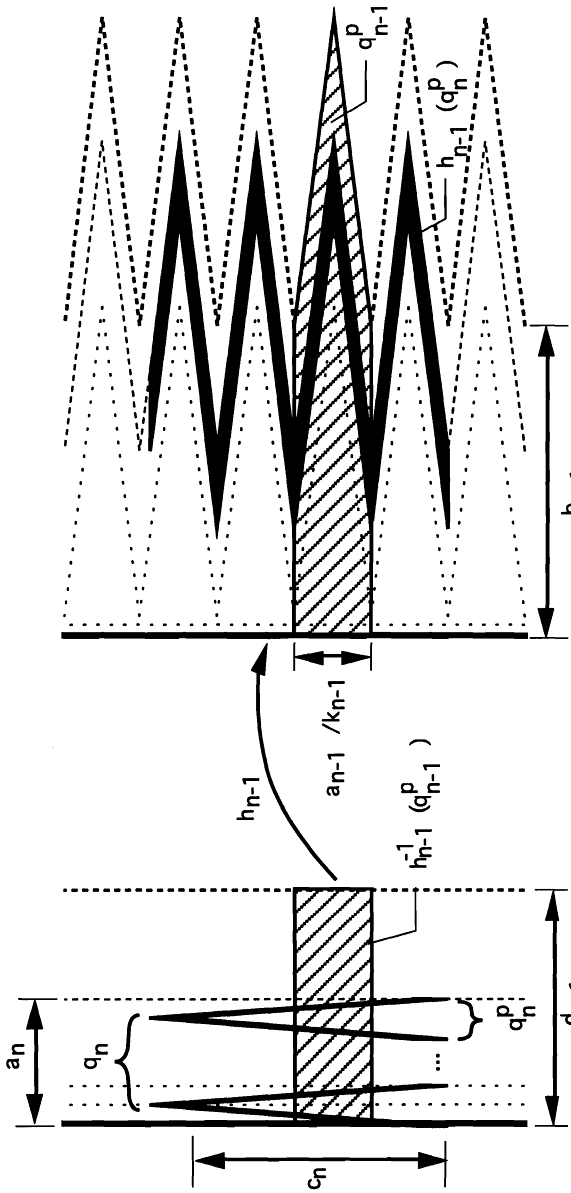

FIGURE 14. Case 2 for Lemma 5.

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