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UNCOVERING SEPARATION PROPERTIES IN THE EASTON MODELS

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ABSTRACT. We will be studying separation properties of $\aleph_1$-paralindelof spaces and countably metacompact spaces in the Easton and reverse Easton models. We prove, among other things, that first countable $\aleph_1$-paralindelof spaces are cwH in many Easton extensions including those violating GCH. This answers a question of F.D. Tall.

1. INTRODUCTION

A space is said to be $\aleph_1$-paralindelof if every open cover of size $\aleph_1$ has a locally countable refinement. There are a number of related results concerning separation properties in countably paracompact, normal, $\aleph_1$-paralindelof and countably metacompact spaces. There are three models where many of these spaces behave nicely: L, the Easton models, and models of PMEA. In each of these models normal spaces and countably paracompact spaces of character $\leq \aleph_1$ are collectionwise Hausdorff. Likewise under $V = L$, $\aleph_1$-paralindelof spaces of character $\leq \aleph_2$ are collectionwise Hausdorff. In fact, Fleissner showed that $\diamondsuit$ for stationary systems, which holds in both L and the GCH Easton model, suffices. Therefore, as pointed out in [T1], we also get that $\aleph_1$-paralindelof spaces of character $\leq \omega_2$ are collectionwise Hausdorff in the Easton model obtained by adding κ^+ many subsets of κ for every κ. Tall asked (in [T1] and [T2]) for a forcing proof of this fact and whether it also holds in the Easton models violating GCH. We show that the answer is yes in the class of first countable $\aleph_1$-paralindelof spaces. The stronger result, even the first step of an inductive proof, remains open.

Question 1.1 (Tall): *In the Easton model obtained by adding $> \kappa^+$ many subsets of each regular κ over a model of GCH, are $\aleph_1$-paralindelof spaces of character $\leq \omega_2$ (or $\leq 2^{\omega_1}$) collectionwise Hausdorff?*

Question 1.2 (Tall): *After adding $\lambda > \omega_2$ Cohen subsets of ω_1 over a model of GCH, are $\aleph_1$-paralindelof spaces of character $\leq \omega_2$ (or $\leq \lambda$) $< \lambda$-collectionwise Hausdorff?*

Related to the question of when normal, countably paracompact or $\aleph_1$-paralindelof spaces are collectionwise Hausdorff is the question of when closed discrete sets are G_δ in first countable, countably metacompact T_1 spaces. In [B] Burke proved that they are always G_δ assuming PMEA.

Theorem 1.3. (PMEA) *In a countably metacompact T_1 space X of character $< \mathfrak{c}$, if points are G_δ's then every closed discrete subset is a G_δ.*

This raised a natural question. Is the large cardinal inherent in the PMEA assumption necessary? In particular, does the result hold in either L or the Easton models? Nyikos was able to show that under V=L (actually $\diamondsuit$ for stationary systems) locally countable, first countable, countably metacompact spaces have closed discrete sets G_δ (see [N]). The full result, however, does not hold in L as a first countable counterexample has recently been constructed by the author assuming $\diamondsuit^*$ ([S]). Whether this example can be constructed in one of the Easton models remains open. By our previous remarks, Nyikos's result also holds in the Easton model obtained by adding κ^+ many subsets of each regular κ. In section 3 we prove that it also holds in the Easton models violating the GCH.

We let E denote the class of regular cardinals. By an Easton indexing function we mean any monotone increasing $\nu : E \rightarrow$

E. In section 2 we are primarily interested in models violating the GCH obtained by adding $\nu(\kappa)$ many Cohen subsets of each regular κ with via product forcing over a model of GCH (Easton class forcing). The main theorem of section 2 and its proof also holds in the model obtained by iteratively adding κ^+ many subsets of each regular κ (reverse Easton forcing). The techniques and proofs closely follow those in [T1] and we assume the reader is somewhat familiar with it. Our notation and terminology are standard and any unfamiliar notions, in particular the basics on both Easton and reverse Easton forcing, can be found in [Ku].

2. $\aleph_1$-PARALINDELOF SPACES

Theorem 2.1. *Let $\nu : E \to E$ be an Easton indexing function satisfying $\nu(\kappa)$ is never the successor of a singular and let P be the Easton forcing for adding $\nu(\kappa)$ many subsets of κ for each regular κ. If $\mathcal{M}$ is a model of GCH and G is P-generic over $\mathcal{M}$, then in $\mathcal{M}[G]$ first countable $\aleph_1$ paralindelof spaces are collectionwise Hausdorff.*

We will prove Theorem 2.1 by induction on the cardinality of the closed discrete sets. For singular strong limit cardinals λ the results of [Ke] (for $cof(\lambda) \geq \omega_1$) and [F] (for $cof(\lambda) = \omega$) imply that $< \lambda$-collectionwise Hausdorff entails λ-collectionwise Hausdorff in $\aleph_1$-paralindelof spaces.

Lemma 2.2. [F] *Assume λ is a singular strong limit cardinal of cofinality ω. If X is regular and $< \lambda$-collectionwise Hausdorff, then it is λ-collectionwise Hausdorff.*

Lemma 2.3. [K] *Assuming the SCH, suppose that λ is a singular strong limit cardinal of cofinality $\geq \omega_1$. Let X be a regular $\aleph_1$-paralindelof space of character $< \lambda$. If X is $< \lambda$-collectionwise Hausdorff, then it is λ-collectionwise Hausdorff.*

Note that in $M[G]$ if λ is a singular strong limit cardinal then $2^\lambda = \lambda^+$, hence SCH holds in $M[G]$. To take care of other λ (including singular λ that are no longer strong limits in $\mathcal{M}[G]$) we prove that for κ regular, $< \kappa$-collectionwise Hausdorff implies $< \nu(\kappa)$-collectionwise Hausdorff.

Lemma 2.4. *With $\mathcal{M}$ and P as above, if G is P-generic over M and in $\mathcal{M}[G]$, X is a first countable $\aleph_1$-paralindelof space, if λ and κ are regular cardinals such that $\nu(\kappa) > \lambda$, then X is $< \kappa$-collectionwise Hausdorff implies it is also λ-collectionwise Hausdorff.*

In order to apply the lemmas into an inductive proof of Theorem 2.1, some assumption on the indexing function is needed. If we proceed as in [T1], we must assume that strong limits are preserved (i.e. $\forall\lambda$ singular and $\forall\kappa < \lambda$, $\nu(\kappa) < \lambda$.) However, once we prove Lemma 2.4 it suffices to assume that for each regular κ, $\nu(\kappa)$ is never the successor of a singular (or even $\forall\lambda$ singular either $\forall\kappa < \lambda$ $\nu(\kappa) < \lambda$ or $\exists\kappa < \lambda$, $\nu(\kappa) > \lambda^+$).

Suppose in $\mathcal{M}[G]$ there is a first countable $\aleph_1$-paralindelof space X which is $< \lambda$-collectionwise Hausdorff but contains an unseparated closed discrete set of size λ. When λ is a singular strong limit cardinal we apply 2.2 and 2.3. If λ is singular but not a strong limit then by the assumption on ν, we may fix $\kappa < \lambda$ with $\nu(\kappa) > \lambda^+$. Then by Lemma 2.4, X is λ^+-collectionwise Hausdorff, hence it is λ-collectionwise Hausdorff. The case λ is regular follows again from Lemma 2.4.

Proof of Lemma 2.4: Let λ be the minimal regular cardinal for which the Lemma is false and fix $\kappa < \lambda$ satisfying $\nu(\kappa) > \lambda$. Let $X \in \mathcal{M}[G]$ be a first countable $\aleph_1$-paralindelof space containing an unseparated closed discrete subset $A = \{a_\alpha : \alpha < \lambda\}$. Note that if we modify the topology by isolating every point in $X \setminus A$, then the resulting space has all the pertinent properties of the original space. Furthermore, we may assume that X has cardinality λ since we need only λ of the isolated points to witness that A is unseparated. We factor P into three pieces: $P = P_{<\kappa} \times Fn(\nu(\kappa), \omega_1, \kappa) \times P_{>\kappa}$. For

$\kappa > \omega$ forcing with $P_{<\kappa} \times Fn(\nu(\kappa), \omega_1, \kappa)$ is equivalent to forcing with $P_{<\kappa} \times Fn(\nu(\kappa), 2, \kappa)$. The initial part of the forcing $\prod_{\xi<\kappa} Fn(\nu(\xi), \omega_1, \xi)$ is denoted by $P_{<\kappa}$. Similarly $P_{>\kappa}$ denotes the tail of the forcing for adding subsets of $\nu(\xi)$ for each $\xi > \kappa$. For any index set $I \in \mathcal{M}$ let Q_I denote $Fn(I, \omega_1, \kappa)$ relativized to $\mathcal{M}$. Since $Q_{\nu(\kappa)}$ has the κ^+-cc in $\mathcal{M}^{P_{<\kappa} \times P_{>\kappa}}$, X, the topology and A all appear at some initial stage $\mathcal{N} = \mathcal{M}^{P_{<\kappa} \times Q_\eta \times P_{>\kappa}}$ where $\eta < \nu(\kappa)$. We work now in the intermediate forcing extension $\mathcal{N}$. For each $a \in A$ let $\{V_n(a) : n < \omega\}$ be a decreasing local neighborhood base at a. Now, consider the next partition $\Gamma : \lambda \to \omega_1$ generic over $\mathcal{N}$. To be more precise, we need to note that Γ is the name for the generic subset of Q_I where $I = \{\eta + \alpha : \alpha < \lambda\}$. Since $\Vdash$ *X is* $\aleph_1$-paralindelof, in the extension there is a locally countable expansion of the partition Γ. By the κ^+-cc and the maximal principle, there is a $J \subseteq \nu(\kappa)$ of size λ and a sequence of $Q_{I \cup J}$-names $\{\tau_\alpha : \alpha < \omega_1\}$ such that

$$\Vdash_{Q_{I\cup J}} \{\tau_\alpha : \alpha < \omega_1\} \text{ is a locally countable expansion of the partition } \Gamma \textit{ of } A.$$

To simplify notation we assume wlog that $I \cup J = \lambda + \lambda$ and that τ_α is the open set containing $\Gamma^{-1}(\alpha)$. Therefore there is a name σ for a function from A to ω coding a neighborhood assignment such that

$$\Vdash_{Q_{\lambda+\lambda}} \forall a \in A,\ V_{\sigma(a)}(a) \subset \tau_{\Gamma(a)} \text{ and } \exists \gamma < \omega_1\ \forall \beta > \gamma,\ (V_{\sigma(a)}(a) \cap \tau_\beta = \emptyset).$$

For each $\alpha \in \lambda$ let $D_\alpha \subset Q_{\lambda+\lambda}$ be defined by

$$D_\alpha = \{p \in Q_{\lambda+\lambda} : \alpha \in dom(p),\ \exists n\ \exists \gamma,\ p \Vdash \sigma(a_\alpha) = n \text{ and } \forall \beta > \gamma\ (V_n(a_\alpha) \cap \tau_\beta = \emptyset)\}.$$

Notice that D_α is dense. For each $\alpha < \lambda$ and for each $p \in Q_{\lambda+\lambda}$ with $\alpha \in dom(p)$ and for each $\beta < \omega_1$, let $p^{\alpha,\beta} = p \setminus \{(\alpha, p(\alpha))\} \cup \{(\alpha, \beta)\}$. So $p^{\alpha,\beta}$ is obtained by changing p's value at α to β. For each $\alpha \in \lambda$ and $\beta \in \omega_1$, let $D_\alpha^\beta = \{p : p^{\alpha,\beta} \in D_\alpha\}$.

Claim: *For each α and β, D_α^β is dense open in $Q_{\lambda+\lambda}$.*

Proof: Suppose $p \in Q_{\lambda+\lambda}$. If $\alpha \in dom(p)$ extend $p^{\alpha,\beta}$ to $q \in D_\alpha$. Then $p' = q^{\alpha,p(\alpha)} < p$ and $p' \in D_\alpha^\beta$. If $\alpha \notin dom(p)$, then extend p to p' with $p'(\alpha) = \beta$. Then any $q < p'$ in D_α is also in D_α^β.

Let G_1 be a $Q_{\lambda+\lambda}$ generic filter over $\mathcal{N}$. We work now in the generic extension $\mathcal{N}[G_1]$. For each α and β let $p_\alpha \in G_1 \cap D_\alpha$ and let $p_{\alpha,\beta} \in G_1 \cap D_\alpha^\beta$. For each α, let $n_\alpha \in \omega$ be such that

(i) $p_\alpha \Vdash \sigma(a_\alpha) < n_\alpha$

(ii) $p_{\alpha,\beta}^{\alpha,\beta} \Vdash \sigma(a_\alpha) < n_\alpha$ for uncountably many β.

We claim that $\{V_{n_\alpha}(a_\alpha) : \alpha \in \lambda\}$ is a locally $< \kappa$ on A cover of the discrete set A. So suppose that $\alpha \in \lambda$ is such that $V_{n_\alpha}(a_\alpha) \cap V_{n_\xi}(a_\xi) \neq \emptyset$ for at least κ many $\xi \in \lambda$. Choose $\xi \in \lambda$ such that $\xi \notin dom(p_\alpha)$ and such that $V_{n_\alpha}(a_\alpha) \cap V_{n_\xi}(a_\xi) \neq \emptyset$. There is a γ such that for each $\beta > \gamma$, $p_\alpha \Vdash V_{n_\alpha}(a_\alpha) \cap \tau_\beta = \emptyset$. Choose $\beta > \gamma$ so that $p_{\xi,\beta}^{\xi,\beta} \Vdash n_\xi > \sigma(a_\xi)$. Then

(1) $p_{\xi,\beta}^{\xi,\beta}$ and p_α are compatible.

(2) $p_{\xi,\beta}^{\xi,\beta} \Vdash V_{n_\xi}(a_\xi) \subset \tau_\beta$.

(3) $p_\alpha \Vdash V_{n_\alpha}(a_\alpha) \cap \tau_\beta = \emptyset$.

This contradicts $V_{n_\alpha}(a_\alpha) \cap V_{n_\xi}(a_\xi) \neq \emptyset$. The following lemma completes the proof.

Lemma 2.5. *Suppose that X is $< \kappa$-collectionwise Hausdorff and suppose that $\mathcal{U} = \{U(a) : a \in A\}$ is a locally $< \kappa$ cover of a closed discrete set $A \subseteq X$. Then A is separated.*

Proof: Define an equivalence relation on A by saying that $a \in A$ is equivalent to $b \in A$ if there is a finite path from a to b via the cover $\mathcal{U}$. More precisely, define $a \sim b$ if there is a sequence $(a_i : i \leq n)$ such that $a_0 = a$, $a_n = b$, and $U(a_i) \cap U(a_{i+1}) \neq \emptyset$ for each $i < n$. As the cover is locally $< \kappa$, equivalence classes are of cardinality $< \kappa$. If $a \not\sim b$ then $U(a) \cap U(b) = \emptyset$. Therefore since X is $< \kappa$-collectionwise Hausdorff, A can be separated.

3. Countably metacompact spaces.

Theorem 3.1. *Let $\nu : E \to E$ be an Easton indexing function satisfying $\nu(\kappa)$ is never a successor of a singular and let P be the Easton forcing for adding $\nu(\kappa)$ subsets of each regular κ. Then in locally countable, countably metacompact spaces, closed discrete sets are G_δ.*

The forcing proof of Theorem 3.1 is almost identical to the proof of 2.1 and Tall's proof that countably paracompact spaces are collectionwise Hausdorff in the Easton models modulo the main inductive lemmas which we present. We leave it to the reader to fill in the rest of the details.

Lemma 3.2. *Let $\lambda < \kappa$ be regular cardinals and force with $Fn(\kappa, \omega, \omega_1)$ over a model of GCH. Then in locally countable, countably metacompact T_1 spaces, closed discrete sets of size λ are G_δ.*

Proof: Suppose $\Vdash$ *$A \subset X$ is closed discrete and* $|A| = \lambda$. As in the proof of Lemma 2.4 if we modify the topology of X by isolating every point of $X \setminus A$, then the resulting space is countably metacompact and A is still not a G_δ. Since X is locally countable, we may assume that $\Vdash |X| = \lambda$. Therefore we may assume that X and A appear after adding the first λ Cohen subset of ω_1. Henceforth we are working in $\mathcal{M}[G]$ where G is $Fn(\lambda, \omega, \omega_1)$ generic over $\mathcal{M}$. Let Γ be the canonical name for the next generic function from $\lambda \to \omega$ added. As in [T1], we may assume that GMA(λ) for Cohen forcing holds in $M[G]$. Considering Γ as a partition of A, if it is forced to have a point finite open expansion in the extension then in $\mathcal{M}[G]$ there is a σ eventually point countable sequence of open covers of A. I.e., there are open covers $\mathcal{W}_n = \{W_n(a) : a \in A\}$ such that for each $x \in X$ there is an N such that $\forall m > N$ $\{a \in A : x \in W_m(a)\}$ is countable. We may also assume that each $\mathcal{W}_n$ witnesses the discreteness of A and that each $W_n(a)$ is countable. The proof of this fact is similar to the proof of Lemma 2.4 and almost identical to the proof of the analogous fact in the proof of Theorem 5 in [T1]. Partition $X \setminus A$ into countably many pieces $\{X_n : n < \omega\}$ where for each n,

$$X_n = \{x : ord(x, \mathcal{W}_n) \le \aleph_0\}.$$

It is easy to see that if for each n, A is a G_δ in the subspace $X_n \cup A$, then A is a G_δ in X. The rest of the proof is standard. For each n, $\{W_n(a) \cap (X_n \cup A); a \in A\}$ is a star countable collection in the subspace $X_n \cup A$. For a and b in A, define $a \sim b$ if there is a finite path from a to b as in the proof of Lemma 2.5. This is an equivalence relation. Each equivalence class is countable and equivalence classes are separated. This implies that A is in fact separated in $X_n \cup A$ hence a G_δ.

The proof for γ regular and larger than ω_1 is similar. Instead of using that countable closed discrete sets are G_δ's we need to use the inductive hypothesis that closed discrete sets of size $< \gamma$ are G_δ's to step up using $Fn(\nu(\gamma), \omega, \gamma)$ to closed discrete sets of regular size $< \nu(\gamma)$ are G_δ. For λ singular, the following lemma generalizes a result of Nyikos who proved the same result assuming GCH.

Lemma 3.3. *Assume SCH and fix κ a singular strong limit cardinal. Suppose X is a T_1 locally countable, countably metacompact space and $A \subset X$ is a closed discrete set of size κ. If subsets of A of size $< \kappa$ are G_δ's then so is A.*

The proof of 3.3 is almost identical to the Nyikos's GCH result; for completeness sake we include the proof.

Proof: If λ is of cofinality ω, partition A into countably many pieces $\{A_n : n < \omega\}$ each of size $< \kappa$. By assumption each is a G_δ and the fact that X is countably metacompact easily implies that A is a G_δ. Otherwise we modify Nyikos's GCH proof much of which we lift verbatim from [N] pp 4-5, 8-10. Let $\{\kappa_\alpha : \alpha < cof(\kappa)\}$ be a club in κ such that for each $\alpha < \kappa$, $2^{\kappa_\alpha} = \kappa_\alpha^+$. That such a sequence exists follows from $cof(\kappa) \ge \omega_1$ and the

SCH (see [J] Lemma 8.1). Fix X countably metacompact and a subset $D \subseteq X$ which is closed discrete, size κ and not a G_δ.

By our previous remarks we may assume that $X \setminus D = \kappa' = \bigcup\{[\kappa_\alpha, \kappa_\alpha^+) : \alpha < cof(\kappa)\}$ and that κ' consists of isolated points. Enumerate D as $\{d_\gamma : \gamma \in \kappa'\}$. For each $\xi \in \kappa$ let $\xi' = \xi \cap \kappa'$.

Definition 3.4. *For each $f : \alpha' \to \omega$ where $\alpha < \kappa$ and each $\xi \in \kappa'$ and $n < \omega$, let*

$$E_n(f, \xi) = \{d_\gamma : \gamma \geq \xi,\ \gamma \in \kappa',\ d_\gamma \in \overline{f^{-1}\{0, ..., n\} \cap \xi'}\}.$$

We say f is thin if $|E_n(f, \xi)| \leq |\xi|$ for all n and all $\xi \in \kappa'$.

Lemma 3.5. *There is a thin $f : \kappa' \to \omega$.*

Proof. List all partial functions $g : \lambda' \to \omega$ where $\lambda \in \kappa'$ as $\{g_\xi : \xi \in \kappa'\}$ so that if $\lambda \in [\kappa_\alpha, \kappa_\alpha^+)$ and $dom(g) = \lambda'$ then $g = g_\xi$ for some $\xi \in [\kappa_\alpha, \kappa_\alpha^+)$. We may do this since $2^{\kappa_\alpha} = \kappa_\alpha^+$.

Define subsets $F_n(\eta)$ of D for $\eta \in \kappa'$ by transfinite recursion, letting $F_n = \bigcup\{F_n(\eta) : \eta \in \kappa'\}$. Assume $F_n(\eta)$ has been defined for all $\eta \in \xi'$ where $\xi \in \kappa'$. If for each n

$$E_n(g_\xi, \xi) \subseteq \bigcup\{F_m(\eta) : m < \omega, \eta \in \xi'\}$$

let $F_n(\xi) = \bigcup\{F_n(\eta) : \eta \in \xi'\}$. If not, fix n minimal such that

$$E_n(g_\xi, \xi) \not\subseteq \bigcup\{F_m(\eta) : m < \omega, \eta \in \xi'\}$$

and fix $\gamma \in \kappa'$ minimal such that $d_\gamma \in E_n(g_\xi, \xi) \setminus \bigcup\{F_m(\eta) : \eta \in \xi'\}$. Let $F_{n+1}(\xi) = \bigcup\{F_{n+1}(\eta) : \eta \in \xi'\} \cup \{d_\gamma\}$. For $k \neq n+1$ let $F_k(\xi) = \bigcup\{F_k(\eta) : \eta \in \xi'\}$.

Let $D_n = \bigcup_{i \geq n} F_n$ then $\bigcap_{n<\omega} D_n = \emptyset$. Therefore since X is countably metacompact, there exist a decreasing sequence of open sets $G_n \supseteq D_n$ such that $\bigcap_{n<\omega} G_n = \emptyset$. Finally define $f : \kappa' \to \omega$ by

$$f(\xi) = min\{n : \xi \notin G_n\}.$$

Claim: *f is thin.*

Proof: If not, fix $\xi \in \kappa'$ minimal such that $|E_n(f,\xi)| > |\xi| = \kappa_\alpha$ for some $n \in \omega$. Fix $\xi_0 \in [\kappa_\alpha, \kappa_\alpha^+)$ such that $f \restriction \xi = g_{\xi_0}$. Then

$$E_n(g_{\xi_0}, \xi_0) \not\subseteq \bigcup\{F_m(\eta) : m < \omega, \eta \in \xi_0'\}$$

since the first set has cardinality $> \kappa_\alpha$ and the latter $\leq \kappa_\alpha$. Assume that n is minimal satisfying the above and choose γ minimal such that

$$d_\gamma \in E_n(g_{\xi_0}, \xi_0) \setminus \bigcup\{F_m(\eta) : \eta \in \xi_0'\}.$$

Therefore, by construction $d_\gamma \in F_{n+1}$. Then $d_\gamma \in \overline{f^{-1}\{0...n\}}$ and $G_{n+1} \cap f^{-1}\{0...n\} = \emptyset$ contradicts $d_\gamma \in G_{n+1}$ and that G_{n+1} is open.

As the above proof didn't depend on how D or the isolated points $X \setminus D$ were enumerated in κ', if we reindex them with injections ρ and $\sigma : \kappa' \to \kappa'$ we may define

$$E_n(f,\xi,\rho,\sigma) = \{d_\gamma : \rho(\gamma) \geq \xi,\ \gamma \in \kappa',\\ d_\gamma \in \overline{f^{-1}\{0,...,n\} \cap \sigma^{-1}(\xi')}\}.$$

We say f is thin with respect to. (ρ, σ) if $|E_n(f,\xi,\rho,\sigma)| \leq |\xi|$ for all n and all $\xi \in \kappa'$. As above we have

Lemma 3.6. *For any injective $\rho, \sigma : \kappa' \to \kappa'$ there is an $f : \kappa' \to \omega$ thin with respect to. (ρ,σ).*

For every $\gamma \in \kappa'$ let W_γ be a countable neighborhood of d_γ such that $W_\gamma \cap D = \{d_\gamma\}$. The following Lemma is a restatement of Lemma 2 of [N] and its proof is identical.

Lemma 3.7. *For each injection $\rho : \kappa' \to \kappa'$, there is an injection $\sigma : \kappa' \to \kappa'$ such that $\rho(\gamma) = \delta$ implies $\sigma(W_\gamma \setminus \{d_\gamma\}) \subseteq \omega\delta + \omega$.*

We follow Nyikos's proof and define recursively sets $D_n \subseteq D$, $A_n \subseteq \kappa'$, injections $\sigma_n,\ \rho_n : \kappa' \to \kappa'$ and functions $f_n : \kappa' \to \omega$. Let C be the set of limit ordinals in $cof(\kappa)$. Therefore for each $\alpha \in C$, $\kappa\alpha'$ has cardinality κ_α. Fix $B_\alpha \subseteq \kappa_\alpha'$ for each $\alpha \in C$ such that $|B_\alpha| = \kappa_\alpha$ and $B_\alpha \cap B_\beta = \emptyset$ for each $\beta < \alpha$ from C.

We can do this by partitioning each interval in κ' into $cof(\kappa)$ many pieces of equal size.
Let $D_0 = D$, $A_0 = \kappa'$, and $\rho_0 = \sigma_0 = id \restriction \kappa'$. Having A_i, D_i, ρ_i and σ_i, let f_i be thin with respect to (ρ_i, σ_i). For $\alpha \in C$ let

$$E(i, \alpha) = D_i \cap \bigcup_{n<\omega} E_n(f_i, \kappa_\alpha, \rho_i, \sigma_i).$$

Then $|E(i, \alpha)| \leq \kappa_\alpha$. Let $D_{i+1} = \bigcup_{\alpha \in C} E(i, \alpha)$ and let $A_{i+1} = \{\gamma : d_\gamma \in D_{i+1}\}$. Let $\rho_{i+1} : \kappa' \to \kappa'$ be such that for each $\alpha \in C$

$$\rho_{i+1} : E(i, \alpha) \setminus \bigcup_{\beta \in C \cap \alpha} E(i, \beta) \to B_\alpha.$$

Pick $\sigma_{i+1} : \kappa' \to \kappa'$ satisfying the conclusion of Lemma: $\rho_{i+1}(\gamma) = \delta$ implies $\sigma_{i+1}(W_\gamma \setminus \{d_\gamma\} \subseteq \omega\delta + \omega$.

Claim: $\bigcap_{n<\omega} A_n = \emptyset$

Proof. Suppose $\xi \in A_{i+1}$, then $D_\xi \in E(i, \alpha)$ for some $\alpha \in C$. By definition of $E(i, \alpha)$ $\rho_i(\xi) \geq \kappa_\alpha$ and by definition of ρ_{i+1}, $\kappa_\alpha > \rho_{i+1}(\xi)$. Therefore any $\xi \in \bigcap_{n<\omega} A_n$ would define an infinite strictly decreasing sequence of ordinals.

As X is countably metacompact, the following claim completes the proof of 3.3.

Claim: *For each $i < \omega$ $D_i \setminus D_{i+1}$ is a G_δ.*

Proof. For each $\alpha \in C$ let $\alpha(+)$ be the successor of α in C (so $\alpha(+) = \alpha + \omega$). For $\alpha \in C$ let

$$D(\alpha) = \{d_\gamma : \kappa_\alpha \leq \rho_i(\gamma) < \kappa_{\alpha(+)}, d_\gamma \in D_i \setminus D_{i+1}\}.$$

Then since $|D(\alpha)| < \kappa$, $D(\alpha)$ is a G_δ. Fix a sequence of open neighborhoods of $D(\alpha)$,

$$\{V_n(\alpha) : n < \omega\}$$

such that $\bigcap_{n<\omega} V_n(\alpha) = D(\alpha)$. Also, $D(\alpha) \cap E(i, \alpha) = \emptyset$ so for each $n < \omega$ there is an open $U_n(\alpha) \supseteq D(\alpha)$ such that

$$U_n(\alpha) \cap f^{-1}\{0, ..., n\} \cap \sigma_i^{-1}(\kappa'_\alpha) = \emptyset.$$

Furthermore assume that for each α, $U_n(\alpha)$ refines both $V_n(\alpha)$ and $\bigcup\{W_\gamma : d_\gamma \in D(\alpha)\}$. Let $U_n = \bigcup_{\alpha\in C} U_n(\alpha)$. We claim that $\bigcap_{n<\omega} U_n = D_i \setminus D_{i+1}$. To see this , fix $\beta \in \kappa'$ and fix $\alpha \in C$ such that $\sigma_i(\beta) \in [\kappa_\alpha, \kappa_{\alpha(+)})$. Fix m and n such that

$\beta \notin V_m(\alpha)$ and

$f_i(\beta) = n$.

Let $N = max\{n, m\}$. We claim that $\beta \notin U_N$. It suffices to show that for each $\eta \in C$, $\beta \notin U_N(\eta)$.

Case 1: $\eta = \alpha$. By choice of m.

Case 2: $\eta < \alpha$. Notice if $d_\gamma \in D(\eta)$, then $\rho_i(\gamma) \in \kappa'_\alpha$ hence $\sigma_i(W_\gamma \setminus \{d_\gamma\}) \subseteq \kappa'_\alpha$ while $\sigma_i(\beta) > \kappa_\alpha$. Therefore $\beta \notin$ any $U_i(\eta)$ for $\eta < \alpha$ since each $U_i(\eta)$ refines $\bigcup\{W_\gamma : \gamma \in [\kappa_\eta, \kappa_{\eta(+)})$ and for each such γ, $\beta \notin W_\gamma$.

Case 3: $\eta > \alpha$. As $f_i(\beta) = n$, we chose $U_n(\eta)$ so that in particular,

$$U_n(\eta) \cap f_i^{-1}(n) \cap \sigma_i^{-1}(\kappa'_\eta) = \emptyset.$$

But $\sigma_i(\beta) \in [\kappa_\alpha, \kappa_{\alpha(+)}) \subseteq \kappa_\eta$. Therefore $\beta \in f_i^{-1}(n) \cap \sigma_i^{-1}(\kappa'_\eta)$ hence $\beta \notin U_n(\eta) \supseteq U_N(\eta)$. This completes the proof.

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