Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

Topology Proceedings
Vol. 20, 1995

ELEMENTARY SUBMODELS AND CARDINAL FUNCTIONS

ALESSANDRO FEDELI (*) AND STEPHEN WATSON (**)

ABSTRACT. The purpose of this article is to illustrate the use of elementary submodels in the theory of cardinal functions.

1. INTRODUCTION AND PRELIMINARIES

In this paper we prove many classical results on cardinal functions in the language of Skolem functions or elementary submodels. This language allows the main ideas to emerge from what would otherwise be a mass of technical details. The key for such applications is the following weak form of the Löwenheim-Skolem theorem (see [W2] Proposition 1, [Ku] Theorem 7.8 and [Je] Lemma 11.2).

Proposition 1. *Let $\phi(x, v_0, ..., v_n)$ be a formula of set-theory with free variables x and the v_i's. If A is any set, then there is a set $\mathcal{M} \supset A$ such that $|\mathcal{M}| \leq |A| + \omega$ and, whenever there are $m_0, ..., m_n \in \mathcal{M}$ such that there is some x such that $\phi(x, m_0, ..., m_n)$, then there is some $x \in \mathcal{M}$ such that $\phi(x, m_0, ..., m_n)$ (we say that $\mathcal{M}$ reflects the formula $\exists x \phi$).*

We can also find a single $\mathcal{M}$ which works for finitely many formulas simultaneously.

(*) Supported by the Consiglio Nazionale delle Ricerche, Italy
(**) Supported by the Natural Sciences and Engineering Research Council of Canada.

Proof: Construct inductively a sequence of sets $\{M_i : i \in \omega\}$ so that $M_0 = A$, $(\forall i \in \omega)\ M_{i+1} \supset M_i$, $|M_i| \leq |M_0| + \omega$ and so that , whenever, there are $m_0, ..., m_n \in M_i$ such that there is some x such that $\phi(x, m_0, ..., m_n)$ then there is some $x \in M_{i+1}$ such that $\phi(x, m_0, ..., m_n)$. Now let $\mathcal{M} = \bigcup\{M_i : i \in \omega\}$, then $|\mathcal{M}| \leq |A| + \omega$ and $\mathcal{M}$ reflects the formula $\exists x\phi$.

An obvious modification of the above proof gives a set $\mathcal{M}$ which works for finitely many formulas simultaneously.

Corollary 2. *There are three formulas so that, if $\mathcal{M}$ satisfies Proposition 1 for these formulas and a set A, then every finite subset of $\mathcal{M}$ is an element of $\mathcal{M}$.*

Proof: Let us consider the following formulas :

$\phi_1(x, v_0)$ says that $x = \{v_0\}$; $\phi_2(x, v_0, v_1)$ says that $x = v_0 \cup v_1$, $\phi_3(x)$ says that $\forall w \in x (w \neq w)$.

Now let $\mathcal{M}$ be a set such that $A \subset \mathcal{M}$, $|\mathcal{M}| \leq |A| + \omega$ and $\mathcal{M}$ reflects $\exists x\phi_1$, $\exists x\phi_2$, $\exists x\phi_3$.

We claim that every finite subset of $\mathcal{M}$ is an element of $\mathcal{M}$. By induction on n, we show that $\forall F \subset \mathcal{M}\ (|F| = n \rightarrow F \in \mathcal{M})$.

If $n = 0$ then $F = \emptyset$ and $F \in \mathcal{M}$ (since $\mathcal{M}$ reflects $\exists x\phi_3$). If we know it for n and $F \subset \mathcal{M}$ has $n+1$ elements, let $m \in F$, then $\{m\} \in \mathcal{M}$ (since $\mathcal{M}$ reflects $\exists x\phi_1$) and $F \setminus \{m\} \in \mathcal{M}$. Now let $m_0 = \{m\}$, $m_1 = F \setminus \{m\}$, so there is some x such that $\phi_2(x, m_0, m_1)$, since $\mathcal{M}$ reflects $\exists x\phi_2$ there exists some $z \in \mathcal{M}$ such that $\phi_2(z, m_0, m_1)$, i.e. $x = z = \{m\} \cup (F \setminus \{m\}) = F \in \mathcal{M}$.

Proposition 3. *Let κ be an infinite cardinal number. If A is a set such that $|A| \leq 2^\kappa$ and ϕ is a formula of set-theory then there is a set $\mathcal{M}$ such that $A \subset \mathcal{M}$, $|\mathcal{M}| = 2^\kappa$, $\mathcal{M}$ reflects $\exists x\phi$ and moreover $\mathcal{M}$ is closed under κ-sequences (i.e. $[\mathcal{M}]^{\leq\kappa} \subset \mathcal{M}$).*

Proof: Let $\mathcal{M}_0 = A$ and for every $\alpha \in (0, \kappa^+)$ let $\mathcal{M}_\alpha$ be a set such that $\mathcal{M}_\beta \cup [\mathcal{M}_\beta]^{\leq\kappa} \subset \mathcal{M}_\alpha\ \forall\beta \in \alpha$, $|\mathcal{M}_\alpha| \leq 2^\kappa$ and $\mathcal{M}_\alpha$

reflects $\exists x\phi$. Let $\mathcal{M} = \bigcup_{\alpha<\kappa^+} \mathcal{M}_\alpha$, clearly $A \subset \mathcal{M}$, $|\mathcal{M}| = 2^\kappa$ and $\mathcal{M}$ reflects $\exists x\phi$ (let $m_0, ..., m_n \in \mathcal{M}$ such that there is some x such that $\phi(x, m_0, ..., m_n)$, then there is an $\alpha \in (0, \kappa^+)$ such that $m_0, ..., m_n \in \mathcal{M}_\alpha$, $\mathcal{M}_\alpha$ reflects $\exists x\phi$ so there is some $x \in \mathcal{M}_\alpha \subset \mathcal{M}$ such that $\phi(x, m_0, ..., m_n)$.

It remains to show thaat $\mathcal{M}$ is closed under κ-sequences. Let $S = \{m_\lambda : \lambda \in \Lambda\} \subset \mathcal{M}$ and $|\Lambda| \leq \kappa$. Since there is a $\beta \in \kappa^+$ such that $m_\lambda \in \mathcal{M}_\beta$ for every $\lambda \in \Lambda$ it follows that $S \in [\mathcal{M}_\beta]^{\leq\kappa} \subset \mathcal{M}_{\beta+1}$. Hence $S \subset \mathcal{M}$.

Another result which is used frequently in elementary submodels arguments is the following

Lemma 4. *Let κ be an infinite cardinal number. There are two formulas so that, if $\mathcal{M}$ satisfies Proposition 1 for these formulas and a set A and if $\kappa \subset A$, $\kappa \in A$, $E \in \mathcal{M}$ and $|E| \leq \kappa$, then $E \subset \mathcal{M}$.*

Proof: The formula $\phi_0(x, v_0, v_1)$ says that x is an onto mapping from v_0 to v_1 and the formula $\phi_1(x, v_0, v_1, v_2, v_3)$ says that v_3 is an onto mapping from v_0 to v_1, $v_2 \in v_0$ and $x = v_3(v_2)$. Suppose $\mathcal{M}$ satisfies Proposition 1 for ϕ_0, ϕ_1, and A. If $E \in \mathcal{M}$ and $|E| \leq \kappa$, since $\kappa \in \mathcal{M}$ we can apply Proposition 1 to ϕ_0 to obtain $\pi \in \mathcal{M}$ which is an onto mapping from κ to E (let $m_0 = \kappa$ and $m_1 = E$, so there is some x such that $\phi_0(x, m_0, m_1)$, $\mathcal{M}$ reflects $\exists x\phi_0$ so there is some $\pi \in \mathcal{M}$ such that $\phi_0(\pi, m_0, m_1)$).

Now if $e \in E$ and $\pi(\alpha) = e$, then we can apply Proposition 1 to ϕ_1 to obtain $x \in \mathcal{M}$ such that $x = \pi(\alpha)$ (let $m_0 = \kappa$, $m_1 = E$, $m_2 = \alpha$ and $m_3 = \pi$, so there is some x such that $\phi_1(x, m_0, m_1, m_2, m_3)$, since $\mathcal{M}$ reflects $\exists x\phi_1$ there is some $x \in \mathcal{M}$ such that $\phi_1(x, m_0, m_1, m_2, m_3)$). Now $e = \pi(\alpha) = x \in \mathcal{M}$. Since e was an arbitrary element of E, we have shown that $E \subset \mathcal{M}$.

Now let us see how we can use Proposition 1 (we refer the reader to the Introduction of [W2] for more extensive informations).

Example 5. ([St]) *Let X be countably compact and B any subset of X of cardinality $\leq 2^{\aleph_0}$. Then there exists a countably compact subset G of X such that $B \subset G$ and $|G| \leq 2^{\aleph_0}$.*

Proof: ("formal") Let $\phi(x, v_0, v_1, v_2)$ be the formula

$$x \in v_0 \wedge \forall w \in v_1[(x \in w) \rightarrow (|w \cap v_2| = |v_2|)]$$

or, in plain language, "x is a complete accumulation point of v_2 in the topological space (v_0, v_1)".

Let $A = B \cup \{X, \tau\}$ (where τ is the topology on X), apply Proposition 3 to obtain a set $\mathcal{M}$ such that $A \subset \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, $\mathcal{M}$ reflects the formula $\exists x\phi$ and is closed under ω-sequences. We claim that $G = \mathcal{M} \cap X$ has the required properties. Clearly $B \subset G$ and $|G| \leq 2^{\aleph_0}$, it remains to show that G is countably compact. Let $S \in [G]^\omega$, then $S \in \mathcal{M}$ (since $S \in [\mathcal{M}]^\omega$ and $\mathcal{M}$ is closed under ω-sequences). Now let $m_0 = X$, $m_1 = \tau$, $m_2 = S$, by the countable compactness of X it follows that there exists some x such that $\phi(x, m_0, m_1, m_2)$. Since $m_0, m_1, m_2 \in \mathcal{M}$ and $\mathcal{M}$ reflects $\exists x\phi$ there is some $x \in \mathcal{M}$ such that $\phi(x, m_0, m_1, m_2)$, i.e. there exists a complete accumulation point of S in G, therefore G is countably compact.

Proof: ("in practice") Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X, each element of B and which is closed under ω-sequences. Let $G = \mathcal{M} \cap X$. We show that G is countably compact. Let S be a countably infinite subset of G. Since $\mathcal{M}$ is closed under ω-sequences we know $S \in \mathcal{M}$. X is countably compact so there is a point $x \in X$ which is a complete accumulation point of S in X. Hence, by elementarity, there is some $x \in X \cap \mathcal{M} = G$ which is a complete accumulation point of S in X (and hence in G).

Example 6. ([Po]) *Every first countable Hausdorff space with a dense subset of cardinality $\leq 2^{\aleph_0}$ has cardinality $\leq 2^{\aleph_0}$.*

Proof: ("formal") Let $\phi(x, v_0, v_1, v_2)$ be the formula

$$x \in v_2 \wedge \forall w \in v_0(x \in w \rightarrow |v_1 \setminus w| < \aleph_0)$$

or, in plain language, "all but finitely many points of v_1 lie inside any open set in (v_2, v_0) which contains x". Let D be a dense subset of X such that $|D| \le 2^{\aleph_0}$, and let $A = D \cup \{X, \tau\}$. Apply Proposition 3 to obtain a set $\mathcal{M}$ such that $A \subset \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, $\mathcal{M}$ reflects $\exists x\phi$, and it is closed under ω-sequences. We claim that $X \cap \mathcal{M}$ is closed. If $x \in \overline{X \cap \mathcal{M}}$ then there is a sequence $\{x_n\}_{n\in\omega}$ in $X \cap \mathcal{M}$ such that $x_n \to x$. Since $\{x_n : n \in \omega\} \in [\mathcal{M}]^{\le\omega}$ and $\mathcal{M}$ is closed under ω-sequences, it follows that $\{x_n : n \in \omega\} \in \mathcal{M}$. Now let $m_0 = \tau$, $m_1 = \{x_n\}_{n\in\omega}$ and $m_2 = X$. Since there is some x such that $\phi(x, m_0, m_1, m_2)$ (i.e., $x_n \to x$), $m_0, m_1, m_2 \in \mathcal{M}$, and $\mathcal{M}$ reflects $\exists x\phi$, there is also some $y \in \mathcal{M}$ such that $\phi(y, m_0, m_1, m_2)$, i.e., $x_n \to y$. We know that X is T_2 and so $y = x \in X \cap \mathcal{M}$. Therefore $X = \overline{D} \subset \overline{X \cap \mathcal{M}} = X \cap \mathcal{M}$ and $X \subset \mathcal{M}$, hence $|X| \le 2^{\aleph_0}$.

Proof: ("in practice") Let D be a dense subset of X such that $|D| \le 2^{\aleph_0}$. Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X, each element of D and which is closed under ω-sequences. We show that $X \cap \mathcal{M}$ is closed. If $x \in \overline{X \cap \mathcal{M}}$, then there is a sequence $\{x_n\}_{n\in\omega} \subset X \cap \mathcal{M}$ such that $x_n \to x$. Clearly $\overline{\{x_n\}}_{n\in\omega} = \{x_n\}_{n\in\omega} \cup \{x\}$. Now $\overline{\{x_n\}}_{n\in\omega}$ is definable in $\mathcal{M}$ (see the remark below) so $\overline{\{x_n\}}_{n\in\omega} \in \mathcal{M}$. Therefore $\overline{\{x_n\}}_{n\in\omega} \subset \mathcal{M}$ (apply Lemma 4), so $x \in X \cap \mathcal{M}$. Hence $X = \overline{D} \subset \overline{X \cap \mathcal{M}} = X \cap \mathcal{M}$, therefore $X \subset \mathcal{M}$ and $|X| \le 2^{\aleph_0}$.

Remark 7. In the above proof we said that $\overline{\{x_n\}}_{n\in\omega} \in \mathcal{M}$ because it is "definable in $\mathcal{M}$". This means, in our case, that $\mathcal{M}$ was implicitly chosen to reflect also the formula $\exists x\psi$ where $\psi(x, v_0, v_1, v_2)$ is

$$x \subset v_2 \wedge \forall w[(w \in x) \leftrightarrow (\forall u(u \in v_1 \wedge w \in u) \to (u \cap v_0 \neq \emptyset))]$$

or, in plain language, "x is the closure of v_0 in the topological space (v_2, v_1)". Now let $m_0 = \{x_n\}_n$, $m_1 = \tau$, and $m_2 = X$. The sentence $\exists x\ \psi(x, m_0, m_1, m_2)$ says that there is a subset x of X which is the closure of $\{x_n\}_n$. Since $m_0, m_1, m_2 \in \mathcal{M}$ ($m_0 \in \mathcal{M}$ because $\mathcal{M}$ is closed under ω-sequences) and $\mathcal{M}$

reflects $\exists x\psi$, there is some $z \in \mathcal{M}$ such that $\psi(z, m_0, m_1, m_2)$. So $x = z = \{x_n\}_n \in \mathcal{M}$.

Henceforth, in all the proofs, we shall assume the basic assumption that any set which is definable in $\mathcal{M}$ and which we will need in the proof is in fact already in $\mathcal{M}$. Moreover, most of our results will use elementary submodels of size $2^{\aleph_0}$, it is worth noting that many of these results (and proofs) generalize to higher cardinals.

For notation and terminology we refer the reader to [Ho] and [E]. For other applications of elementary submodels the reader is referred to [Do1], [Do2], [W1], [W2], and [FW].

2. The results

First we state some results which will allow us to simplify a certain number of proofs.

Lemma 8. *Let X be a T_1-space and let $\mathcal{M}$ be an elementary submodel which reflects sufficiently many formulas and such that $X \in \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, and which is closed under ω-sequences. If $\psi(X) \leq 2^{\aleph_0}$ and $X \cap \mathcal{M}$ is Lindelöf, then $|X| \leq 2^{\aleph_0}$.*

Proof: For each $x \in X$ let $\mathcal{B}_x$ be a pseudobase for x such that $|\mathcal{B}_x| \leq 2^{\aleph_0}$. By elementarity it follows that $\mathcal{B}_x \in \mathcal{M}$ for every $x \in X \cap \mathcal{M}$, therefore $\mathcal{B}_x \subset X \cap \mathcal{M}$ for every $x \in X \cap \mathcal{M}$ (apply Lemma 4). We claim that $X \subset \mathcal{M}$. Suppose not, choose a point $x \in X \setminus \mathcal{M}$. For every $y \in X \cap \mathcal{M}$ there is a $B_y \in \mathcal{B}_y$ such that $x \notin B_y$. $X \cap \mathcal{M}$ is Lindelöf so there is a $C \subset X \cap \mathcal{M}$ such that $|C| \leq \aleph_0$ and $\{B_y\}_{y \in C}$ covers $X \cap \mathcal{M}$. Since $\{B_y : y \in C\} \in \mathcal{M}$ (observe that $\mathcal{M}$ is closed under ω-sequences) it must cover X (suppose not, then there is a $p \in X$ such that $p \notin \cup_{y \in C} B_y$, so by elementarity there is a point in $X \cap \mathcal{M}$ which is not covered by $\{B_y\}_{y \in C}$), a contradiction.

Observe that every hereditarily Lindelöf T_2-space has countable pseudocharacter so an immediate consequence of the above lemma is the de Groot inequality : "Every hereditarily Lindelöf T_2-space has cardinality not greater than $2^{\aleph_0}$".

Lemma 9. *Let X be a topological space and let $\mathcal{M}$ be an elementary submodel which reflects sufficiently many formulas and such that $X \in \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, and which is closed under ω-sequences. If $t(X) = \aleph_0$ and $|\overline{A}| \leq 2^{\aleph_0}$ for every countable subset A of X, then $X \cap \mathcal{M}$ is closed in X. Moreover if X is a space such that $|\overline{A}| \leq 2^{\aleph_0}$ for every $A \subset X$ such that $|A| \leq 2^{\aleph_0}$, κ is an infinite regular cardinal such that $\kappa \leq 2^{\aleph_0}$, and for every $A \subset X$ and $x \in \overline{A}$ there is a $B \subset A$ such that $B < \kappa$ and $x \in \overline{B}$, then there is an elementary submodel $\mathcal{M}$ such that $X \in \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, which is closed under ω-sequences and such that $X \cap \mathcal{M}$ is closed in X.*

Proof: Let $x \in \overline{X \cap \mathcal{M}}$. By hypothesis there is an $A \subset X \cap \mathcal{M}$ such that $|A| \leq \aleph_0$, $x \in \overline{A}$ and $|\overline{A}| \leq 2^{\aleph_0}$. Since $A \in \mathcal{M}$ it follows that $\overline{A} \in \mathcal{M}$, so $\overline{A} \subset \mathcal{M}$ (apply Lemma 4) and therefore $x \in X \cap \mathcal{M}$. For the second part of the lemma consider a chain of elementary submodels $\{\mathcal{M}_\alpha : \alpha < \kappa\}$ such that $X \in \mathcal{M}_0$, $\mathcal{M}_\alpha \in \mathcal{M}_{\alpha+1}$, $|\mathcal{M}_\alpha| = 2^{\aleph_0}$ and $\mathcal{M}_\alpha$ is closed under ω-sequences for every $\alpha < \kappa$. Let $\mathcal{M} = \cup\{\mathcal{M}_\alpha : \alpha < \kappa\}$; $\mathcal{M}$ is the required elementary submodel. Let us show that $X \cap \mathcal{M}$ is closed. If $x \in \overline{X \cap \mathcal{M}}$ then there is a $B \subset X \cap \mathcal{M}$ such that $|B| < \kappa$, $x \in \overline{B}$, and $|\overline{B}| \leq 2^{\aleph_0}$. Since $|B| < \kappa$ there is an $\alpha < \kappa$ such that $B \subset \mathcal{M}_\alpha$, so $\overline{B} \subset \overline{\mathcal{M}_\alpha \cap X} \in \mathcal{M}_{\alpha+1}$. Since $|\overline{\mathcal{M}_\alpha \cap X}| \leq 2^{\aleph_0}$ it follows that $\overline{\mathcal{M}_\alpha \cap X} \subset \mathcal{M}_{\alpha+1}$, hence $x \in \mathcal{M} \cap X$.

Theorem 10. *Let X be a space and let $\mathcal{M}$ be an elementary submodel such that $X \in \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, $\mathcal{M}$ is closed under ω-sequences and $X \cap \mathcal{M}$ is closed in X. Then for every $A \subset X$ such that $|A| \leq 2^{\aleph_0}$, it follows that $|\overline{A}| \leq 2^{\aleph_0}$.*

Proof: Suppose not, then there is a subset A of X such that $|A| \leq 2^{\aleph_0}$ and $|\overline{A}| > 2^{\aleph_0}$. By elementarity there is a subset A of X such that $A \in \mathcal{M}$, $|A| \leq 2^{\aleph_0}$ and $|\overline{A}| > 2^{\aleph_0}$. Since $A \subset \mathcal{M}$

it follows that $\overline{A} \subset \overline{X \cap \mathcal{M}} = X \cap \mathcal{M} \subset \mathcal{M}$. Hence $|\overline{A}| \leq 2^{\aleph_0}$, a contradiction.

Problem. Give necessary and sufficient conditions on a space X for the existence of an elementary submodel $\mathcal{M}$ such that $X \cap \mathcal{M}$ is closed in X.

Corollary 11. *If X is a Lindelöf T_1-space, $t(X) = \aleph_0$, $\psi(X) \leq 2^{\aleph_0}$, and $|\overline{A}| \leq 2^{\aleph_0}$ for every countable subset A of X then $|X| \leq 2^{\aleph_0}$.*

Recall that if X is a Hausdorff space then $\psi_c(X) = \aleph_0$ means that for every $x \in X$ there is a family $\{G_n\}_{n\in\omega}$ of open sets such that $\{x\} = \cap_{n\in\omega} G_n = \cap_{n\in\omega} \overline{G_n}$. Note that if X is a Lindelöf T_2-space such that $\psi(X) = \aleph_0$ then $\psi_c(X) = \aleph_0$.

Proposition 12. *Let X be a Hausdorff space.*

(i) *If $\psi_c(X)d(X) = \aleph_0$ then $|X| \leq 2^{\aleph_0}$.*
(ii) *If $\psi_c(X) = \aleph_0$ then $|\overline{A}| \leq 2^{\aleph_0}$ for every countable subset A of X.*

Proof: (i) Let $D \subset X$ such that $\overline{D} = X$ and $|D| \leq \aleph_0$. Take an elementary submodel $\mathcal{M}$ such that $|\mathcal{M}| = 2^{\aleph_0}$, $X \in \mathcal{M}$, $D \subset \mathcal{M}$ and which is closed under ω-sequences. We show that $X \subset \mathcal{M}$. Let $x \in X$ and take a family $\{G_n\}_{n\in\omega}$ of open sets such that $\{x\} = \cap_{n\in\omega} G_n = \cap_{n\in\omega} \overline{G_n}$. It is enough to observe that $\{x\} = \cap_{n\in\omega} \overline{G_n \cap D} \in \mathcal{M}$ (notice that we have shown that the points of X are definable in $\mathcal{M}$).
(ii) Let A be a countable subset of X. Since $\psi_c(\overline{A})d(\overline{A}) = \aleph_0$ it follows that $|\overline{A}| \leq 2^{\aleph_0}$.

Remark 13. Observe that the well-known Arhangel'skiĭ result: "if X is a Lindelöf T_2-space such that $t(X)\psi(X) = \aleph_0$, then $|X| \leq 2^{\aleph_0}$" is an easy consequence of the above results. In fact by Proposition 12 (ii), it follows that $|\overline{A}| \leq 2^{\aleph_0}$ for every countable subset A of X, so applying corollary 11 we have $|X| \leq 2^{\aleph_0}$.

Recall that a cover $\mathcal{A}$ of a set E is separating if for every $p \in E$, $\cap\{A : A \in \mathcal{A}, p \in A\} = \{p\}$. The point-separating weight of a T_1-space X, denoted $psw(X)$, is the smallest infinite cardinal κ such that X has a separating open cover $\mathcal{V}$ with $ord(p, \mathcal{V}) \leq \kappa$ for every $p \in X$.

The next proposition collects two results having a common feature: the elementary submodel $\mathcal{M}$ can be taken in such a way that $X \cap \mathcal{M}$ is dense in X.

Proposition 14. (i) ([Sa2]) *If X is a regular space with the countable chain condition and the π-character of X is $\leq 2^{\aleph_0}$ then $d(X) \leq 2^{\aleph_0}$.*
(ii) *[Ch] If X is a Lindelöf T_1-space and $psw(X) \leq 2^{\aleph_0}$, then $d(X) \leq 2^{\aleph_0}$.*

Proof: (i) Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X and which is closed under ω-sequences. We claim that $X \cap \mathcal{M}$ is dense in X. Suppose not, choose a non-empty open set R such that $\overline{R} \cap (X \cap \mathcal{M}) = \emptyset$. For each $y \in X \cap \mathcal{M}$, let $\mathcal{B}_y$ be a local π-base at y such that $\mathcal{B}_y \subset \mathcal{M}$. Let $\mathcal{G} = \{V : V \in \mathcal{B}_y,\ y \in X \cap \mathcal{M},\ V \cap R = \emptyset\}$, clearly $\mathcal{G} \subset \mathcal{M}$. X is c.c.c. so there is a $\mathcal{W} \subset \mathcal{G}$ such that $\cup\mathcal{G} \subset \overline{\cup\mathcal{W}}$ and $|\mathcal{W}| \leq \aleph_0$ ([Ho], 3.4). $\mathcal{M}$ is closed under ω-sequences so $\mathcal{W} \in \mathcal{M}$, therefore $W = \cup\mathcal{W} \in \mathcal{M}$ and hence $\overline{W} \in \mathcal{M}$. Now $X \cap \mathcal{M} \subset \overline{W}$ so $\overline{W} = X$, a contradiction since $W \cap R = \emptyset$.
(ii) Let $\mathcal{B}$ be a separating open cover of X such that $ord(x, \mathcal{B}) \leq 2^{\aleph_0}$ for every $x \in X$. Let $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ and let $f : X \to \mathcal{P}(\mathcal{B})$ be the map defined by $f(x) = \mathcal{B}_x$ for every $x \in X$. Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains $X, \mathcal{B}, f$ and which is closed under ω-sequences. Observe that $\mathcal{B}_x \subset \mathcal{M}$ for every $x \in X \cap \mathcal{M}$ (since $\mathcal{B}_x \in \mathcal{M}$ and $|\mathcal{B}_x| \leq 2^{\aleph_0}$). Now $X \cap \mathcal{M}$ is dense in X. Suppose not, choose $x \in X \setminus \overline{X \cap \mathcal{M}}$, for every $y \in \overline{X \cap \mathcal{M}}$ there is a $B_y \in \mathcal{B}_y$ such that $x \notin B_y$. Since $B_y \cap (X \cap \mathcal{M}) \neq \emptyset$ there is a $p \in X \cap \mathcal{M}$ such that $B_y \in \mathcal{B}_p \subset \mathcal{M}$, so $B_y \in \mathcal{M}$. Now $\overline{X \cap \mathcal{M}}$ is Lindelöf so there is a $C \subset \overline{X \cap \mathcal{M}}$ such that $|C| \leq \aleph_0$ and $\{B_y\}_{y \in C}$ covers $\overline{X \cap \mathcal{M}}$, since $\{B_y\}_{y \in C} \in \mathcal{M}$ it must cover X, which is a contradiction.

Corollary 15. [Ar] *If X is a compact sequential c.c.c. T_2-space then $|X| \leq 2^{\aleph_0}$.*

Proof: X has countable tightness so $\pi_\chi(X) = \aleph_0$ (see e.g., Theorem 7.13 in [Ho]). By Proposition 14 (i) it follows that there is an elementary submodel $\mathcal{M}$ such that $X \in \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, which is closed under ω-sequences, and $\overline{X \cap \mathcal{M}} = X$. Since X is Hausdorff and sequential we can apply Lemma 9 and so $X \cap \mathcal{M}$ is closed in X. Therefore $X \subset \mathcal{M}$ and $|X| \leq 2^{\aleph_0}$.

Problem. Characterize those X for which there is an elementary submodel $\mathcal{M}$ for which $X \cap \mathcal{M}$ is dense in X.

The next four results show a different aspect of the use of elementary submodels (see also Example 5).

Theorem 16. *Let X be a normal space of countable tightness and let $\mathcal{M}$ be an elementary submodel which reflects sufficiently many formulas and such that $X \in \mathcal{M}$, $|\mathcal{M}| = 2^{\aleph_0}$, and which is closed under ω-sequences. Then $X \cap \mathcal{M}$ is C^*-embedded in X.*

Proof: Let C, F be two subsets of $X \cap \mathcal{M}$ which are completely separated in $X \cap \mathcal{M}$. We claim that C and F are completely separated in X (and hence $X \cap \mathcal{M}$ is C^*-embedded in X). Clearly $cl_{X\cap\mathcal{M}}(C) \cap cl_{X\cap\mathcal{M}}(F) = \emptyset$.

Let us show that $cl_X(C) \cap cl_X(F) = \emptyset$. Suppose there is a point $x \in cl_X(C) \cap cl_X(F)$. As X has countable tightness there are sets $A \in [C]^{\leq\omega}$ and $B \in [F]^{\leq\omega}$ such that $x \in cl_X(A) \cap cl_X(B)$. Since $A, B \in \mathcal{M}$, by elementarity there is a $x \in cl_X(A) \cap cl_X(B) \cap \mathcal{M}$, so $x \in cl_{X\cap\mathcal{M}}(C) \cap cl_{X\cap\mathcal{M}}(F)$, which is a contradiction. So $cl_X(C) \cap cl_X(F) = \emptyset$ and by the normality of X it follows that C and F are completely separated in X.

Corollary 17. [Gr2] *Let X be a compact T_2-space of countable tightness such that $d(X) \leq 2^{\aleph_0}$. Then there is a countably compact normal subset Y of X such that $|Y| \leq 2^{\aleph_0}$ and $\beta(Y) = X$.*

Proof: Let D be a dense subset of X such that $|D| \leq 2^{\aleph_0}$. Take an elementary submodel $\mathcal{M}$ as in Theorem 16 such that $D \subset \mathcal{M}$ and let $Y = X \cap \mathcal{M}$. Clearly $|Y| \leq 2^{\aleph_0}$ and Y is countably compact (see Example 5). Moreover Y is dense and C^*-embedded in X so it is normal and $\beta(Y) = X$.

Problem. Give necessary and sufficient conditions on a space X for the existence of an elementary submodel $\mathcal{M}$ such that $X \cap \mathcal{M}$ is C^*-embedded in X.

Theorem 18. [Sa1] *If X has countable spread and for every $x \in X$, U_x is an open neighbourhood of x, then there is a countable subset A of X such that $X = \overline{A} \cup \bigcup\{U_x : x \in A\}$.*

Proof: Let $\{\mathcal{M}_\alpha : \alpha \in \omega_1\}$ be an increasing chain of countable elementary submodels such that $X, \{U_x\}_{x\in X} \in \mathcal{M}_0$ and $\mathcal{M}_\alpha \in \mathcal{M}_{\alpha+1}$ for every $\alpha \in \omega_1$. If for each $\alpha \in \omega_1$, $X \cap \mathcal{M}_\alpha$ does not have the required property then take a point $x_\alpha \in \mathcal{M}_{\alpha+1}$ such that $x_\alpha \notin \overline{X \cap \mathcal{M}_\alpha} \cup \bigcup\{U_x : x \in X \cap \mathcal{M}_\alpha\}$ for every $\alpha \in \omega_1$. We claim that $\{x_\alpha : \alpha \in \omega_1\}$ is a discrete subset of X. In fact $x_\alpha \notin \overline{\{x_\beta : \beta < \alpha\}}$ because $\overline{\{x_\beta : \beta < \alpha\}} \subset \overline{X \cap \mathcal{M}_\alpha}$; moreover U_{x_α} is an open neighbourhood of x_α such that $U_{x_\alpha} \cap \{x_\beta : \beta > \alpha\} = \emptyset$, so $x_\alpha \notin \overline{\{x_\beta : \beta > \alpha\}}$ Therefore $\{x_\alpha : \alpha \in \omega_1\}$ is an uncountable discrete subset of X, a contradiction.

Theorem 19. [Sa1] *If X is a Hausdorff space with countable spread then there is a subset S of X such that $|S| \leq 2^{\aleph_0}$ and $X = \cup\{\overline{A} : A \subset S, |A| \leq \aleph_0\}$.*

Proof: Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ such that $X \in \mathcal{M}$ and which is closed under ω-sequences. $X \cap \mathcal{M}$ is the subset with the required properties. Let $x \in X$, we may assume $x \notin X \cap \mathcal{M}$. Since $\psi(X) \leq 2^{\aleph_0}$ ([Ho], 4.11), it follows that for every $y \in X \cap \mathcal{M}$ there is an open neighbourhood U_y of y such that $U_y \in \mathcal{M}$ and $x \notin U_y$. $\{U_y\}_{y\in X\cap\mathcal{M}}$ covers $X \cap \mathcal{M}$ and $X \cap \mathcal{M}$ has countable spread, so there is a countable subset A of $X \cap \mathcal{M}$ such that $\mathcal{U} = \{\overline{A}\} \cup \{U_y\}_{y\in A}$ covers $X \cap \mathcal{M}$. Since $\mathcal{U} \in \mathcal{M}$ it follows that $\mathcal{U}$ must cover X, so $x \in \overline{A}$.

Now we give a series of results showing a way to obtain a bound of the cardinality of some object: the idea is to produce a "suitable" elementary submodel $\mathcal{M}$ having that object as a subset (note that this idea has already been applied in Lemma 8).

Theorem 20. ([Ho]) *If X is c.c.c. and the π-weight of X is $\leq 2^{\aleph_0}$ then $|RO(X)| \leq 2^{\aleph_0}$.*

Proof: Let $\mathcal{B}$ be a π-base for X such that $|\mathcal{B}| \leq 2^{\aleph_0}$. Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X and each member of $\mathcal{B}$, and which is closed under ω-sequences. We show that $RO(X) \subset \mathcal{M}$. Let $R \in RO(X)$ and let $\mathcal{G}_R = \{B \in \mathcal{B} : B \subset R\}$. Now X is c.c.c. so there exists $\mathcal{H}_R \subset \mathcal{G}_R$ such that $\cup\mathcal{G}_R \subset \overline{\cup\mathcal{H}_R}$ and $|\mathcal{H}_R| \leq \aleph_0$. Since $\mathcal{H}_R \subset \mathcal{M}$ and $\mathcal{M}$ is closed under ω-sequences then $\mathcal{H}_R \in \mathcal{M}$, so $H = \overline{\cup\mathcal{H}_R} \in \mathcal{M}$. Since R is open and $\mathcal{B}$ is a π-base we have $R \subset \overline{\cup\mathcal{G}_R}$, therefore $\overline{R} = H$, so $R = \mathrm{Int}(H) \in \mathcal{M}$.

Theorem 21. ([HJ]) *If X is a T_1-space with countable spread and countable pseudocharacter then $|X| \leq 2^{\aleph_0}$.*

Proof: Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X and which is closed under ω-sequences. We claim that $X \subset \mathcal{M}$. Suppose not, choose a point $x \in X - \mathcal{M}$, so there is a family of closed sets $\{C_i\}_{i \in \omega}$ such that $X - \{x\} = \cup C_i$. For every $i \in \omega$ and for every $y \in X \cap \mathcal{M} \cap C_i$ take an open neighbourhood U_y of y such that $U_y \in \mathcal{M}$ and $x \notin U_y$. Then $\mathcal{U}_i = \{U_y\}_{y \in X \cap \mathcal{M} \cap C_i}$ covers $X \cap \mathcal{M} \cap C_i$ and $X \cap \mathcal{M} \cap C_i$ has countable spread, so by Theorem 31 there is a countable subset A_i of $X \cap \mathcal{M} \cap C_i$ such that $\mathcal{V}_i = \{\overline{A_i}\} \cup \{U_y\}_{y \in A_i}$ covers $X \cap \mathcal{M} \cap C_i$. Now let $\mathcal{V} = \cup\{\mathcal{V}_i : i \in \omega\}$; $\mathcal{V}$ covers $X \cap \mathcal{M}$ and $\mathcal{V} \in \mathcal{M}$ so it must cover X, a contradiction.

Theorem 22. ([HJ]) *Any first countable Hausdorff space with the countable chain condition must have cardinality at most $2^{\aleph_0}$.*

Proof: Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X and which is closed under ω-sequences. We show that $X \subset \mathcal{M}$. Suppose not, choose a point $x \in X - \mathcal{M}$ and let $\{U_n\}_{n\in\omega}$ be a countable local base at x. For every $y \in X \cap \mathcal{M}$ let $\mathcal{B}_y$ be a countable local base at y such that $\mathcal{B}_y \in \mathcal{M}$ (so $\mathcal{B}_y \subset \mathcal{M}$). For each $n \in \omega$ let $\mathcal{G}_n = \{U : U \in \mathcal{B}_y$ for some $y \in X \cap \mathcal{M}$, $U \cap U_n = \emptyset\}$, let $\mathcal{F}_n$ be a family such that $\mathcal{F}_n \subset \mathcal{G}_n$, $|\mathcal{F}_n| \leq \omega$ and $\bigcup\mathcal{G}_n \subset \overline{\bigcup\mathcal{F}_n}$. $\mathcal{M}$ is closed under ω-sequences so $\mathcal{F}_n \in \mathcal{M}$, hence $\overline{\bigcup\mathcal{F}_n} \in \mathcal{M}$ for each $n \in \omega$. Let $\mathcal{F} = \{\overline{\bigcup\mathcal{F}_n} : n \in \omega\}$, again $\mathcal{F} \in \mathcal{M}$. Now $X \cap \mathcal{M} \subset \bigcup_{n\in\omega}(\bigcup\mathcal{G}_n) \subset \bigcup_{n\in\omega}(\overline{\bigcup\mathcal{F}_n}) = \bigcup\mathcal{F}$, so $\mathcal{F}$ must cover X, a contradiction.

Theorem 23. ([BGW]) *If X is first countable, weakly Lindelöf and T_4 then $|X| \leq 2^{\aleph_0}$.*

Proof: Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X and which is closed under ω-sequences. Now $X \cap \mathcal{M}$ is closed (by Lemma 9). We claim that $X \subset \mathcal{M}$, suppose not, choose $x \in X - \mathcal{M}$. Since X is regular there is an open set R such that $X \cap \mathcal{M} \subset R$ and $x \notin \overline{R}$. For every $y \in X \cap \mathcal{M}$ take an open neighbourhood U_y of y such that $U_y \in \mathcal{M}$ and $U_y \subset R$. Let $G = \bigcup_{y\in X\cap\mathcal{M}} U_y$, clearly $X \cap \mathcal{M} \subset G$ and $x \notin \overline{G}$. Since X is normal there is an open set L such that $X \cap \mathcal{M} \subset L \subset \overline{L} \subset G$. Thus $\mathcal{G} = \{U_y\}_{y\in X\cap\mathcal{M}} \cup \{X - \overline{L}\}$ covers X, and X is weakly Lindelöf so there is a $C \subset X \cap \mathcal{M}$ such that $|C| \leq \aleph_0$ and $X = \overline{\bigcup_{y\in C} U_y \cup (X - \overline{L})}$. Since $(X \cap \mathcal{M}) \cap \overline{(X - \overline{L})} = \emptyset$ we have $X \cap \mathcal{M} \subset \overline{\bigcup_{y\in C} U_y}$. Let $\mathcal{V} = \{U_y\}_{y\in C}$. Thus $\mathcal{V} \subset \mathcal{M}$ and $|\mathcal{V}| \leq \aleph_0$ so $\mathcal{V} \in \mathcal{M}$, and therefore $\overline{\bigcup\mathcal{V}} \in \mathcal{M}$. Hence $X = \overline{\bigcup\mathcal{V}}$, which is a contradiction.

Lemma 24. *Let X be a T_1-space such that there is a dense subset D of X which does not contain uncountable closed discrete subsets of X and there is a separating open cover $\mathcal{B}$ of X which is point-countable on D. If there is an elementary submodel $\mathcal{M}$ which reflects sufficiently many formulas and such that $|\mathcal{M}| = 2^{\aleph_0}$, $X, \mathcal{B}, D \in \mathcal{M}$, $\mathcal{M}$ is closed under ω-sequences* *and $D \cap \mathcal{M}$ is dense in $X \cap \mathcal{M}$, then $X \cap \mathcal{M}$ is dense in X. Moreover, both $D, \mathcal{B} \subset \mathcal{M}$, so $sw(X) \leq 2^{\aleph_0}$ and $|D| \leq 2^{\aleph_0}$.*

Proof: Observe that $\{B \in \mathcal{B} : y \in B\} \subset \mathcal{M}$ for every $y \in D \cap \mathcal{M}$. Let us show that $D \subset \overline{X \cap \mathcal{M}}$. If not, there is a point $x \in D \setminus \overline{X \cap \mathcal{M}}$. For every $y \in \overline{X \cap \mathcal{M}}$ there is a $B_y \in \mathcal{B}$ such that $y \in B_y$ and $x \notin B_y$. Since $B_y \cap X \cap \mathcal{M} \neq \emptyset$ and $D \cap \mathcal{M}$ is dense in $X \cap \mathcal{M}$ it follows that $B_y \cap D \cap \mathcal{M} \neq \emptyset$, so $B_y \in \mathcal{M}$. Now $\mathcal{G} = \{B_y\}_{y \in \overline{X \cap \mathcal{M}}}$ covers $\overline{X \cap \mathcal{M}}$ and there is a $\mathcal{H} \in [\mathcal{G}]^{\leq \omega}$ such that $\overline{X \cap \mathcal{M}} \cap D \subset \cup \mathcal{H}$ (otherwise it is easy to see that D would contain an uncountable closed discrete subset of X, see [Ho], 9.2). Since $\mathcal{H} \in \mathcal{M}$ it must cover D, a contradiction. So $D \subset \overline{X \cap \mathcal{M}}$ and $\overline{X \cap \mathcal{M}} = X$. Now $B \cap (X \cap \mathcal{M}) \neq \emptyset$ for every $B \in \mathcal{B}$ so $\mathcal{B} \subset \mathcal{M}$ and $sw(X) \leq 2^{\aleph_0}$. Moreover, for every $x \in D$ it follows that $\{x\} = \cap \{B \in \mathcal{B} : x \in B\} \in \mathcal{M}$, hence $D \subset \mathcal{M}$ and $|D| \leq 2^{\aleph_0}$.

A space X is called ω_1-compact if it has countable extent.

Corollary 25. ([Ho]) *If X is an ω_1-compact T_1-space and $psw(X) = \aleph_0$ then $X \leq 2^{\aleph_0}$.*

Problem. Give necessary and sufficient conditions on a space X for the existence of an elementary submodel $\mathcal{M}$ such that if D is a dense subset of X and $D \in \mathcal{M}$ then $D \cap \mathcal{M}$ is dense in $X \cap \mathcal{M}$.

Remark 26. Let X be a space such that $\pi_\chi(X) \leq 2^{\aleph_0}$ and let D be a dense subset of X. If $\mathcal{M}$ is an elementary submodel such that $|\mathcal{M}| = 2^{\aleph_0}$, $X, D \in \mathcal{M}$ and $\mathcal{M}$ is closed under ω-sequences then $D \cap \mathcal{M}$ is dense in $X \cap \mathcal{M}$. For every $y \in X \cap \mathcal{M}$ let $\mathcal{B}_y$ be a local π-base at y such that $\mathcal{B}_y \in \mathcal{M}$. If $D \cap \mathcal{M}$ is not dense in $X \cap \mathcal{M}$ then there is a $y \in X \cap \mathcal{M}$ and an open set U such that $y \in U$ and $U \cap \mathcal{M} \cap D = \emptyset$. Take a $B \in \mathcal{B}_y$ such that $B \subset U$, then $B \cap \mathcal{M} \cap D = \emptyset$. Since $B, D \in \mathcal{M}$ it follows that $B \cap D = \emptyset$, a contradiction.

Theorem 27. ([Gr]) *If X is a compact T_1-space with countable pseudocharacter then $|X| \leq 2^{\aleph_0}$.*

Proof: Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ such that $X \in \mathcal{M}$ and which is closed under ω-sequences. $Y = X \cap \mathcal{M}$ is countably compact (see Example 5). We claim that Y is compact. Take a maximal family $\mathcal{F}$ of closed sets of Y with the f.i.p. As X is compact there is a point $x \in \cap\{cl_X(F) : F \in \mathcal{F}\}$, we claim that $x \in Y$ (so Y is compact). Suppose that $x \notin Y$ and consider a family of open sets $\{G_n\}_{n\in\omega}$ of X such that $\cap_{n\in\omega} G_n = \{x\}$. For every $n \in \omega$, $Y \setminus G_n$ is a closed set of Y such that $Y \setminus G_n \notin \mathcal{F}$. By the maximality of $\mathcal{F}$, it follows that there is a $F_n \in \mathcal{F}$ such that $(Y \setminus G_n) \cap F_n = \emptyset$, i.e., $F_n \subset G_n$. Y is countably compact so $\emptyset \neq \cap_{n\in\omega} F_n \subset \cap_{n\in\omega} G_n \cap Y = \emptyset$, a contradiction and therefore $x \in Y$. By Lemma 8 it follows that $X \subset \mathcal{M}$, i.e., $|X| \leq 2^{\aleph_0}$.

For any space X, $K(X)$ denotes the collection of all compact subsets of X. In the next four results we will be showing that the compact subsets are elements of $\mathcal{M}$.

Theorem 28. ([J]) *If X is T_1 and $sw(X) = \aleph_0$ then $|K(X)| \leq 2^{\aleph_0}$.*

Proof: Clearly $|X| \leq 2^{\aleph_0}$ (if $\mathcal{B}$ is a countable separating open cover of X then $|X| \leq |[\mathcal{B}]^{\leq\omega}| = |\mathcal{P}(\mathcal{B})| \leq 2^{\aleph_0}$). Now let $\mathcal{V}$ be a countable separating open cover of X closed under finite aunions. Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains X, each member of $\mathcal{V}$ and which is closed under ω-sequences.

We show that $K(X) \subset \mathcal{M}$. Let $K \in K(X)$. For every $p \in X - K$ there is a $B \in \mathcal{V}$ such that $K \subset B \subset X - \{p\}$. Let $\mathcal{B}_K = \{B \in \mathcal{V} : K \subset B\}$, $\mathcal{B}_K \subset \mathcal{M}$ and $|\mathcal{B}_K| \leq \aleph_0$, hence $\mathcal{B}_K \in \mathcal{M}$. So $K = \cap\mathcal{B}_K \in \mathcal{M}$.

Theorem 29. ([BH]) *If X is a ω_1-compact T_1-space and $psw(X) = \aleph_0$, then $|K(X)| \leq 2^{\aleph_0}$.*

Proof: First observe that $|X| \leq 2^{\aleph_0}$ (by Corollary 25). Now let $\mathcal{V}$ be a point-countable separating open cover of X. Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ which contains each element of X, each member of $\mathcal{V}$, and which is closed under ω-sequences. We show $K(X) \subset \mathcal{M}$: let $K \in K(X)$ and let $\{\mathcal{A}_n : n \in \omega\}$ be the set of all finite minimal covers of K by elements of $\mathcal{V}$ (by Miščenko's lemma [M], the number of such covers is at most $\aleph_0$). For every $n \in \omega$ let $W_n = \cup \mathcal{A}_n$. Now $W_n \in \mathcal{M}$ and $K = \cap_n W_n \in \mathcal{M}$.

Theorem 30. ([BH]) *If X is T_2 and hereditarily Lindelöf, then $|K(X)| \leq 2^{\aleph_0}$.*

Proof: $|X| \leq 2^{\aleph_0}$ (see the comment after Lemma 8). For every $x \in X$ let $\mathcal{B}_x$ be a family of open neighbourhoods of x such that $|\mathcal{B}_x| \leq \aleph_0$ and $\cap\{\overline{B} : B \in \mathcal{B}_x\} = \{x\}$. Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ such that $X \subset \mathcal{M}, \mathcal{B}_x \subset \mathcal{M}$ for every $x \in X$, and which is closed under ω-sequences. We claim that $K(X) \subset \mathcal{M}$. Take $K \in K(X)$, for every $p \in X - K$ there is an open neighbourhood G_p of p such that $G_p \in \mathcal{M}$ and $G_p \subset X - K$. So $X - K = \cup_{p \in X-K} G_p$. Now $X - K$ is Lindelöf so there is a countable subset A of $X - K$ such that $X - K = \cup_{p \in A} G_p$. Since $\{G_p\}_{p \in A} \in \mathcal{M}$ we have $X - K \in \mathcal{M}$ and so $K \in \mathcal{M}$.

Theorem 31. ([BH]) *If X is a Hausdorff space with countable spread and every compact subset of X is a G_δ-set, then $|K(X)| \leq 2^{\aleph_0}$.*

Proof: $|X| \leq 2^{\aleph_0}$ (by Theorem 21). For every $p \in X$ take a family $\mathcal{V}_p$ of open neighbourhoods of p such that $|\mathcal{V}_p| \leq 2^{\aleph_0}$ and $\cap\{\overline{V} : V \in \mathcal{V}_p\} = \{p\}$. Take an elementary submodel $\mathcal{M}$ of cardinality $2^{\aleph_0}$ such that $X \subset \mathcal{M}, \mathcal{V}_p \subset \mathcal{M}$ for every $p \in X$ and which is closed under ω-sequences. We claim that $K(X) \subset \mathcal{M}$. Let $K \in K(X)$; by hypothesis there is a family $\{F_n\}_{n \in \omega}$ of closed sets such that $X - K = \cup_n F_n$. For every n and for every $p \in F_n$ there is an open set $G_p \in \mathcal{M}$ such that $p \in G_p \subset X - K$. Now F_n has countable spread so there are $A_n \subset F_n$ and $\mathcal{G}_n \subset \{G_p\}_{p \in F_n}$ such that $|A_n| \leq \aleph_0$, $|\mathcal{G}_n| \leq \aleph_0$ and $\{\overline{A}_n\} \cup \{G : G \in \mathcal{G}_n\}$ covers F_n. For every n we have

$\mathcal{B}_n = \overline{A}_n \cup \cup \mathcal{G}_n \in \mathcal{M}$ and $\overline{A}_n \cap K = \emptyset$ so $X - K = \cup_n \mathcal{B}_n \in \mathcal{M}$ and $K \in \mathcal{M}$.

In the proofs of Lemma 9 and Theorem 18 we saw that we can use Proposition 1 to build chains of elementary submodels. We conclude this paper with two other applications of this kind (see also [Do1]).

Theorem 32. *Let X be an initially ω_1-compact T_3-space of countable tightness. If Y is a hereditarily Lindelöf subspace of X then $d(Y) \leq \aleph_1$.*

Proof: Let $\{\mathcal{M}_\alpha : \alpha \in \omega_1\}$ be an increasing chain of countable elementary submodels such that $X, Y \in \mathcal{M}_0$ and $\mathcal{M}_\alpha \in \mathcal{M}_{\alpha+1}$ for every $\alpha \in \omega_1$. Let $\mathcal{M}_{\omega_1} = \bigcup\{\mathcal{M}_\alpha : \alpha \in \omega_1\}$. We claim that $Y \subset \overline{Y \cap \mathcal{M}_{\omega_1}}$ (so $d(Y) \leq \aleph_1$). X has countable tightness so $\overline{\mathcal{M}_{\omega_1} \cap Y} = \bigcup\{\overline{Y \cap \mathcal{M}_\alpha} : \alpha \in \omega_1\}$. Now for every $\alpha \in \omega_1$ there is an open collection $\mathcal{V}_\alpha$ in X such that $|\mathcal{V}_\alpha| \leq \aleph_0$, $\overline{\mathcal{M}_\alpha \cap Y} \subset \cap \mathcal{V}_\alpha$ and $Y \cap (\overline{Y \cap \mathcal{M}_\alpha}) = \bigcap \mathcal{V}_\alpha \cap Y$ ([Ho], 7.16). By elementarity we can take each $\mathcal{V}_\alpha$ in $\mathcal{M}_{\omega_1}$, so $\mathcal{V}_\alpha \subset \mathcal{M}_{\omega_1}$ for every $\alpha \in \omega_1$. Suppose there is a $y \in Y \setminus \overline{Y \cap \mathcal{M}_{\omega_1}}$. Then for every $\alpha \in \omega_1$ there is a $V_{\alpha(y)} \in \mathcal{V}_\alpha$ such that $y \notin V_{\alpha(y)}$. $\{V_{\alpha(y)} : \alpha \in \omega_1\}$ covers $\overline{Y \cap \mathcal{M}_{\omega_1}}$ so there is a subfamily $\{V_{\alpha_i(y)} : i = 1, ..., n\}$ which covers $\overline{Y \cap \mathcal{M}_{\omega_1}}$. Since $\{V_{\alpha_i(y)} : i = 1, ..., n\} \in \mathcal{M}_{\omega_1}$ it must cover Y, which is a contradiction.

Corollary 33. *Let X be an initially ω_1-compact T_3-space with countable spread. Then $hd(X) \leq \aleph_1$.*

Proof: Observe that X has countable tightness (if $t(X) > \aleph_0$, then X has an uncountable free sequence and this contradicts $s(X) = \aleph_0$). Let $Z \subset X$. As X has countable spread there is a hereditarily Lindelöf dense subspace Y of $\overline{Z}$. Hence $d(Z) \leq d(\overline{Z})t(\overline{Z}) = d(\overline{Z}) \leq d(Y)$ and $d(Y) \leq \aleph_1$ by the above theorem.

Corollary 34. [Sa2] *If X is a compact Hausdorff space with countable spread then $hd(X) \leq \aleph_1$.*

Theorem 35. *Let X be a c.c.c. non-separable space of countable tightness with a dense set P of points of character $\leq \aleph_1$. Then X contains a closed c.c.c. subspace of density $\aleph_1$.*

Proof: Let $\{\mathcal{M}_\alpha : \alpha \leq \omega_1\}$ be a continuous increasing chain of elementary submodels such that $X, \omega_1 \in \mathcal{M}_0$ and such that for each $\alpha \in \omega_1$, $\mathcal{M}_\alpha$ is countable and $\mathcal{M}_\alpha \in \mathcal{M}_{\alpha+1}$. Let $F = \overline{\mathcal{M}_{\omega_1} \cap X}$. First we show $d(F) = \aleph_1$. Suppose there is a $D \subset F$ such that $|D| \leq \aleph_0$ and $\overline{D} = F$. For every $d \in D$ there exists an $A_d \subset X \cap \mathcal{M}_{\omega_1}$ such that $|A_d| \leq \aleph_0$ and $d \in \overline{A}_d$. Let $\alpha \in \omega_1$ such that $\cup_{d \in D} A_d \subset \mathcal{M}_\alpha$. Thus $\mathcal{M}_{\alpha+1} \cap X \subset \overline{\mathcal{M}_{\omega_1} \cap X} = F \subset \overline{\mathcal{M}_\alpha \cap X}$. We reach a contradiction if we show that $\overline{\mathcal{M}_\alpha \cap X} = X$. Suppose not, choose a point $x \in X \backslash \overline{\mathcal{M}_\alpha \cap X}$, by elementarity there is a $x \in X \cap \mathcal{M}_{\alpha+1}$ such that $x \notin \overline{\mathcal{M}_\alpha \cap X}$. Now we show that $c(F) = \aleph_0$. Suppose that $\{U_\alpha : \alpha \in \omega_1\}$ is a family of non-empty open sets in X such that $U_\alpha \cap U_{\alpha'} \cap F = \emptyset$ whenever $\alpha, \alpha' \in \omega_1, \alpha \neq \alpha'$, and $U_\alpha \cap F \neq \emptyset$ for every $\alpha \in \omega_1$.

Let $x \in \mathcal{M}_{\omega_1} \cap X$. Now P is dense in X and X has countable tightness so there exists an $A_x \subset P$ such that $|A_x| \leq \aleph_0$ and $x \in \overline{A}_x$, by elementarity we can take such A_x in $\mathcal{M}_{\omega_1}$. Therefore we have $A_x \subset P \cap \mathcal{M}_{\omega_1}$, hence $x \in \overline{A}_x \subset \overline{P \cap \mathcal{M}_{\omega_1}}$ and $\mathcal{M}_{\omega_1} \cap X \subset \overline{P \cap \mathcal{M}_{\omega_1}}$, so $P \cap \mathcal{M}_{\omega_1}$ is dense in F. Now for every $p \in P \cap \mathcal{M}_{\omega_1}$ let $\mathcal{B}_p$ be a local base at p such that $|\mathcal{B}_p| \leq \aleph_1$ and $\mathcal{B}_p \in \mathcal{M}_{\omega_1}$ (so $\mathcal{B}_p \subset \mathcal{M}_{\omega_1}$). For every $\alpha \in \omega_1$ take $p_\alpha \in U_\alpha \cap P \cap \mathcal{M}_{\omega_1}$ and $V_\alpha \in \mathcal{B}_{p_\alpha}$ such that $V_\alpha \subset U_\alpha$. Now $\{V_\alpha : \alpha \in \omega_1\}$ is a family of non-empty open subsets of X such that $V_\alpha \cap V_p \cap \mathcal{M}_{\omega_1} \subset U_\alpha \cap U_\beta \cap F = \emptyset$ whenever $\alpha \neq \beta$. Since $V_\alpha \in \mathcal{M}_{\omega_1}$ for every $\alpha \in \omega_1$ then $\{V_\alpha : \alpha \in \omega_1\}$ is a cellular family of X, which is a contradiction.

The authors wish to thank the referee for many helpful suggestions which improved the exposition of the paper.

References

[Ar1] A.V. Arhangel'skiĭ, *On the cardinality of bicompacta satisfying the* *first axiom of countability*, Soviet Math. Doklady, **10** (1969), 951-955.

[Ar2] A.V. Arhangel'skiĭ, *The Suslin number and cardinality. Characters of points in sequential compact Hausdorff spaces*, Soviet Math. Doklady, **12** (1971), 597-601.

[BGW] M. Bell, J. Ginsburg, and G. Woods, *Cardinal inequalities for topological spaces involving the weak Lindelöf number*, Pacific J. Math., **79** (1978), 37-45.

[BH] D.K. Burke and R.E. Hodel, *The number of compact subsets of a topological space*, Proc. Amer. Math. Soc., **58** (1976), 363-368.

[Ch] A. Charlesworth, *On the cardinality of a topological space*, Proc. Amer. Math. Soc., **66** (1977), 138-142.

[DeG] J. De Groot, *Discrete subspaces of Hausdorff spaces*, Bull. Acad. Polon. Sci., **13** (1965), 537-544.

[Do1] A. Dow, *An introduction to applications of elementary submodels to topology*, Top. Proc., **13** (1988), 17-72.

[Do2] A. Dow, *More set-theory for topologists*, Top. Appl., **64** (1995), 243-300.

[E] R. Engelking, General Topology, Heldermann Verlag, Berlin 1989.

[FW] A. Fedeli and S. Watson, *Continuous images of products of separable spaces*, Top.Appl., (1996).

[Gr] A. A. Gryzlov, *Two theorems on the cardinality of topological spaces*, Soviet Math. Dokl., **21** (1980), 506-509.

[Gr2] A. A. Gryzlov, *Cardinal invariants and compactifications*, Comment. Math. Univ. Carolinae, **35, 2** (1994), 403-408.

[HJ] A. Hajnal and I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math., **29** (1967), 343-356.

[Ho] R. Hodel, *Cardinal Functions I*, in: K. Kunen and J.E. Vaughan eds., Handbook of Set Theoretic Topology, North Holland 1984, 1-61.

[Je] T. Jech, Set Theory, Academic Press, 1978.

[Ju] I. Juhàsz, *Cardinal functions in topology - ten years later*, Mathematisch Centrum Amsterdam 1980.

[Ku] K. Kunen, Set theory, North Holland 1980.

[M] A. Miščenko, *Spaces with point-countable bases*, Soviet Math. Dokl., **3** (1962), 855-858.

[Po] B. Pospišil, *Sur la puissance d'un espace contenant una partie dense de puissance donnée*, Căsopis Pro Pěstováni Matematiky a Fysiky, **67** (1937), 89-96.

[Sa1] B. Šapirovskiĭ, *On discrete subspaces of topological spaces; weight, tightness and Suslin number*, Soviet Math. Dokl., **13** (1972), 215-219.

[Sa2] B. Šapirovskiĭ, *Canonical sets and character. Density and weight in compact spaces*, Soviet Math. Dokl., **15** (1974), 1282-1287.

[St] R.M. Stephenson Jr., *Initially κ-compact and related spaces*, in: K. Kunen and J.E. Vaughan eds., Handbook of Set Theoretic Topology, North Holland, 1984, 603-632.

[W1] S. Watson, *The construction of topological spaces: Planks and Resolutions*, in M. Hušek and J. Van Mill eds., Recent Progress in General Topology Elsevier Science Publishers, 1992, 675-757.

[W2] S. Watson, *The Lindelöf number of a power; an introduction to the use of elementary submodels in general topology*, Top. Appl., **58** (1994), 25-34.

University of L'Aquila
67100 L'Aquila, Italy

York University
North York, Ontario, M3J 1P3 Canada