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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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THE DIMENSION OF PARACOMPACT NORMAL κ-FRAMES

M.G. CHARALAMBOUS

ABSTRACT. In the category of paracompact normal κ-frames, we prove that direct limits do not raise covering dimension (dim) and establish the direct sum formula $\dim(L \oplus M) \leq \dim L + \dim M$.

1. INTRODUCTION

In this paper, lattices contain 0 and 1 and lattice homomorphisms preserve finite joins and meets, inclucling 0 and 1. A *frame* L is a lattice satisfying the distributive law $x \wedge \bigvee(x_\lambda : \lambda \in \Lambda) = \bigvee(x \wedge x_\lambda; \lambda \in \Lambda)$ for every set Λ. If this distributive law holds for sets Λ with cardinality $|\Lambda| \leq \kappa$, where κ is an infinite cardinal, then L is called a *κ-frame.* It is convenient here to adjoin a maximum (∞) and a minimum (finite) to the range of values of κ, interpreting the relations $|\Lambda| \leq \infty$, and $|\Lambda| \leq$ finite in the obvious way, so that κ-frame for these extreme values of κ means frame and distributive lattice, respectively. An $\aleph_0$-frame is better known as a *σ-frame.* A *κ-map (frame map, σ-map* when $k = \infty, \aleph_0$, respectively) is a lattice homomorphism between κ-frames that preserves joins of sets of cardinality at most κ. The archetype of a frame is the topology $\Omega(X)$ of a space X, a fact that provided the original motivation for the study of frames. The set of cozero sets of X is a sub-σ-frame of $\Omega(X)$, and if $f : X \to Y$ is a continuous function, then $\Omega(f) = f^{-1} : \Omega(Y) \to \Omega(X)$, is an example of a frame map. Ω is in fact a cofunctor from **Top**, the category of topological spaces and continuous functions, to **κFrm (σ Frm, Frm** when

$\kappa = \aleph_0, \infty$, respectively), the category of κ-frames and κ-maps. However, coproducts or sums of frames behave much more satisfactorily than products of topological spaces, preserving for regular frames the property of being Lindelöf, paracompact or metacompact [5, 7]. The purpose of this paper is to show that direct limits and sums of paracompact normal frames behave well with respect to covering dimension. Our main results are as follows.

Theorem 1. *Let L be the colimit of a direct system $(L_\alpha, q_{\alpha\beta}, A)$ of* $\boldsymbol{\kappa}$***Frm***, *where L_α is paracompact normal and $\dim L_\alpha \leq n$ for each α in A. Then L is paracompact normal and $\dim L \leq n$.*

Theorem 2. *In the category of κ-frames, if L and M are non-trivial, paracompact and normal, then $L \oplus M$ is paracompact normal and*

$$\dim(L \oplus M) \leq \dim L + \dim M.$$

It should be noted that the corresponding propositions for topological spaces are far from valid. Thus, it is possible to have an inverse sequence of zero-dimensional Lindelöf spaces, or an inverse system of spaces homeomorphic with N, the space of natural numbers, with infinite-dimensional limit space [3]. A similarly unsatisfactory state of affairs prevails as regards products of topological spaces, where the product theorem for dim is valid only under very restrictive conditions [17,15].

Section 2 contains the definitions of most notions and several observations that are needed in this paper as well as proposition 1, a version of theorem 1 for compact κ- frames. The proofs of the main results are contained in section 5 and are based on propositions 5 and 6, the main results of sections 3 and 4, respectively. Proposition 6 supplies the information needed in proposition 5 that covers of colimits of paracompact normal κ-frames L_α have locally finite refinements consisting of finite meets of canonical images of cozero elements of L_α. Proposition 5 is deduced from known results on topological spaces with the help of a cofunctor $\mathcal{P}$: $\boldsymbol{\kappa}$**Frm** $\rightarrow$ **Top** that preserves normality and dimension and is discussed in section 4. Theorem 2 for $\kappa = \aleph_0$ extends a result of Banaschewski and Gilmour [1, theorem 7 (3)] for regular σ-frames and, when interpreted for topological spaces, the results contained in this paper yield (generalizations of) several known non-trivial results about κ-paracompact spaces. Some of these are noted in section 6, whose main result, theorem 4, gives sufficient conditions for the topology of the limit of an inverse system $(X_\alpha, \pi_{\alpha\beta})$ in **Top** to be the colimit of the direct system $(\Omega(X_\alpha), \Omega(\pi_{\alpha\beta}))$ in $\boldsymbol{\kappa}$**Frm**.

For standard results in dimension theory, we refer to [6, 16], and to [8] or [10] for basic facts concerning frames or κ-frames.

From section 3 onward, κ is restricted to infinite values.

2. Preliminaries

There is a functor $\mathcal{I} : \boldsymbol{\kappa}\mathbf{Frm} \to \mathbf{Frm}$ which is described as follows. A *κ-ideal* of a κ- frame L is a *lower set* I of L (which means that $x \in I$ whenever $x \leq y$ and $y \in I$) such that $\bigvee J \in I$ for every subset J of I with $|J| \leq \kappa$. $\mathcal{I}(L)$ is the set of all κ-ideals of L partially ordered by inclusion. In $\mathcal{I}(L), I_1 \wedge I_2 = I_1 \cap I_2$ and $\bigvee\{I_\alpha : \alpha \in A\}$ consists of all $x \in L$ such that $x \leq \bigvee\{x_\alpha : \alpha \in B\}$, where $x_\alpha \in I_\alpha$ for each element α of some subset B of A with $|B| \leq \kappa$. For a κ-map $\varphi : L \to M$, $\mathcal{I}(\varphi) : \mathcal{I}(L) \to \mathcal{I}(M)$ is the map that sends a κ-ideal I of L to the κ-ideal of M generated by $\varphi(I)$. There is also an injective κ-map from L to $\mathcal{I}(L)$ that sends each x in L to $I_x = \{y \in L : y \leq x\}$, and it is frequently useful to identify L with the sub-κ-frame $\{I_x : x \in L\}$ of $\mathcal{I}(L)$.

A *system* $(L_\alpha, q_{\alpha\beta}, A)$ of $\boldsymbol{\kappa}$**Frm** consists of a partially ordered set A, κ-frames $L_\alpha, \alpha \in A$, and κ-maps $q_{\alpha\beta} : L_\alpha \to L_\beta$ for $\alpha \leq \beta$ such that q_α is the identity and $q_{\alpha\gamma} = q_{\alpha\beta}q_{\beta\gamma}$ whenever $\alpha \leq \beta \leq \gamma$. It is a *direct system* if A is directed. A *target* (M, r_a) of the system consists of a κ-frame M and κ-maps $r_a : L_\alpha \to M$, $\alpha \in A$, such that $r_\alpha = r_\beta q_{\alpha\beta}$ whenever $\alpha \leq \beta$. A target (L, q_α) is called the *colimit* of the system if for each target (M, r_α) there is a unique κ-map $r : L \to M$ with $r_\alpha =$

rq_α. If $\alpha \leq \beta$ means $\alpha = \beta$, the colimit is the coproduct or sum $\sum L_\alpha$.

Certain properties of colimits of κ-frames, e.g. existence, follow from the corresponding properties of colimits of frames. Let $(L_a, q_{\alpha\beta}, A)$ be a system in $\boldsymbol{\kappa}\mathbf{Frm}$. Let (L, q_α) be the colimit of the system $(\mathcal{I}(L_\alpha), \mathcal{I}(q_{\alpha\beta}), A)$ in $\mathbf{Frm}$. Then it is readily verified that the sub-κ-frame of L generated by $\bigcup\{q_\alpha(L_\alpha) : \alpha \in A\}$ is the colimit in $\boldsymbol{\kappa}\mathbf{Frm}$ of $(L_\alpha, q_{\alpha\beta}, A)$ with canonical maps the restrictions of the maps q_α to $L_\alpha, \alpha \in A$.

For our first result, a version of theorem 1 for compact κ-frames, we need the following definitions. Let L be a κ-frame. A subset M is a *cover* of L if $|M| \leq \kappa$ and $\bigvee M = 1$. L is *compact* if every cover has a finite subcover. If M_1, M_2 are covers of L, M_1 *refines* M_2 if for each x in M_1, there is y in M_2 with $x \leq y$. For $n = -1, 0, 1, 2, \ldots\infty$ and $M \subset L$, order $M \leq n$ if the meet of any $n+2$ elements of M is 0, and dim$L \leq n$ if every finite cover of L has a finite refinement of order $\leq n$. Here by standard argument the requirement of finiteness on the refinement may be dropped. Note that L is compact iff $\mathcal{I}(L)$ is and dim$L$ = dim$\mathcal{I}(L)$. If $f : L \to M$ is a frame map, its right adjoint $f^\# : M \to L$, usually denoted by f_*, is defined by

$$f^\#(y) = \bigvee\{x \in L : f(x) \leq y\}.$$

Proposition 1. *Let (L, q_α) be the colimit of a direct system $(L_\alpha q_{\alpha\beta}, A)$ in κFrm, where L_α is compact and dim$L_\alpha \leq n$ for each a in A. Then L is compact and dim $\leq n$.*

Proof: The proof for frames is such that the general case follows from the preceeding remarks concerning the functor $\mathcal{I}$. We work then in $\mathbf{Frm}$ and consider a cover M of L. It clearly suffices to prove $x_\alpha = \bigvee\{q_\alpha^\#(m) : m \in M\}$ equals 1 for some α in A. Assume on the contrary that $x_\alpha < 1$ for each α in A. Consider the family $\mathcal{F}$ of all (y_α) in ΠL_α such that $x_\alpha \leq y_\alpha < 1$ and $q_{\alpha\beta}(y_a) \leq y_\beta$ for $\alpha \leq \beta$. Note that we have $q_\alpha = q_\beta q_{\alpha\beta}$, and hence $q_{\alpha\beta}(q_\alpha^\#(m)) \leq q_\beta^\#(m)$ and $(x_\alpha) \in \mathcal{F}$. $\mathcal{F}$ is partially ordered by defining $(y_\alpha) \leq (z_\alpha)$ iff $y_\alpha \leq z_\alpha$ for each

α. By compactness of each L_α each chain of $\mathcal{F}$ has an upper bound and, by Zorn's lemma, $\mathcal{F}$ has a maximal element (z_α). If $z_\alpha = a \wedge b$ and $a, b \neq z_\alpha$, then by the maximality of (z_α) for some successors γ and δ of α in A we must have $q_{\alpha\gamma}(a) = 1$ and $q_{\alpha\delta}(b) = 1$. For a common successor β of γ and δ in the directed set A, this would imply $q_{\alpha\beta}(z_\alpha) = 1 = z_\beta$. Consequently z_α is prime in L_α for each α in A, so that we can define a frame map p_α from L_α to the two-element frame $\mathbf{2}$ by $p_\alpha(x) = 0$ iff $x \leq z_\alpha$. For $\alpha \leq \beta$, if $w_\gamma = q_{\alpha\gamma}(q^{\#}_{\alpha\beta}(z_\beta))$ for $a \leq \gamma$ and $w_\gamma = z_\gamma$ for $\alpha \not\leq \gamma$, it is readily seen that $(w_\gamma) \in \mathcal{F}$. The maximality of (z_α) implies that $q^{\#}_{\alpha\beta}(z_\beta) = z_\alpha$. Hence $p_\alpha = p_\beta q_{\alpha\beta}$ and there is a frame map $p : L \to \mathbf{2}$ with $p_\alpha = pq_\alpha$. Finally, because A is directed, $m = \bigvee\{q_\alpha(q^{\#}_\alpha(m)) : \alpha \in A\}$ for each m in L, and in the frame $\mathbf{2}$ we have

$$1 = p(\bigvee M) = p(\bigvee\{q_\alpha(x_\alpha) : \alpha \in A\}) =$$
$$\bigvee pq_\alpha(x_\alpha) = \bigvee p_\alpha(x_\alpha) = 0$$

This contradiction shows that $x_\alpha = 1$ for some α in A and completes the proof. $\square$

A different proof of the compactness of L can be found in [7].

We recall the following definitions for a κ-frame L. L is *normal* if whenever $a \vee b = 1$, there exist c and d with $c \wedge d = 0$ and $a \vee c = b \vee d = 1$. The "well-iside" relation $\lessdot$ on L is defined by $a \lessdot b$ iff there exists c with $a \wedge c = 0$ and $c \vee b = 1$. If $a \leq b \lessdot c \leq d$, then $b \leq c$ and $a \lessdot d$; $a \lessdot b$ and $c \lessdot d$ imply $a \wedge c \lessdot b \wedge d$ and $a \vee c \lessdot b \vee d$; and $\lessdot$ is preserved by κ-maps. For subsets of L, we write $G \lessdot H$ if for each a in G there is b in H with $a \lessdot b$. L is *cover regular* if every cover G has a refinement H with $G \lessdot H$. L is *regular* if every member x is the join of at most κ members y with $y \lessdot x$. A subset G of L is *locally finite* if there is a cover H such that $\{b \in G : a \wedge b \neq 0\}$ is finite for each a in H. Note that a locally finite subset of L has cardinality at most κ. Also, a locally finite subset G of a frame L with $|G| \leq \kappa$ is locally finite as a κ-frame subset. For if H is the frame cover of the definition, then

$$\{\bigvee\{x \in H : x \wedge y = 0 \text{ iff } y \notin a\} : \alpha \text{ a finite subset of } G\}$$

is a κ-frame cover each of whose members meets only finitely many elements of G. L is *paracompact* if every cover has a locally finite refinement. Recall that a space is *κ-paracompact* if every open cover of cardinality $\leq \kappa$ has a locally finite open refinement, which by standard argument may be assumed to have cardinality $\leq \kappa$. Thus, a space is κ-paracompact iff its topology is paracompact as a κ-frame. The following result is readily verified.

Proposition 2. *A κ-frame L is compact, cover regular, normal or paracompact iff $\mathcal{I}(L)$ is.*

Two results that are needed in the sequel can now be deduced from the corresponding results for frames [4, 5]. For paracompact κ-frames normality is equivalent to cover regularity, and in a normal κ-frame, every locally finite cover $\{x_\gamma\}$ is *shrinkable*, i.e. there is a cover $\{y_\lambda\}$ with $y_\lambda \lessdot x_\lambda$.

3. The space of prime ideals

There is a useful cofunctor $\mathcal{P} : \boldsymbol{\kappa}\mathbf{Frm} \to \mathbf{Top}$ defined as follows. $\mathcal{P}(L)$ consists of all prime filters (dual ideals) of a κ-frame L with topology the one generated by the sets $h(x) = \{I \in \mathcal{P}(L) : x \in I\}, x \in L$. Note that $h = h_L : L \to \Omega\mathcal{P}(L)$ is an injective lattice homomorphism. For a κ-map $\varphi : L \to M$, $\mathcal{P}(\varphi) : \mathcal{P}(M) \to \mathcal{P}(L)$ is defined by $\mathcal{P}(\varphi)(I) = \{x \in L : \varphi(x) \in I\}$. It is shown in [2] that $\mathcal{P}(L)$ (in fact, every basic open set $h(x)$) is compact, $\dim L = \dim \mathcal{P}(L)$, and L is normal iff $\mathcal{P}(L)$ is. For $Y \subset \mathcal{P}(L)$, let κY be the sub-κ-frame of $\Omega(Y)$ consisting of all sets of the form $\bigcup\{h(x) \cap Y : x \in M\}$ where $M \subset L$ and $|M| \leq \kappa$. By [2, proposition 3], we have a κ-map $r = r_L : \kappa\mathcal{P}(L) \to L$ defined by $r(\bigcup\{h(x) : x \in M\}) = \bigvee\{x : x \in M\}$. For a κ-map $\varphi : L \to M$, one verifies that $\Omega(\mathcal{P}(\varphi))h_L = h_M\varphi$ and hence $r_M\Omega(\mathcal{P}(\varphi)) = \varphi r_L$.

Henceforth κ denotes either an infinite cardinal or ∞. This restriction is in general necessary only when cozero elements are considered, and several of the results that follow can be seen to be valid even for κ finite.

The *cozero* elements of a κ-frame L are the images of open sets of the space of real numbers under κ-maps (equivalently, σ-maps) into L. The set of all such elements is denoted by coz$L$. Let X be a compact metrizable space and $\varphi : \Omega(X) \to L$ a κ-map. Then coz$\mathcal{P}(L)$ =coz$\Omega\mathcal{P}(L)$ is a sub-σ-frame of $\kappa\mathcal{P}(L)$ and there is a unique continuous $f : \mathcal{P}(L) \to X$ such that $\varphi = r\Omega(f)$ [2, proposition 10]. Hence coz$L$ is a quotient of coz$\mathcal{P}(L)$ and a regular, and therefore normal [1, corollary 2], sub-σ-frame of L. We can also deduce from the corresponding property for normal spaces that for any finite cover $\{g_i\}$ of a normal κ-frame L, there is a cover $\{U_i\}$ of a compact polyhedron X and a κ-map $\varphi : \Omega(X) \to L$, such that $\varphi(U_i) \leq g_i$.

Proposition 3. dimcoz$L$ = dimcoz$\mathcal{P}(L)$ *and, if* L *is normal,* dim$L$ = dimcoz$L$.

Proof: If $\{G_i\}$ is finite cozero cover of coz$\mathcal{P}(L)$, then $\{r(G_i)\}$ is a finite cover of coz$L$ of the same order. Also, given a finite cover $\{h_i\}$ of the normal σ-frame coz$L$, there is a cover $\{U_i\}$ of a compact polyhedron X and a κ-map $\varphi : \Omega(X) \to L$ such that $\varphi(U_i) \leq h_i$. As $\varphi = r\Omega(f)$ for some continuous $f : \mathcal{P}(L) \to X$, $\{\Omega(f)(U_i)\}$ is a cover of coz$\mathcal{P}(L)$ with $r\Omega(f)(U_i) \leq h_i$. These observations immediately imply dimcoz$L$ $\leq$ dim coz$\mathcal{P}(L)$. To complete the proof of the first equality, given a finite cozero cover $\{G_i\}$ of $\mathcal{P}(L)$, let $\{H_i\}$ be a cozero and $\{E_i\}$ a zero cover with $H_i \subset E_i \subset G_i$. Choose a cover $\{x_i\}$ of coz$L$ of order $\leq$ dimcoz$L$ such that $x_i \leq r(H_i)$, and a cozero cover $\{U_i\}$ of $\mathcal{P}(L)$ with $r(U_i) \leq x_i$. Then $U_i \subset E_i \subset G_i$ and $\{U_i\}$ has order $\leq$ dimcoz$L$. Hence dimcoz$\mathcal{P}(L) \leq$ dimcoz$L$.

Let L be normal. The equality dim$L$ = dimcoz$L$ follows from the fact that every finite cover of L has a cozero refinement.

Proposition 4. *Let* $G = \bigcap\{G_n : n \in N\}$, *where* $G_n \in \kappa\mathcal{P}(L)$ *and* $r(G_n) = 1$. *Then (i)* G *is dense in* $h(y) - h(x)$ *for all* *elements x, y of L, (ii) L is a quotient of κG, (iii) G is C^*-embedded in $\mathcal{P}(L)$ and (iv) $cozG \subset \sigma G$.*

Proof: Write $G_n = \bigcup\{h(m) : m \in M_n\}$, where $M_n \subset L$ and $|M_n| \leq \kappa$, and suppose $y \not\leq x$. We have $z = \bigvee\{z \wedge m : m \in M_n\}$ and therefore we can pick inductively y_n in M_n such that $y \wedge y_1 \ldots \wedge y_n \not\leq x$. By Zorn's lemma, there is a maximal filter I that contains each $y \wedge y_1 \wedge \ldots \wedge y_n$ but not x. Then I is an element of $G \cap (h(y) - h(x))$, which is therefore non-empty. This property implies (i).

Let $H_i = \bigcup\{h(x) : x \in A_i\}, i = 1, 2$, where $A_i \subset L$ and $|A_i| \leq \kappa$. If $r(H_1) \not\leq r(H_2)$, for some x in $A_1, x \not\leq r(H_2)$ so that $h(x) - hr(H_2) \neq \emptyset$; by (i), $G \cap (h(x) - hr(H_2)) \neq \emptyset$ and therefore $G \cap (H_1 - H_2) \neq \emptyset$. It follows that $G \cap H_1 = G \cap H_2$ implies $r(H_1) = r(H_2)$. Thus, can define a surjective κ-map $s : \kappa G \to L$ by $s(G \cap H) = r(H), H \in \kappa\mathcal{P}(L)$, which proves (ii).

Consider disjoint closed sets E, F of G. Write $\mathcal{P}(L) - \bar{E} \cap \bar{F}$ in the form $\cup\{h(x_a) : \alpha \in A\}$, and suppose there exists x_n in M_n such that $h(x_1 \wedge \ldots \wedge x_n)$ intersects $\bar{E} \cap \bar{F}$ for each n in N. Then, by Zorn's lemma, there is a maximal filter I such that $x_n \in I$, for n in N, and I is disjoint from the ideal generated by $\{x_\alpha : \alpha \in A\}$ and I is a member of $\bar{E} \cap \bar{F} \cap G$ (cf. [8, lemma I.2.3 and theorem I.2.4]). It follows that each point of E is contained in one of the compact sets $h(x_1 \wedge \ldots \wedge x_n), n \in N, x_n \in M_n$, that does not intersect $\bar{F}$, and hence there is U in κG such that $E \subset U \subset G - F$. Therefore coz$G \subset \kappa G$ and, given a continuous $e : G \to X$, where X is compact and Hausdorff, we have lattice homomorphisms $\varphi :$ coz$X \to \Omega\mathcal{P}(L)$ and $\psi :$ coz$X \to \Omega(G)$ defined by $\varphi(U) = \text{hse}^{-1}(U)$ and $\psi(U) = G \cap \text{hse}^{-1}(U)$. Then, by lemma 4 of [2], there are unique maps $f : \mathcal{P}(L) \to X$ and $g : G \to X$ such that $f^{-1}(U) \subset \varphi(U)$ and $g^{-1}(U) \subset \psi(U)$. Then clearly e, g and the restriction of f to G coincide, and (iii) and (iv) follow.

Proposition 5. *Let (L, q_α) be a target of a direct system $(L_\alpha, q_{\alpha\beta}, A)$ of κ-frames with $dimcozL_\alpha \leq n$ for each α in A.*

Let U be a σ-locally finite cover of L each member of which is of the form $q_\alpha(x)$ for some α in A and x in cozL_α. Then there is a metric space M with dim $\leq n$ and weight $\leq \kappa$, an open cover V of M and a κ-map $\varphi : \Omega(M) \to L$ such that $\varphi(V) = U$. Hence U has a refinement of order $\leq n$.

Proof: Let Λ_n be sets and K_n covers of L such that $U = \{u_\lambda : \lambda \in \Lambda\}$, where $\Lambda = \bigcup\{\Lambda_n : n \in N\}$, and each member of K_n meets only finitely many members of $\{u_\lambda : \lambda \in \Lambda_n\}$. For each λ, fix $\alpha(\lambda)$ in A, x_λ, in coz$L_{\alpha(\lambda)}$ and H_λ in coz$\mathcal{P}(L_{\alpha(\lambda)})$ such that $u_\lambda = q_{\alpha(\lambda)}(x_\lambda) = r\Omega\mathcal{P}(q_{\alpha(\lambda)})(H_\lambda)$. Let $G_0 = \bigcup\{\Omega\mathcal{P}(q_{\alpha(\lambda)})(H_\lambda) : \lambda \in \Lambda\}, G_n = \bigcup\{h(x) : x \in K_n\} \cap G_o$ and $G = \bigcap\{G_n : n \in N\}$. Then $Z = \{G \cap \Omega\mathcal{P}(q_{\alpha(\lambda)})(H_\lambda) : \lambda \in \Lambda\}$ is a σ-locally finite cover of G.

Let $f_\alpha : G \to \mathcal{P}(L_\alpha)$ be the restriction of $\mathcal{P}(q_\alpha)$ to G. We have an inverse system $(\mathcal{P}(L_\alpha), \mathcal{P}(q_{\alpha\beta}), A)$ of topological spaces with $f_\alpha = \mathcal{P}(q_{\alpha\beta)} f_\beta$ for $\alpha \leq \beta$ and $Z = \{f^{-1}_{\alpha(\lambda)}(H_\lambda) : \lambda \in \Lambda\}$, where H_λ is a cozero set of $\mathcal{P}(L_{\alpha(\lambda)})$. Also, by proposition 3, dimcoz$\mathcal{P}(L_\alpha) \leq n$. Therefore, by [15, proposition 1, or 14, proposition 9], there is a map f from G to a metric space M with dim$M \leq n$ and an open cover $V = \{V_\lambda : \lambda \in \Lambda\}$ of M such that $f^{-1}(V_\lambda) = f^{-1}_{\alpha(\lambda)}(H_\lambda), \lambda \in \Lambda$. We can of course assume that $|A| \leq \kappa$, in which case the construction is such that $wM \leq \kappa$. This at any rate follows by an easy application of Pasynkov's factorization theorem [14, theorem 2]. We let φ be the composite of f^{-1} with the κ-map $s : \kappa G \to L$ of proposition 4, which applies in the present circumstances. Then $\varphi(V) = U$, and if W is a refinement of V of order $\leq n$, then $\varphi(W)$ is a refinement of U of order $\leq n$.

4. The cozero elements of colimits

In this section, we consider a fixed system $(L_\alpha, q_{\alpha\beta}, A)$ in $\boldsymbol{\kappa}$**Frm** with colimit (L, q_α). We will assume that any two successors of an element of A have a common successor. This condition holds for coproducts as well as direct limits. Our aim is to prove

Proposition 6. *Let L_α be paracompact and normal for each α in A. Then L is paracompact normal and every cover of L has a locally finite refinement consisting of meets of finite subsets of $\bigcup\{q_\alpha(cozL_\alpha) : \alpha \in A\}$.*

This is a corollary of another result, proposition 7, which sharpens [5, theorem 7]. We need first two more definitions for a κ-frame M. A *system of subsets* of M is a family $\mathcal{F}$ of subsets of cardinality at most κ that satisfies

(i). $F_1 \subset F_2 \in \mathcal{F} \Rightarrow F_1 \in \mathcal{F}$.

(ii). $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \wedge F_2 = \{x \wedge y : x \in F_1, y \in F_2\} \in \mathcal{F}$.

and (iii). If $F \in \mathcal{F}$ and for each x in F there is $E_x \in \mathcal{F}$ with $x = \bigvee E_x$, then $\bigcup\{E_x : x \in F\} \in \mathcal{F}$.

M is *$\mathcal{F}$-regular* if for every cover H, there is a cover G in $\mathcal{F}$ with $G \preccurlyeq H$. One readily proves that in an $\mathcal{F}$-regular κ-frame if $x \preccurlyeq \bigvee G$, where $|G| \leq \kappa$, then there is F in $\mathcal{F}$ with $x \preccurlyeq \bigvee F$ and $F \preccurlyeq G$.

Proposition 7. *Let $\mathcal{F}_\alpha, \alpha \in A$, and $\mathcal{F}$ be systems of subsets of L_a and L, respectively, such that $q_\alpha(\mathcal{F}_\alpha) \subset \mathcal{F}$ and $q_{\alpha\beta}(\mathcal{F}_\alpha) \subset \mathcal{F}_\beta$ for $\alpha \leq \beta$. Let each L_α be $\mathcal{F}_\alpha$-regular. Then L is $\mathcal{F}$-regular.*

Proof: This differs little from the proof of [5, theorem 7], but is sketched here for the convenience of the reader.

Let D be the family of lower sets of L and $M = \{\psi(G) : G \in D\}$, where $\psi(G) = \{\bigvee F : F \in \mathcal{F}$ and $F \preccurlyeq G\}$. Partially ordered by inclusion, D and M are frames and $\psi : D \to M$ is a frame map. Also, $F \subset \psi(G)$ and $F \in \mathcal{F}$ imply $\bigvee F \in \psi(G)$.

Define $\sigma_\alpha : L_\alpha \to D$ and $\varphi_\alpha : L_\alpha \to M$ by $\sigma_\alpha(x) = \{y \in L :$ for some $\beta \geq \alpha$ and $z \preccurlyeq q_{\alpha\beta}(x)$ and $y \leq q_\beta(z)\}$ and $\varphi_\alpha = \psi\sigma_\alpha$. One readily proves that σ_α, preserves finite meets, 0 and 1, and φ_α is a κ-map, the non-trivial part being the inequality $\varphi_\alpha(\bigvee x_\lambda) \leq \bigvee \varphi_\alpha(x_\lambda) = \psi(\bigcup \sigma_\alpha(x_\lambda))$, where λ ranges over a set of cardinality $\leq k$. Consider an element y of the left hand side and fix F in $\mathcal{F}$ with $y = \bigvee F$ and $F \preccurlyeq \sigma_\alpha(\bigvee x_\lambda)$. It suffices to show $F \subset \psi(\bigcup \sigma_\alpha(x_\lambda))$. Let $x \in F$ and pick $\beta \geq \alpha$ and z in L_β such that $x \leq q_\beta(z)$ and $z \preccurlyeq q_{\alpha\beta}(\bigvee x_\lambda) = \bigvee q_{\alpha\beta}(x_\lambda)$.

As L_β is $\mathcal{F}_\beta$-regular, there are G, H in $\mathcal{F}_\beta$ such that $z \leqslant \bigvee G$ and $G \leqslant H \leqslant \{q_{\alpha\beta}(x_\lambda)\}$ Then $\{x\} \wedge q_\beta(G) \leqslant \bigcup \sigma_\alpha(x_\lambda)$ and $\{x\} \wedge q_\beta(G)$ is an element of $\mathcal{F}$ with join x. It follows that $F \subset \psi(\bigcup \sigma_\alpha(x_\lambda))$ and φ_α is a κ-map. Evidently (M, φ_α) is a target of the system and so it induces a κ-map $\varphi : L \to M$.

Any cover of L has a refinement U consisting of at most κ elements of the form $\bigwedge\{q_\alpha(x_\alpha) : \alpha \in \lambda\}, \lambda$ a finite subset of A. Then

$$\psi(L) = \varphi(1) = \varphi(\bigvee(\bigwedge\ q_\alpha(x_\alpha) : \alpha \in \lambda\})) = \bigvee(\bigwedge \varphi q_\alpha(x_\alpha)) =$$

$$\bigvee(\bigwedge \varphi_\alpha(x_\alpha)) = \bigvee(\bigwedge \psi\sigma_\alpha(x_\alpha)) = \psi(\bigcup(\bigwedge\{\sigma_\alpha(x_\alpha) : \alpha \in \lambda\})).$$

Now 1 is in $\psi(\sigma_\alpha(1)) = \psi(L)$ because L_α is $\mathcal{F}_\alpha$-regular. Hence there is a cover F in $\mathcal{F}$ with $F \leqslant \bigcup(\bigwedge\{\sigma_\alpha(x_\alpha) : \alpha \in \lambda\})$ and therefore $F \leqslant U$. Thus, L is $\mathcal{F}$-regular.

Proof of proposition 6: Let $\mathcal{F}_\alpha$ consist of all locally finite in L_α subsets of $\mathrm{coz}L_\alpha$ of cardinality at most κ. In a normal κ-frame, every locally finite cover is shrinkable, and $a \leqslant b$ implies $a \leqslant z \leqslant b$ for some cozero element z. As L_α is paracompact and normal, it follows that it is $\mathcal{F}_\alpha$-regular. Let $\mathcal{F}$ be the set of all locally finite subsets of L consisting of at most κ meets of finite subsets of $\bigcup\{q_\alpha(\mathrm{coz}L_\alpha) : \alpha \in A\}$. Then, by proposition 7, L is $\mathcal{F}$-regular and therefore paracompact and cover regular. The result follows from the fact that cover regularity in the presence of paracompactness implies normality.

5. The main theorems

The main results follow from propositions 5 and 6.

Proof of theorem 1: Let q_α denote the canonical map from L_a to L for each α in A. As $(L_\alpha, q_{\alpha\beta}, A)$ is a direct system, $\bigcup\{q_a(\mathrm{coz}L_\alpha) : \alpha \in A\}$ is a sub-lattice of L, and by proposition 6 a cover G of L has a locally finite refinement U consisting of sets of the form $q_\alpha(x), x \in \mathrm{coz}L_\alpha$. As each L_α is normal, by proposition 3, $\dim\mathrm{coz}L_\alpha \leq n$, and proposition 5 provides a cover of L of order $\leq n$ that refines U and therefore G. Hence $\dim L \leq n$.

For further applications of proposition 5, we need some definitions and two elementary lemmas. Let L be a κ-frame. A subset B of L is called a *base* if every element of L is the join of a subset of B of cardinality at most κ. The *weight* of L, wL, is at most λ if it has a base of cardinality at most λ. If $wL \leq \kappa$, then one readily checks that joins exist and the distributive law holds for arbitrary subsets of L, so that L is in fact a frame. We call L *metrizable* if it is regular and has a σ-locally finite base. As locally finite subsets of a κ- frame have cardinality at most κ, if L is metrizable, then $wL \leq \kappa$ and L is actually a frame. $P_\lambda(L)$ denotes the set of all metrizable sub-κ-frames of L of weight at most λ ordered by inclusion, and $D_\lambda(L)$ the subset of $P_\lambda(L)$ consisting of members M with $\dim M \leq \dim \mathrm{coz} L$.

Lemma 1. *In a κ-frame, if $u \lessdot x$ for each member u of a locally finite subset U, then $\vee U \lessdot x$.*

Proof: Let K be a cover every element of which meets only finitely many members of U. For each u in U fix u^* with $u \wedge u^* = 0$ and $u^* \vee x = 1$. Put

$$z = \bigvee\{k \wedge \bigwedge\{u^* : u \wedge k \neq 0\} : k \in K\}$$

Then $\bigvee U \wedge z = 0$ and $z \vee x = 1$. Hence $\bigvee U \lessdot x$.

Lemma 2. *Let L be a metrizable frame. Then L is normal and $L = \mathrm{coz} L$.*

Proof: Let $\{x_{i\alpha} : i \in N, a \in A\}$ be a σ-locally finite base and y an element of L. Put $y_i = \bigvee\{x_{i\alpha} : x_{i\alpha} \lessdot y, \alpha \in A\}$. Then $y_i \lessdot y$ by lemma 1 and $\bigvee y_i = y$ by regularity. Thus, L is regular as a σ-frame and is therefore normal [1, corollary 2]. It follows that there is a z_i in $\mathrm{coz} L$ with $y_i \leq z_i \leq y$ and hence $y = \bigvee z_i \in \mathrm{coz} L$.

Proposition 8. *A metrizable frame L is the quotient of the topology of a metric space M of the same weight and dimension.*

Proof. In view of lemma 2, proposition 5 applied to an obvious direct system consisting of copies of L supplies a metric space M with $wM \leq wL$ and $\dim M \leq \dim L$ and a frame map $\varphi : \Omega(M) \to L$ whose image includes a σ-locally finite base of L. Hence φ is surjective and therefore $wL \leq wM$ and $\dim L \leq \dim M$ [2, proposition 15].

Proposition 9. *$D_\lambda(L)$ is cofinal in $P_\lambda(L)$ for any κ-frame L.*

Proof: Given K in $P_\lambda(L)$, proposition 5 supplies a metric space M with $wM \leq \lambda$ and $\dim M \leq \dim \operatorname{coz} L$ and a frame map $\varphi : \Omega(M) \to L$ whose image includes a σ-locally finite base of K. Then by [2, proposition 15] the image of φ is an element of $D_\lambda(L)$ that follows K.

The following result is needed in the proof of theorem 2.

Proposition 10. *For non-zero countably generated metrizable frames L, M,*

$$\dim(L \oplus M) \leq \dim L + \dim M.$$

Proof: By proposition 8, L, M are quotients of the topologies of separable metrizable spaces X, Y, respectively, with $\dim X \leq \dim L$ and $\dim Y \leq \dim M$. As separable metric spaces have metric compactifications of the same dimension, we can assume that X and Y are compact. Then by [5, theorem 8], $\Omega(X \times Y) = \Omega(X) \oplus \Omega(Y)$ and hence $L \oplus M$ is a quotient of $\Omega(X \times Y)$. Finally by [2, proposition 15] and the product theorem for metric spaces,

$$\dim(L \oplus M) \leq \dim \Omega(X \times Y) \leq \dim X + \dim Y \leq \dim L + \dim M.$$

Proof of theorem 2: Let q, r denote the canonical maps from L, M, respectively, into $L \oplus M$. A cover G of this sum has by proposition 6 a σ-locally finite refinement $U = \{q(x_\alpha) \wedge r(y_\alpha) : \alpha \in A\}$, where $|A| \leq \kappa, x_\alpha \in \operatorname{coz} L$ and $y_\alpha \in \operatorname{coz} M$. Let A_f be the set of finite subsets of A partially ordered by inclusion. For each α in A_f, using proposition 9, we construct by induction on $|a|$ countably generated metrizable sub-κ-frames L_α, M_α of

L, M, respectively, such that $\dim L_a \leq \dim L$, $\dim M_a \leq \dim M$, $x_\alpha \in L_\alpha$, $y_\alpha \in M_\alpha$, and for $\alpha \leq \beta, L_\alpha \subset L_\beta$ and $M_\alpha \subset M_\beta$. These inclusions induce κ-maps $q_{\alpha\beta}$ from $L_\alpha \oplus M_\alpha$ into $L_\beta \oplus M_\beta$ and we have a direct system $(L_\alpha \oplus M_\alpha, q_{\alpha\beta}, A_f)$ that has a target $(L \oplus M, r_\alpha)$, where r_α is induced by the inclusions $L_\alpha \subset L$, and $M_\alpha \subset M$. Also, x_α and y_α map to elements of $L_\alpha \oplus M_\alpha$ whose meet maps by r_α to $q(x_\alpha) \wedge r(y_\alpha)$. Let $n = \dim L + \dim M$. Then, by proposition 10, $\dim(L_\alpha \oplus M_\alpha) \leq n$. Thus, by proposition 5, U and therefore G has a refinement of order $\leq n$. This shows that $\dim(L \oplus M) \leq n$ and completes the proof. $\square$

The following interesting result has a similar proof.

Theorem 3. *For a paracompact normal κ-frame L the following are equivalent.*

(i) *$\dim L \leq n$.*

(ii) *Every cover of L has a refinement of order $\leq n$.*

Proof: Evidently (ii) implies (i). So assume (i) and consider a cover G of L. As L is paracompact and normal, G has a locally finite cozero refinement $U = \{x_\alpha : \alpha \in A\}$ where $|A| \leq \kappa$. As in the proof of theorem 2, for each α in A_f, we construct by induction on $|a|$ countably generated metrizable sub-κ-frames L_α of L, such that $\dim L_\alpha \leq n, x_\alpha \in L_\alpha$, and for $\alpha \leq \beta, L_\alpha \subset L_\beta$. Now we have a direct system $(L_\alpha, q_{\alpha\beta}, A_f)$ where all maps $q_{\alpha\beta}$, and q_α are inclusions. By proposition 5, U and therefore G has a refinement of order $\leq n$. This completes the proof. $\square$

6. Direct limits of topologies

The purpose of this section is to show that several known theorems for topological spaces are special cases of results obtained in this paper for κ-frames. We need the following theorem, for which we recall some definitions. A *perfect map* is a closed continuous function with compact fibers. A non-empty closed subset of a topological space X is called *irreducible* if it is not the union of two proper closed subsets. X is *sober* if every irreducible closed set of X is the closure of a unique point. Hausdorff spaces are sober and sober spaces are T_0.

Theorem 4. *Let $(X_\alpha, \pi_{\alpha\beta}, A)$ be an inverse system of topological spaces and perfect bonding maps with limit (X, π_α), where $|A| \leq \kappa$. Let (L, q_α) be the colimit of the direct system $(\Omega(X_\alpha), \Omega(\pi_{\alpha\beta}), A)$ in κ**Frm**. If $A = N$ or each X_α, is sober, then L and $\Omega(X)$ are isomorphic.*

Proof: As $(\Omega(X), \Omega(\pi_\alpha))$ is a target of $(\Omega(X_\alpha), \Omega(\pi_{\alpha\beta}), A)$, there is a κ-map $r : L \to \Omega(X)$ such that $\Omega(\pi_\alpha)) = rq_\alpha$. As $|A| \leq \kappa$, every element of L can be written in the form $\bigvee\{q_\alpha(G_\alpha) : \alpha \in A\}$ and, of course, every element of $\Omega(X)$ can be written in the form $\bigcup\{\pi_\alpha^{-1}(G_\alpha) : \alpha \in A\}$, where $G_\alpha \in \Omega(X_\alpha)$. It follows that r is surjective and it remains to prove that it is also injective.

For $\alpha \leq \beta$ and $S \subset X_\beta$, we adopt the notation S^α for the set $X_\alpha - \pi_{\alpha\beta}((X_\beta - S) = \{x \in X_\alpha : \pi_{\alpha\beta}^{-1}(x) \subset S\}$. As $\pi_{\alpha\beta}$ is closed, S^α is open if S is.

Consider elements G, H of L with $G < H$. Write $G = \bigvee\{q_\alpha(V_\alpha) : \alpha \in A\}$ and $H = \bigvee\{q_\alpha(H_\alpha) : \alpha \in A\}$, where $V_\alpha, H_\alpha \in \Omega(X_\alpha)$. For $\alpha \leq \beta$, we can assume $\pi_{\alpha\beta}^{-1}(V_\alpha) \subset V_\beta$. Then, for $\alpha \leq \beta \leq \gamma, V_\beta^\alpha \subset V_\gamma^\alpha$ and $\pi_{\alpha\beta}^{-1}(V_\gamma^\alpha) \subset V_\gamma^\beta$. Define $G_\alpha = \bigcup\{V_\gamma^\alpha : \alpha \leq \gamma\}$. Then $G = \bigvee\{q_\alpha(G_\alpha) : \alpha \in A\}$ and, for $\alpha \leq \beta, \pi_{\alpha\beta}^{-1}(G_\alpha) \subset G_\beta$ and $G_\beta^\alpha = G_\alpha$. To prove the equality, if $\pi_{\alpha\beta}^{-1}(x) \subset G_\beta$, by compactness of $\pi_{\alpha\beta}^{-1}(x), \pi_{\alpha\beta}^{-1}(x) \subset V_\gamma^\beta$ for some $\gamma \geq \beta$. Hence $\pi_{\alpha\beta}^{-1}(x) \subset V_\gamma$ so that $x \in V_\gamma^\alpha \subset G_\alpha$. Write $D_\alpha = X_\alpha - G_\alpha$. Then $\pi_{\alpha\beta}(D_\beta) \subset D_\alpha$ and, as $G < H$, there is μ in A such that $H_\mu \cap D_\mu \neq \emptyset$. Hence H_μ contains a point z such that $\pi_{\mu\alpha}^{-1}(z) \cap D_\alpha \neq \emptyset$ for $\mu \leq \alpha$, otherwise, $z = G_\alpha^\mu = G_\mu$.

Consider now the family $\mathcal{F}$ consisting of all $(F_\alpha) \in \Pi Y_\alpha$, where Y_α is the collection of all closed subsets of D_α with $\pi_{\mu\alpha}^{-1}(z) \cap F_\alpha \neq \emptyset$ and $\pi_{\alpha\beta}(F_\beta) \subset F_\alpha$ for $\mu \leq \alpha \leq \beta$. $\mathcal{F}$ is partially ordered by declaring $(E_\alpha) \leq (F_\alpha)$ iff $F_\alpha \subset E_\alpha$ for $\mu \leq \alpha$. By compactness of $\pi_{\mu\alpha}^{-1}(z)$, each chain in $\mathcal{F}$ has an upper bound and, by Zorn's lemma, $\mathcal{F}$ contains a maximal element (F_α). It is readily seen that $z \in F_\mu, F_\alpha$ is irreducible, $F_\alpha \cap G_\alpha = \emptyset$ and

$\pi_{\alpha\beta}(F_\beta) = F_\alpha$ for $\mu \leq \alpha \leq \beta$. If each X_α is sober, let x_α be the unique point with closure F_α. Then, $x_\mu \in H_\mu$, and, because $\pi_{\alpha\beta}(F_\beta) = F_\alpha, \pi_{\alpha\beta}(x_\beta) = x_\alpha$ for $\mu \leq \alpha \leq \beta$. In case $A = N$, we let $x_\mu = z$ and, using induction, we pick points x_α in F_α with $\pi_{\alpha\beta}(x_\beta) = x_\alpha$ for $\mu \leq \alpha \leq \beta$. In both cases, for $\mu \not\leq \gamma$, we let $x_\gamma = \pi_{\gamma\beta}(x_\beta)$ for any $\beta \geq \gamma, \mu$. Then (x_α) is a point of $r(H) - r(G)$. Hence r is injective as wanted.

The following two results are immediate consequences of theorems 1 and 4. For Tychnoff spaces, these were first proved by Katuta [9, theorem 1.1].

Proposition 11. *Let X be the limit space of an inverse system $(X_\alpha, \pi_{\alpha\beta}, A)$ of sober κ-paracompact normal topological spaces and perfect bonding maps, where $|A| \leq \kappa$ and $dimX_\alpha \leq n$ for each α in A. Then X is κ-paracompact normal and $dimX \leq n$.*

Proposition 12. *Let X be the limit space of an inverse sequence (X_i, π_{ij}, N) of countably paracompact normal topological spaces and perfect bonding maps such that $dimX_i \leq n$ for each i in N. Then X is countably paracompact normal and $dimX \leq n$.*

The frames $\Omega(X \times Y)$ and $\Omega(X) \oplus \Omega(Y)$ are isomorphic if X is locally compact (i.e. compact neighbourhoods form a base) or if both factors are Čech-complete [5, theorems 8 and 10]. In fact, suppose X is locally compact with $wX \leq \kappa$. Let $\{G_\alpha : \alpha \in A\}$, where $|A| \leq \kappa$, be a base of X, and let q, r be the canonical maps from $\Omega(X), \Omega(Y)$, respectively, into their sum in κ**Frm**. Then every element of the sum is of the form $\bigvee\{q(G_\alpha) \wedge r(H_a) : \alpha \in A\}$, where H_α is open in Y, and the proof of [5, theorem 8] can be adjusted to show that $\Omega(X) \oplus \Omega(Y)$ and $\Omega(X \times Y)$ are isomorphic. We therefore have the following corollaries of theorem 2. Proposition 13 is due to Pasynkov [13].

Proposition 13. *For non-empty spaces X and Y that are Čech-complete, paracompact and normal,*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

Proposition 14. *Let X and Y be non-empty, κ-paracompact and normal spaces with X locally compact and $wX \leq \kappa$. Then $X \times Y$ is κ-paracompact and normal and*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

That $X \times Y$ is κ-paracompact and normal under the additional conditions that both spaces are Hausdorff and X is compact is due to Morita [11]. The product formula holds if both spaces are Tychonoff and one of them is locally compact and paracompact [12, theorem 1].

REFERENCES

1. B. Banaschewski and C. Gilmour, *Stone-Čech compactification and dimension theory for regular σ-frames*, J. London Math. Soc., **(2) 39** (1989), 1-8.
2. M. G. Charalambous, *Dimension theory for σ-frames*, J. London Math. Soc., **(2) 8** (1974), 149-160.
3. M. G. Charalambous, *The dimension of inverse limit and N-compact spaces*, Proc. Amer. Math. Soc., **85** (1982), 648-652.
4. C. H. Dowker and D. Strauss, *Paracompact frames and closed maps*, Symp. Math., **16** (1975), Convegno di Topologia Isiemistica e Generale, 93-116.
5. C. H. Dowker and D. Strauss, *Sums in the category of frames*, Houston J. Math., **3** (1977), 17-32.
6. R. Engelking, Dimension theory, Polish Scientific Publishers, Warsaw, 1978.
7. J. R. Isbell, *Atomless parts of spaces*, Math. Scand., **31** (1972), 5-32.
8. P. T. Johnstone, Stone spaces, Cambridge University Press, 1982.
9. Y. Katuta, *On the covering dimension of inverse limits*, Proc. Amer. Math. Soc., **84** (1982), 588-592.
10. J. J. Madclen, *κ-frames*, J. Pure Appl. Algebra, **70** (1991), 107 -127.
11. K. Morita, *Paracompactness and product spaces*, **50** (1961), 223-236.
12. K. Morita, *On the dimension of the product of Tychonoff spaces*, General Topology and its Appls., **3** (1973), 125-133.
13. B. A. Pasynkov, *On the dimension of products of normal spaces*, Soviet Math. Dokl., **14** (1973), 530-533.

14. B. A. Pasynkov, *Factorization theorems in dimension theory*, Russian Math. Surveys, **36** (1981),175-209.
15. B. A. Pasynkov, On dimension theory, Aspects of topology, edited by I. M. James and E. H. Kronheimer, Cambridge University Press, 1985, 227-250.
16. A. R. Pears, Dimension theory of general spaces, Cambridge University Press, 1976.
17. T. C. Przymusiński, Product spaces, Surveys in general topology, edited by G. M. Reed, Academic Press, 1980, 399-429.

University of the Aegean
Karlovassi 83200, Samos GREECE