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PERIODIC POINTS FOR AN ORIENTATION PRESERVING HOMEOMORPHISM OF THE PLANE NEAR AN INVARIANT IMMERSED LINE

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ABSTRACT. If F is an orientation preserving homeomorphism of the plane that leaves invariant a one-to-one immersed line Γ, Γ bounded, nowhere dense and with non-separating closure, then the set of rotation numbers of periodic orbits in the closure of Γ about a fixed point in Γ is a rational interval or a rational interval less a point.

In this paper we investigate the existence and rotational behavior of periodic points for an orientation preserving homeomorphism of the plane near an invariant one-to-one image of the reals. A familiar situation to which our results apply occurs for diffeomorphisms of the plane. Suppose that F is an orientation preserving diffeomorphism of the plane with fixed hyperbolic saddle p. Then $W^u(p) = \{z \in \mathbb{R}^2 \mid F^{-n}(z) \to p \text{ as } n \to \infty\}$ and $W^s(p) = \{z \in \mathbb{R}^2 \mid F^n(z) \to p \text{ as } n \to \infty\}$ are continuous one-to-one images of $\mathbb{R}$ invariant under F. If $W^u(p)$ and $W^s(p)$ intersect transversely at a point other than p then, according to the Smale Homoclinic Theorem ([S]), there is an $n \geq 1$ such that F^n has periodic points of all periods in the closure of $W^u(p)$. The proof of this theorem relies on the stretching and compressing done by F near p in the direction of $W^u(p)$ and $W^s(p)$, respectively. Assuming only that F is an orientation preserving homeomorphism with a fixed point

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p, and replacing $W^u(p)$ by a continuous one-to-one image Γ of $\mathbb{R}$, invariant under F, one can still (with mild topological assumptions on Γ–see Theorem 1.1) say quite alot about periodic points in the closure of Γ. The main theroem of this paper, Theorem 1.1, states that the collection of rotation numbers (about p) of periodic points in the closure of Γ is a rational interval (less, perhaps, 0 or $\frac{1}{2}$). It follows, for example, that if p is a Möbius saddle ($DF(p)$ has eigenvalues $\lambda_2 < -1 < \lambda_1 < 0$) and there is a second fixed point $q \neq p$ in the closure of $W^u(p)$, then F must have periodic points of all periods, except possibly 2, in the closure of $W^u(p)$ (assuming $W^u(p) = M$ satisfies (i)-(iii) of Theorem 1.1–see the remarks following the proof of Corollary 2). The proofs of the theorems in this paper are accomplished by splitting open the closure of Γ along Γ and applying results of [BG1] to the resulting irreducible plane separating continuum.

1. Preliminaries and Statement of The Main Theorem

In all that follows, $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ will be an orientation preserving homeomorphism of the plane. Given $p = (p_1, p_2) \in \mathbb{R}^2$, let $\Pi\colon \mathbb{R}^2 \to \mathbb{R}^2 \backslash \{p\}$, defined by $\Pi(x,y) = (p_1 + e^{-y}\cos(2\pi x), p_2 + e^{-y}\sin(2\pi x))$, be the universal cover of $\mathbb{R}^2\backslash\{p\}$. If p is a fixed point of F and z is periodic under F, $z \neq p$, the rotation number of z about p under F, $R_p(z;F)$, is defined to be the fractional part of $\frac{m}{n}$ (that is, $\frac{m}{n} - [\![\frac{m}{n}]\!]$) where n is the period of z under F, m is such that $\tilde{F}^n(\tilde{z}) = \tilde{z} + (m, 0)$, $\tilde{z} \in \Pi^{-1}(z)$, and $\tilde{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of F. If S is a subset of $\mathbb{R}^2\backslash\{p\}$, let $R_p(S;F) = \{R_p(z;F) \mid z \in S \text{ is periodic under } F\}$.

Suppose now that $\Gamma = \gamma(\mathbb{R})$ is the image of a continuous one-to-one map $\gamma\colon \mathbb{R} \to \mathbb{R}^2$. Let $\Gamma^+ = \gamma(\mathbb{R}^+ \cup \{0\})$, $\Gamma^- = \gamma(\mathbb{R}^- \cup \{0\})$, $\{p\} = \{\gamma(0)\} = \Gamma^+ \cap \Gamma^-$, and let $\Lambda = cl(\Gamma)$.

Theorem 1.1. *Suppose that:*

(i) Λ *is compact and nowhere dense in* $\mathbb{R}^2$*;*

(ii) Λ *does not separate the plane; and*

(iii) $F(\Gamma) = \Gamma$ *and* $F(p) = p$.

There is then a rational interval $J \subset \mathbb{Q}$ *such that*

(iv) *if* $F(\Gamma^+) = \Gamma^+$ *then* $0 \in J$ *and* $R_p(\Lambda\backslash\{p\}; F)\backslash\{0\} = J\backslash\{0\}(\mod 1)$*; and*

(v) *if* $F(\Gamma^+) = \Gamma^-$ *then* $\frac{1}{2} \in J$ *and* $R_p(\Lambda\backslash\{p\}; F)\backslash\{\frac{1}{2}\} = J\backslash\{1/2\}(\mod 1)$.

Moreover, if $\frac{m}{n} \in J$*,* m *and* n *relatively prime,* $\frac{m}{n} \neq 0$ *in case (iv) and* $\frac{m}{n} \neq \frac{1}{2}$ *in case (v), then* $F|_\Lambda$ *has a periodic orbit of (least) period* n *and rotation number the fractional part of* $\frac{m}{n}$.

Before taking up the proof of Theorem 1.1 we will state and prove an easy corollary and note a common situation to which the theorem applies.

Corollary 1.2. *Suppose that* F *and* Γ *satisfy (i)-(iii) of Theorem 1.1, that* $F(\Gamma^+) = \Gamma^-$*, and, in addition, that there is a periodic point* $z \in \Lambda\backslash\{p\}$*, of odd period* $2m+1 \geq 1$*. For each positive integer* n *let* $\eta(n)$ *be the number of integers* ℓ *relatively prime with* n *such that* $\frac{m}{2m+1} \leq \frac{\ell}{n} < \frac{1}{2}$*. Then* $F|_\Lambda$ *has at least* $\eta(n)$ *distinct periodic orbits of period* n *for each integer* $n \geq 1$ *and* $\eta(n) > 0$ *in case:* $n = 2k+1$ *and* $k \geq m$*;* $n = 2(2k+1)$ *and* $k \geq 2m+1$*; or* $n = 4k$ *and* $k \geq m+1$.

In particular, note that $F|_\Lambda$ must have a periodic point of period n for all $n \geq 8m+6$.

Proof of Corollary 1.2: According to Theorem 1.1, there is a rational interval J such that $\frac{1}{2} \in J$ and $R_p(\Lambda\backslash\{p\}; F)\backslash\{\frac{1}{2}\} = J(\mod 1)$. Thus if $z \in \Lambda\backslash\{p\}$ is periodic of period $2m+1$, with $R_p(z; F) = \frac{a}{b}$, where a and b are relatively prime, then b divides $2m+1$ so that J must contain either $\left[\frac{m}{2m+1}, \frac{1}{2}\right] \cap \mathbb{Q}$ or $\left[\frac{1}{2}, \frac{m+1}{2m+1}\right] \cap \mathbb{Q}$. Say $J \supset \left[\frac{m}{2m+1}, \frac{1}{2}\right] \cap \mathbb{Q}$ (the argument in the other case being the same). By the "moreover" part of Theorem 1.1, for each pair of integers ℓ and n with ℓ and n relatively prime and $\frac{m}{2m+1} \leq \frac{\ell}{n} < \frac{1}{2}$, $F|_\Lambda$ has a periodic orbit of period n and rotation number $\frac{\ell}{n}$. It is an easy matter to check that $\eta(n) > 0$ exactly in the cases listed. □

A commonly occuring situation in which Theorem 1.1 has application is the following. Suppose that $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 diffeomorphism with fixed point p. In addition, suppose that: $p \in \text{int}(D)$, D a bounded topological disk with $F(D) \subset D$; $0 < \det(DF(z)) < 1$ for all $z \in D$ (here $DF(z)$ is the derivative matrix of F at z) and $DF(p)$ has eigenvalues λ_1 and λ_2 with $0 < |\lambda_1| < 1 < |\lambda_2|$. Let $\Gamma = W^u(p) = \{z \in \mathbb{R}^2 \mid F^{-n}(z) \to p \text{ as } n \to \infty\}$ be the unstable manifold at p. Then Γ and F satisy (i)-(iii) of Theorem 1.1 and we have case (iv) of Theorem 1.1 if λ_1 and λ_2 are positive, case (v) if λ_1 and λ_2 are negative.

2. Proof of Theorem 1.1 and Ancillary Results

The method of proof of Theorem 1.1 is to split Λ open along Γ to produce a continuum Δ that separates the plane irreducibly into exactly two domains. The fixed point p will split into a pair of points, p_- and p_+, and F will lift to an orientation preserving homeomorphism G with $G(\Delta) = \Delta$ and either $G(p_+) = p_+$ and $G(p_-) = p_-$ (case (iv)) or $G(p_+) = p_-$ and $G(p_-) = p_+$ (case (v)). We will then appeal to the following adaptaion of Theorem 2.3 of [BG1] and push the information back down into Λ.

Theorem 2.1. *Suppose that $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving homeomorphism of the plane and that Δ is a continuum invariant under G that separates the plane into exactly two complementary domains. Suppose further that that no proper subcontinuum of Δ separates the plane. If q is a fixed point of G that is contained in the bounded complementary domain of Δ then there is a rational interval J such that*

$R_q(\Delta; G) = J($ mod $1)$. Moreover, if $\frac{m}{n} \in J$, m and n relatively prime, then there is a periodic point in Δ of (least) period n with rotation number the fractional part of $\frac{m}{n}$.

The splitting open of Γ is reminiscent of the "derived from Anosov" construction (see [W]) and in fact can be carried out very nicely using inverse limits in case Γ is an unstable manifold. In our more general setting, the process is a bit more tedious.

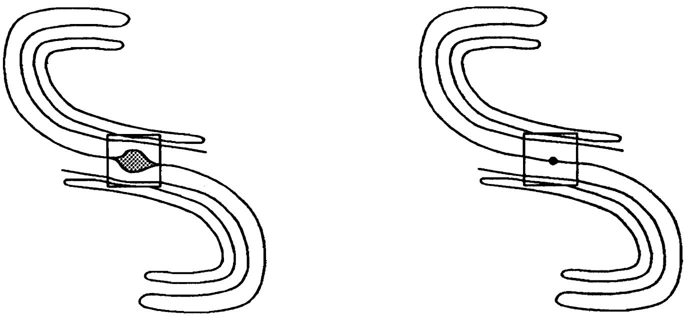

Figure 1

Proof of Theorem 1.1: We will describe the splitting of the entirety of Γ as a limit of splits of subarcs. Recall that $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ is continuous and one- to-one with $\gamma(0) = \{p\}$ and $\gamma(\mathbb{R}) = \Gamma$. Let $0 = t_0 < t_1 < t_2 < \cdots$ and $\cdots t'_2 < t'_1 < t'_0 = 0$ be sequences with $t_n \to \infty$ and $t'_n \to -\infty$ as $n \to \infty$. For each $n \geq 0$ let $t_{n+\frac{1}{2}}$ and $t'_{n+\frac{1}{2}}$ be such that $t_n < t_{n+\frac{1}{2}} < t_{n+1}$ and $t'_n > t'_{n+\frac{1}{2}} > t'_{n+1}$. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ be a sequence of positive numbers, let $C_1 = [t'_1, t_1] \times [-1, 1]$, and, for $n \geq 2$, let $C_n = ([t'_n, t'_{n-\frac{3}{2}}] \cup [t_{n-\frac{3}{2}}, t_n]) \times [-1, 1]$. Choose an embedding $\beta_1 : C_1 \to \mathbb{R}^2$ with the properties: $\beta_1(t, 0) = \gamma(t)$ for $t'_1 \leq t \leq t_1$ and $\beta_1(C_1) \cap \gamma([t'_2, t'_1) \cup (t_1, t_2]) = \emptyset$. Let $h_1 \colon C_1 \to C_1$ be a continuous surjection such that: $h_1 = id$ on ∂C_1; $h_1^{-1}(t, 0)$ is a nondegenerate arc contained in $\{t\} \times [-1, 1]$ for $t'_1 < t < t_1$; h_1 is one-to-one off $h_1^{-1}((t'_1, t_1) \times \{0\})$; and $\|h_1 - id\| < \varepsilon_1$. Let $P_+ \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $P_+ = id$ off $\beta_1(C_1)$ and $P_+ = \beta_1 \circ h_1 \circ \beta_1^{-1}$

on $\beta_1(C_1)$. Let $\alpha_1 : \mathbb{R} \to \mathbb{R}^2$ by $\alpha_1 = \gamma$ on $[t'_1, t_1]$ and $\alpha_1 = P_+^{-1} \circ \gamma$ otherwise. Then α_1 is continuous and one-to-one and P_+^{-1} "splits open" $\gamma([t'_1, t_1])$. We will next split open $\alpha_1([t'_2, t_2])$, and so on.

Suppose, for the purpose of recursion, that α_k, β_k, h_k, and P_k have been defined for $1 \leq k \leq n$. Let $\beta_{n+1} : C_{n+1} \to \mathbb{R}^2$ be an embedding with the properties: $\beta_{n+1} = \beta_n$ on $C_{n+1} \cap C_n$; $\beta_{n+1}(t, 0) = \alpha_n(t)$ for $(t, 0) \in C_{n+1}$; $\beta_{n+1}(C_{n+1}) \cap (P_+ \circ \cdots \circ P_n)^{-1}(\gamma(t'_n, t_n)) = \emptyset$; and $\beta_{n+1}(C_{n+1}) \cap \alpha_n([t'_{n+2}, t'_{n+1}) \cup (t_{n+1}, t_{n+2}]) = \emptyset$. Let $h_{n+1} : C_{n+1} \to C_{n+1}$ be a continuous surjection such that: $h_{n+1} = id$ on ∂C_{n+1}; $h_{n+1}(\{t\} \times [-1, 1]) = \{t\} \times [-1, 1]$ and $h_{n+1}^{-1}(t, 0)$ is a nondegenerate arc for each $t \in (t'_{n+1}, t'_{n-\frac{1}{2}}) \cup (t_{n-\frac{1}{2}}, t_{n+1})$; h_{n+1} is one-to-one off $h_1^{-1}(((t'_{n+1}, t'_{n-\frac{1}{2}}) \cup (t_{n-\frac{1}{2}}, t_{n+1})) \times \{0\})$; and $\|h_{n+1} - id\| < \varepsilon_{n+1}$. Define $P_{n+1} : \mathbb{R}^2 \to \mathbb{R}^2$ by $P_{n+1} = id$ off $\beta_{n+1}(C_{n+1})$ and $P_{n+1} = \beta_{n+1} \circ h_{n+1} \circ \beta_{n+1}^{-1}$ on $\beta_{n+1}(C_{n+1})$. Finally, let $\alpha_{n+1} : \mathbb{R} \to \mathbb{R}^2$ by $\alpha_{n+1} = \alpha_n$ on $[t'_{n+1}, t_n]$ and $\alpha_{n+1} = P_{n+1}^{-1} \circ \alpha_n$ otherwise. Maps $\alpha_n : \mathbb{R} \to \mathbb{R}^2$, $\beta_n : C_n \to \mathbb{R}^2$, $h_n : C_n \to C_n$ and $P_n : \mathbb{R}^2 \to \mathbb{R}^2$ are thus defined recursively for all $n \geq 1$. One can check that α_n is continuous and one-to-one for each n, β_n is an embedding for each n with $\beta_{n+1} = \beta_n$ on the common domain, and P_n is a monotone surjection for each n. The map $\bar{P}_n = P_+ \circ P_2 \circ \cdots \circ P_n$ collapses the disk $(\bar{P}_n)^{-1}(\gamma([t'_n, t_n]))$ (the "split open" $\gamma([t'_n, t_n])$) onto $\gamma([t'_n, t_n])$ and is otherwise one-to-one. Note also that $\bar{P}_n \circ \alpha_n = \gamma$ and that $\|\bar{P}_{n+1} - \bar{P}_n\| \leq \delta_n$ and $\|\alpha_{n+1} - \alpha_n\| \leq \delta_n$ where $\delta_n = \delta_n(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$. Thus by choosing the ε_n sufficiently small (so that, say, $\delta_n \leq \frac{1}{2^n}$) the sequences $\{\bar{P}_n\}$ and $\{\alpha_n\}$ are made to converge uniformly to $P : \mathbb{R}^2 \to \mathbb{R}^2$ and $\alpha : \mathbb{R} \to \mathbb{R}$, respectively.

For later use we extend each h_n to a map $h_n : \mathbb{R} \times [-1, 1] \to \mathbb{R} \times [-1, 1]$ by setting $h_n = id$ off C_n. Let $\bar{h}_n : \mathbb{R} \times [1, 1] \to \mathbb{R} \times [-1, 1]$ by $\bar{h}_n = h_1 \circ \cdots \circ h_n$ and let $h = \lim_{n\to\infty} \bar{h}_n$. Also, let $\bar{\beta}_n : ([t'_n, t_n] \times [-1, 1]) \cup (((-\infty, t'_n) \cup (t_n, \infty)) \times \{0\}) \to \mathbb{R}$ by $\bar{\beta}_n = \beta_k$ on C_k for $1 \leq k \leq n$ and $\bar{\beta}_n(t, 0) = \alpha_{n-1}(t)$ for

$t < t'_n$ or $t > t_n$ (in case $n = 1$, let $\alpha_0 = \gamma$). Finally, let $\beta\colon \mathbb{R} \times [-1,1] \to \mathbb{R}^2$ by $\beta = \beta_k$ on C_k for $1 \leq k \leq \infty$.

Now $h^{-1}(\mathbb{R}\times\{0\})$ is a strip (homeomorphic with $\mathbb{R} \times [-1,1]$) with boundary components $B_1 \subset \mathbb{R}\times(0,1)$ and $B_{-1} \subset \mathbb{R}\times(-1,0)$. Each of these boundary components is the graph of a function from $\mathbb{R}$ to $(-1,1)$. Let $b_1\colon \mathbb{R} \to (0,1)$ have graph B_1 and let $b_{-1}\colon \mathbb{R} \to (-1,0)$ have graph B_{-1}. Let U be the topological disk $U = (h^{-1}(\mathbb{R} \times \{0\}))\backslash(B_{-1} \cup B_1) = \{(t,s) \mid b_{-1}(t) < s < b_1(t)\}$. Then $\beta|_U$ is an embedding and $\beta(U)$ is the interior of $P^{-1}(\Gamma)$, the "split open" Γ. Furthermore, $P^{-1}(\Gamma) = \beta(U) \cup \beta(B_{-1}) \cup \beta(B_1)$ and each of the maps $\beta|_{B_{-1}}$ and $\beta|_{B_1}$ is continuous and one-to-one. Let $\Delta = \partial U = cl(\beta(B_{-1}) \cup \beta(B_1))$.

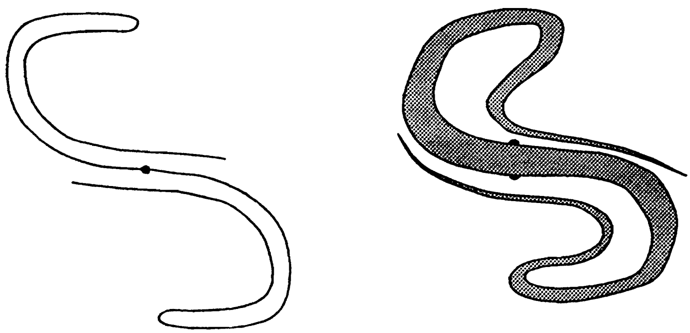

Figure 2

Recall that $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving homeomorphism with $F(\Gamma) = \Gamma$ and $F(p) = F(\gamma(0)) = p$. Either $F(\Gamma^+) = \Gamma^+ = \gamma(\mathbb{R}^+ \cup \{0\})$ (case(iv)) or $F(\Gamma^-) = \Gamma^-$ (case(v)). We will assume the latter, the proof of Theorem 1.1 in case $F(\Gamma^+) = \Gamma^+$ is similar. We now define an orientation preserving homeomorphism $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ with the properties that $G(\Delta) = \Delta$ and $F \circ P = P \circ G$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(t) = \gamma^{-1} \circ F \circ \gamma(t)$. Then f is a homeomorphism, $f(\mathbb{R}^+) = \mathbb{R}^-$, and $f(\mathbb{R}^-) = \mathbb{R}^+$. There is then a sequence $0 = t_0 < t_1 < t_2 < \cdots$, $t_n \to \infty$ as $n \to \infty$, such that either $f^2(t_n) \geq t_n$ for all n or $f^{-2}(t_n) \geq t_n$ for all n. We will assume that $f^2(t_n) \geq t_n$ for all n (if $f^{-2}(t_n) \geq t_n$

for all n, replace F by F^{-1}). In the definition of the α_n, β_n, h_n, and P_n, let these be the t_n and let $t'_n = f(t_n)$ so that $\cdots < t'_2 < t'_1 < t'_0 = 0$, $t'_n \to \infty$ as $n \to \infty$ and $f^2(t'_n) \le t'_n$ for all n.

For each $n \ge 1$ and each $t \in (t'_n, t_n)$, $(\bar{h})^{-1}(t,0) = \{t\} \times [c_n(t), d_n(t)]$ is a nondegenerate arc (note that for $n \ge 2$, and $t'_{n-\frac{3}{2}} \le t \le t_{n-\frac{3}{2}}$, $c_n(t) = b_{-1}(t)$ and $d_n(t) = b_1(t)$). Let $t''_n = f^{-1}(t_n)$. Then $t'_n < t''_n < 0$ for $n \ge 1$ and $f([t''_n, t_n]) = [t'_n, t_n]$. Now, for $n \ge 1$, define $f_n\colon (\bar{h}_n)^{-1}(\mathbb{R} \times \{0\}) \to (\bar{h}_n)^{-1}(\mathbb{R} \times \{0\})$ by $f_n(t,s) = (f(t), g_n(t,s))$ where

$$g_n(t,s) = \begin{cases} 0, & \text{if } t \le t''_n \text{ or } t \ge t_n \\ \left(\frac{c_n(f(t)) - d_n(f(t))}{d_n(t) - c_n(t)}\right)(s - d_n(t)) + c_n(f(t)), & \text{otherwise} \end{cases}$$

(f_n is an affine flip on vertical segments in $(\bar{h}_n)^{-1}(\mathbb{R} \times \{0\})$ that extends "f" on $\mathbb{R} \times \{0\}$). Now let $G_n\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$G_n = \begin{cases} (\bar{P}_n)^{-1} \circ F \circ \bar{P}_n, & \text{on } \mathbb{R}^2 \backslash \bar{\beta}_n((\bar{h}_n)^{-1}(\mathbb{R} \times \{0\})) \\ \bar{\beta}_n \circ f_n \circ (\bar{\beta}_n)^{-1}, & \text{on } \bar{\beta}_n((\bar{h}_n)^{-1}(\mathbb{R} \times \{0\})) \end{cases}$$

Then G_n is a continuous surjection that is one-to-one off $\beta_n((\bar{h}_n)^{-1}((t'_n, t''_n] \times \{0\}))$. The G_n converge uniformly to the desired $G\colon \mathbb{R}^2 \to \mathbb{R}^2$.

Being the uniform limit of continuous functions, G is continuous. To see that G is one-to-one, consider the following simpler description of G. Let g be the homeomorphism of the strip $h^{-1}(\mathbb{R} \times \{0\})$ given by $g(t,s) = f_{n+2}(t,s)$ provided $t'_n \le t \le t_n$. Then

$$G = \begin{cases} P^{-1} \circ F \circ P, & \text{on } \mathbb{R}^2 \backslash \beta(h^{-1}(\mathbb{R} \times \{0\})) \\ \beta \circ g \circ \beta^{-1}, & \text{on } \beta(h^{-1}(\mathbb{R} \times \{0\})). \end{cases}$$

Since P is one-to-one off $P^{-1}(\Gamma) = \beta(h^{-1}(\mathbb{R} \times \{0\}))$, β is one-to-one on $h^{-1}(\mathbb{R} \times \{0\})$, and, since F and g are homeomorphisms, G is one-to-one. Thus G is an orientation preserving homeomorphism of the plane such that $P \circ G = F \circ P$ and

$G(\Delta) = \Delta$. Let $p_+ = \beta(0, b_1(0))$ and $p_- = \beta(0, b_{-1}(0))$. Then p_+ and p_- are in Δ and $G(p_+) = p_-$, $G(p_-) = p_+$. Also, $(0, (b_1(0) + b_{-1}(0))/2)$ is a fixed point of g so that $\beta(0, (b_1(0) + b_{-1}(0))/2) = q \in \beta(U)$ is a fixed point of G. Furthermore, $R_q(p_+; G) = R_{(0,(b_1(0)+b_{-1}(0))/2)}((0, b_1(0)); g) = \frac{1}{2}$.

We now verify that Δ has the properties required for application of Theorem 2.1. Since Δ is the boundary of the bounded open topological disk $\beta(U)$, Δ is a plane separating continuum. Let $W_i = \beta(U)$ and let W_e be the unbounded component of the complement of Δ. To show that Δ irreducibly separates the plane into exactly two domains it suffices to show that $\Delta = \partial W_i = \partial W_e$ and $\mathbb{R}^2 \backslash \Delta = W_i \cup W_e$ (see for example [BG2] or [CL]). Clearly $\Delta = \partial W_i$.

To see that $\Delta = \partial W_e$, let $z \in \Delta$. Then $P(z) \in \Lambda = cl(\Gamma)$ and since Λ is nowhere dense in the plane, there are points $z'_n \in \mathbb{R}^2 \backslash \Lambda$, $z'_n \to P(z)$ as $n \to \infty$, and half-lines $\eta'_n : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^2 \backslash \Lambda$ with $\eta'_n(0) = z'_n$. Since Λ does not separate the plane we may choose these half-lines so that $\|\eta'_n(t)\| \to \infty$ as $t \to \infty$. Let $\eta_n = P^{-1} \circ \eta'_n$ and let $z_n = \eta_n(0) = P^{-1}(z'_n)$. Then $\eta_n(\mathbb{R}^+ \cup \{0\}) \subset \mathbb{R}^2 \backslash \Delta$ and $\|\eta'_n(t)\| \to \infty$ as $t \to \infty$ so $z_n \in W_e$. Let $\bar{z}$ be an accumulation point of $\{z_n\}_{n=1}^{\infty}$. Then $P(\bar{z}) = P(z)$ so that either $\bar{z} = z$ or, since P is one-to-one on $\Delta \backslash \beta(B_{-1})$ and on $\Delta \backslash \beta(B_1)$, one of z and $\bar{z}$ is in $\beta(B_{-1})$ and the other is in $\beta(B_1)$. We see from this that for each $t \in \mathbb{R}$ at least one of the two points $\beta(t, b_{-1}(t)) \in \beta(B_{-1})$ and $\beta(t, b_1(t)) \in \beta(B_1)$ is in $cl(W_e)$.

If $z = \bar{z}$ then $z \in cl(W_e)$. Suppose that $z \neq \bar{z}$ so that, say, $z = \beta(t, b_1(t))$ and $\bar{z} = \beta(t, b_{-1}(t))$. Let $A \subset B_1$ be a compact arc with $(t, b_1(t)) \in A$, $(t, b_1(t))$ not an endpoint of A. Then $\beta(A)$ locally separates the plane: on one side are points of $\beta(U) = W_i$ and on the other side there are points arbitrarily close to z that are not in $\beta(U)$. If there are points of $\beta((B_1 \cup B_{-1}) \backslash A)$ that are arbitrarily close to z then there is a sequence $\tau_n \to \infty$ (or $\tau_n \to -\infty$) and points $(\tau_n, s_n) \in B_{-1} \cup B_1$ (so $s_n = b_{-1}(\tau_n)$ or $s_n = b_1(\tau_n)$ for each n) such that $\beta(\tau_n, s_n) \to z$ as $n \to \infty$. But $\|\beta(t, b_{-1}(t)) - \beta(t, b_1(t))\| \to 0$

as $t \to \pm\infty$ and at least one of the two points $\beta(t, b_{-1}(t))$ and $\beta(t, b_1(t))$ is in $cl(W_e)$. Thus, in this case, $z \in cl(W_e)$. Finally, if there are no points of $\beta((B_{-1} \cup B_1)\backslash A)$ close to z then there are points $z_n \in \mathbb{R}^2\backslash(\Delta \cup \beta(U))$, $z_n \to z$. Then $P(z_n) \in \mathbb{R}^2\backslash\Lambda$ so that each $P(z_n)$ can be joined by a half-line to ∞ in $\mathbb{R}^2\backslash\Lambda$. Applying P^{-1} to these half-lines as before, we see that $z_n \in W_e$ so that $z \in cl(W_e)$.

Thus $\Delta = \partial W_i = \partial W_e$. If $W \neq W_i$ is a component of the complement of Δ then $P(W) \subset \mathbb{R}^2\backslash\Lambda$ and, by considering half-lines to ∞ as above, we see that $W = P^{-1}(P(W)) \subset W_e$ so that $W = W_e$. Thus $\mathbb{R}^2\backslash\Delta = W_i \cup W_e$ and we have shown that Δ irreducibly separates the plane into exactly two domains.

According to Theorem 2.1 there is a rational interval $J \subset \mathbb{Q}$ such that $R_q(\Delta; G) = J(\mod 1)$. Moreover, $\frac{1}{2} = R_q(p_+; G) \in R_q(\Delta; G)$ so $\frac{1}{2} \in J$ (at least after translating J by an integer, if necessary). Since $P \circ G = F \circ P$, $z \in \Delta$ is periodic under G if and only if $P(z) \in \Lambda$ and, if $P(z) \neq p$, $R_p(P(z); F) = R_q(z; G)$. Thus $R_p(\Lambda\backslash\{p\}; F)\backslash\{\frac{1}{2}\} = J\backslash\{\frac{1}{2}\}(\mod 1)$. The "moreover" part of Theorem 1.1 follows from the observation that P doesn't change (least) periods except on the orbit $\{p_+, p_-\}$. □

Suppose that F and $\Lambda = cl(\Gamma)$ are as in (i)-(iii) of Theorem 1.1. There is then a circle of prime ends, $\mathcal{S}$, associated with Λ, and an orientation preserving homeomorphism $\hat{F}\colon \mathcal{S} \to \mathcal{S}$ induced by F (cf. [CL], [AY], [BG1]). If we give $\mathcal{S}$ the natural counterclockwise orientation when viewed from the complement of Λ, where the prime ends live, then the rotation number of $\hat{F}$ with respect to this orientation is the prime end rotation number $\rho_e(\Lambda; F)$. To adjust to p's point of view, let $\rho = 1 - \rho_e(\Lambda; F)(\mod 1)$.

Theorem 2.2. *Suppose that F and Λ satisfy (i)-(iii) of Theorem 1. Let J be the rational interval of (iv) or (v) in Theorem 1 and let ρ be the adjusted prime end rotation number defined above. Then $\rho \in cl(J)$ and, if ρ is rational, $\rho \in J$.*

In case ρ is rational, Theorem 2.2 has some overlap with

Proposition 4.2 of [AY]. Note that if $\rho \neq 0$ and $F(\Gamma^+) = \Gamma^+$ or $\rho \neq \frac{1}{2}$ and $F(\Gamma^+) = \Gamma^-$ then Theorem 2.2, in concert with Theorem 1.1, guarantees a rich periodic structure for $F|_\Lambda$.

Proof of Theorem 2.2: Let Λ, P, G and W_e = unbounded component of $\mathbb{R}^2 \backslash \Delta$ be as in the proof of Theorem 1.1. Let $\mathcal{S}_e$ be the circle of prime ends associated with Δ from W_e, let $\hat{G}_e \colon \mathcal{S}_e \to \mathcal{S}_e$ be the homeomorphism induced by G on $\mathcal{S}_e$, and let $\rho_e(\Delta; G)$ be the rotation number of $\hat{G}_e$ (with counterclockwise orientation on $\mathcal{S}_e$ taken from W_e). By Theorem 2.6 of [BG1], $1 - \rho_e(\Delta; G)(\bmod 1) \in cl(R_q(\Delta; G))$. Since $R_q(\Delta; G)$ is relatively closed in $[0,1) \cap \mathbb{Q}$ (see Theorem 2.5 of [BS] or section 1 of [BG1]), if $1 - \rho_e(\Delta; G)$ is a rational then $1 - \rho_e(\Delta; G)(\bmod 1) \in R_q(\Delta; G)$ by Theorem 2.1. The map P induces an orientation preserving homeomorphism $\hat{P}$ from $\mathcal{S}_e$ onto $\mathcal{S}$, the circle of prime ends associated with Λ, such that $\hat{F} \circ \hat{P} = \hat{P} \circ \hat{G}_e$. Thus $\rho_e(\Delta; F) = \rho_e(\Delta; G)$ and the theorem follows. □

In order for Λ to accomodate the rich periodic structure implied by Theorem 1.1, the topology of Λ must be rather complicated. Recall that a continuum is indecomposable if it is not the union of any two proper subcontinua.

Theorem 2.3. *Suppose that F and Λ satisfy (i)-(iii) of Theorem 1. If the rational interval J of (iv) or (v) is nondegenerate then Λ is an indecomposable continuum.*

In particular, note that if the adjusted prime end rotation number ρ of Theorem 2.2 is not 0 in case (iv) or not $\frac{1}{2}$ in case (v) then Λ is indecomposable.

Proof of Theorem 2.3: Let P, Δ, G, B_{-1}, B_1, β be as in the proof of Theorem 1.1. Let

- $B_{-1}^- = ((\mathbb{R}^- \cup \{0\}) \times [-1,1]) \cap B_{-1}$,
- $B_{-1}^+ = ((\mathbb{R}^+ \cup \{0\}) \times [-1,1]) \cap B_{-1}$,
- $B_1^- = ((\mathbb{R}^- \cup \{0\}) \times [-1,1]) \cap B_1$, and
- $B_1^+ = ((\mathbb{R}^+ \cup \{0\}) \times [-1,1]) \cap B_1$.

Then $\Delta = cl(\beta(B^-_{-1}) \cup \beta(B^+_{-1}) \cup \beta(B^-_1) \cup \beta(B^+_1))$, $\Gamma^- = P(\beta(B^-_{-1})) = P(\beta(B^-_1))$, and $\Gamma^+ = P(\beta(B^+_{-1})) = P(\beta(B^+_1))$. If J is nondegenerate then $R_q(\Delta; G)$ is nondegenerate so that, by Theroem 2.7 of [BG1], Δ is an indecomposable continuum. Thus no proper subcontinuum of Δ has nonempty interior (relative to Δ-see,for example, [HY]) and at least one of $cl(\beta(B^-_{-1}))$, $cl(\beta(B^-_1))$, and $cl(\beta(B^+_1))$ must be all of Δ. Then either $cl(\Gamma^-) = \Lambda$ or $cl(\Gamma^+) = \Lambda$.

Suppose, without loss of generality, that $cl(\Gamma^+) = \Lambda$. For each $t \geq 0$, let $\Gamma^+_t = \gamma([t, \infty))$. We will argue that $cl(\Gamma^+_t) = \Lambda$ for all $t \geq 0$. First note that $\gamma(-1) \in cl(\Gamma^+)$ and $\gamma(-1) \notin \Gamma^+$ implies that $\gamma(-1) \in cl(\Gamma^+_t)$ for all $t \geq 0$. Since Λ is nowhere dense in $\mathbb{R}^2$ and Λ does not separate the plane, no subcontinuum of Λ separates the plane. It follows that $\gamma([-1, t]) \cap cl(\Gamma^+_t)$ must be connected for all $t \geq 0$ (cf, [N]). Thus, since $\{\gamma(-1), \gamma(t)\} \subset \gamma([-1, t]) \cap cl(\Gamma^+_t)$, $\gamma([-1, t]) \subset cl(\Gamma^+_t)$. It follows that $\Gamma^+ \subset cl(\Gamma^+_t)$ so that $\Lambda = cl(\Gamma^+_t)$ for all $t \geq 0$.

Now suppose that H is a subcontinuum of Λ with nonempty interior (relative to Λ). Then $\gamma(t_1) \in H$ for some $t_1 \geq 0$. Suppose that $\gamma(t_2) \notin H$ for some $t_2 > t_1$. Since $cl(\Gamma^+_{t_2}) = \Lambda$, there must be a $t_3 > t_2$ such that $\gamma(t_3) \in H$. But $\gamma([t_1, t_3]) \cap H$ must be connected so, in fact, $\gamma(t_2) \in H$. Thus $\gamma([t, \infty)) = \Gamma^+_{t_1} \subset H$ so $\Lambda = cl(\Gamma^+_{t_1}) \subset H$. We have proved that Λ contains no proper subcontinua with nonempty interior (relative to Λ). It follows that Λ is indecomposable. □

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