Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

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Topology Proceedings
Vol. 20, 1995

ON FINITE PRODUCTS OF MENGER SPACES AND 2-HOMOGENEITY

DENNIS J. GARITY

ABSTRACT. G. S. Ungar has shown that homogeneous metric continua that are 2–homogeneous are locally connected. K. Kuperberg, W. Kuperberg and W. R. R. Transue gave examples of homogeneous metric continua that were locally connected but were not 2-homogeneous. In this paper, examples are produced that show that adding the additional requirement of local n–connectivity is not enough to produce a converse to Ungar's theorem. For every positive integer n, a homogeneous metric continua of dimension (n+1) that is locally (n-1) connected is produced. These spaces are shown not to be 2–homogenous. The examples are produced by taking products of the universal Menger n–dimensional space with S^1. Other examples are produced by taking finite products of Menger spaces. An analysis of Čech homology properties of Menger spaces is needed in the examples.

1. INTRODUCTION

G. S. Ungar has shown that homogeneous metric continua having the stronger homogeneity property of being 2–homogeneous are necessarily locally connected [Un]. This result leads to the question of whether imposing local connectivity or local n–connectivity conditions on homogeneous continua would imply that they possessed stronger homogeneity properties such as 2–homogeneity. In 1980, K. Kuperberg, W. Kuperberg and W. R. R. Transue showed that $\mu_1 \times \mu_1$ and $\mu_1 \times S^1$ were not 2–homogeneous [KKT]. Here, μ_1 is the universal curve, defined below. Their results gave examples of homogeneous spaces that were locally connected, but not 2–homogeneous. They also ask whether finite or countable products of μ_1 with itself are 2–homogeneous.

The results in this paper were first presented in a talk at the Workshop in Geometric Topology in Colorado Springs in June, 1992. A summary of the results appeared in the Proceedings of that conference [Ga]. Since that time, it has been pointed out that arguments of J. Kennedy Phelps [Ke1] , [Ke2] that showed that $\mu_1 \times X$, where X is an arbitrary continuum, is not 2–homogeneous, can be generalized to obtain the same result for the other Menger spaces. K. Kuperberg, W. Kuperberg and W. R. R. Transue also have a more recent paper on 2–homogeneity [KKT2].

The techniques used in this paper obtaining similar results are different enough to be of general interest. In this paper, we are able to show that imposing higher local connectivity conditions on metric homogeneous continua does not lead to a converse to Ungar's Theorem. For each positive integer n, we give examples of homogeneous $(n+2)$–dimensional metric continua that are that are locally n–connected, but are not 2–homogeneous. We also show that finite products with each factor a Menger space are not 2–homogeneous. It remains open whether there is a homogeneous metric continuum that is n–connected for all n, and is not 2–homogeneous.

The results in [KKT] depend on certain one–dimensional facts from Curtis and Fort [CF] that do not generalize directly to higher dimensions. In this paper, we replace the one–dimensional arguments with higher dimensional Čech homology arguments that allow us to generalize the results in [KKT]. As a special case, we are able to show that $\mu_m \times \mu_n$ is not 2–homogeneous for all values of n and m where $\max\{m, n\} \geq 1$.

Section 2 contains the necessary definitions. Section 3 contains results on the Čech homology of Menger spaces. The main results are in Section 4.

The author would like to thank John J. Walsh and Krystyna

M. Kuperberg for numerous helpful conversations.

2. Definitions

All spaces under consideration are separable metric spaces. We begin with definitions of the Menger Spaces. These spaces were originally defined by Menger in 1932 [Mg]. An inductive definition is as follows. Let M_n^0 be $I^{2n+1} \subset R^{2n+1}$, where $I = [0,1]$. Inductively assume that M_n^k is a union of $(2n+1)$–dimensional cells with sides of length $(1/3)^k$. Subdivide each cell in M_n^k into 3^{2n+1} smaller cells by subdividing each side in thirds. Then M_n^{k+1} is the union of all of those smaller cells that intersect the n–skeleton of M_n^k. The Menger n–dimensional space, μ_n, is then defined as $\bigcap_{i=0}^{\infty} M_n^i$. The space μ_0 is the standard middle thirds Cantor set and the space μ_1 is the universal curve characterized by R. D. Anderson [An1], [An2].

In 1984, M. Bestvina characterized all the remaining Menger Spaces [Be]. The following theorem gives Bestvina's characterization.

Theorem 2.1. [Be] *The Menger universal n-dimensional space μ_n is the unique space satisfying the following conditions:*

(1) *μ_n is a compact n-dimensional metric space.*
(2) *μ_n is locally $(n-1)$–connected (LC^{n-1}).*
(3) *μ_n is $(n-1)$–connected (C^{n-1}).*
(4) *μ_n satisfies the Disjoint n-cells Property (DD^nP).*

Definition 2.2. A space X is *k–connected* if every map of $S^j, 0 \leq j \leq k$ into X extends to a map of B^{j+1} into X. A space X is *locally k-connected* if for each point $p \in X$ and for each neighborhood U of p, there exists a neighborhood V of p so that each map of $S^j, 0 \leq j \leq k$ into V extends to a map of B^{j+1} into U. A space X satisfies the *Disjoint k-Cells Property* if for each $\epsilon > 0$ and for each pair of maps f_1 and f_2 from I^k into X, there are maps g_1 and g_2 from I^k into X with $g_1(I^k) \cap g_2(I^k) = \emptyset$ and $d(g_i, f_i) < \epsilon$.

Note that the spaces M_n^i used in the construction of the n –dimensional Menger space μ_n are both C^{n-1} and LC^{n-1}. The conditions in Theorem 2.1 yield the result that μ_n is a universal n –dimensional separable metric space, i.e. μ_n is n-dimensional and contains a homeomorphic copy of every separable metric n–dimensional space. The details are given in [Be]. We are interested in the homogeneity properties of Menger Spaces and of products of Menger Spaces. The relevant definitions are provided next.

Definition 2.3. A space X is *homogeneous* if and only if for each pair of points p and q in X, there is a homeomorphism $h : X \to X$ with the property that $h(p) = q$. A space X is *n-homogeneous* if for each pair of n–point subsets of X, A and B, there is a homeomorphism $h : X \to X$ with the property that $h(A) = B$. A space X is *countable dense homogeneous* if and only if for each pair of countable dense subsets A and B of X, there is a homeomorphism $h : X \to X$ with the property that $h(A) = B$.

It is well known that the Cantor set (μ_0), the Hilbert Cube, and all manifolds satisfy these types of homogeneity. R. D. Anderson established that μ_1 also satisfies these types of homogeneity [An1, An2]. M. Bestvina established the analogous results for the higher dimensional Menger spaces.

Theorem 2.4. [Be, pg. 73]. *Each Menger Space μ_n is k–homogeneous for each k, and is countable dense homogeneous.*

3. Čech Homology of Menger Spaces

We need some preliminary results about embeddings and maps of S^n into μ_n. For these computations, we use singular and Čech homology with coefficients in the rationals. See [ES] for the properties of Čech homology. A map f from S^n into a space X is said to be essential with respect to n–th Čech homology if the induced homomorphism on n–th Čech homology groups is nontrivial. The map is said to be essential with respect to homotopy if it is not homotopic to a constant map.

Lemma 3.1. *For each μ_n, the following results hold:*

(1) *Any embedding of S^n into μ_n is essential both with respect to homotopy and with respect to n-th Čech homology.*

(2) *If f_1 and f_2 are any maps from S^n into μ_n that are essential with respect to n-th Čech homology, and if f_1 and f_2 have disjoint images, then f_1 and f_2 are not homotopic.*

Proof: Consider the exact Čech homology sequence of the pair (μ_n, E) where E is either $e(S^n)$ or $f_1(S^n) \cup f_1(S^n)$. Since μ_n is n–dimensional, it follows that $\check{H}_{n+1}(\mu_n, E)$ is trivial and thus the inclusion induced homomorphism $\check{H}_n(E) \to \check{H}_n(\mu_n)$ is a monomorphism. It follows that if E is $e(S^n)$, $\check{H}_n(E)$ is nontrivial and thus the map e is essential with respect to n -thČech homology. It follows from this that the map is essential with respect to homotopy.

If E is $f_1(S^n) \cup f_1(S^n)$, then $\check{H}_n(E) \simeq \check{H}_n(E_1) \oplus \check{H}_n(E_2)$ where $E_i = f_i(S^n)$. The hypotheses imply that each $\check{H}_n(E_i)$ is nontrivial. If f_1 and f_2 were homotopic, they would induce the same homomorphism on Čech homology, contradicting the fact that the inclusion induced homomorphism $\check{H}_n(E) \to \check{H}_n(\mu_n)$ is a monomorphism.

Lemma 3.2. *For each $\epsilon > 0$ and for each point $p \in \mu_n$, there is an embedding $f : S^n \to \mu_n$ with image contained in the ϵ neighborhood of p.*

Proof: This follows from the construction of μ_n since each point in μ_n has arbitrarily small neighborhoods that are homeomorphic to μ_n and since μ_n itself contains embedded copies of S^n. □

Theorem 3.3. *Let X be a compact subspace of some ANR Z. Fix $n \geq 1$. Let $Y = \mu_n \times X$. Let p and q be projections from*

Y onto μ_n and X respectively. If $f : S^k \to Y, 1 \leq k \leq n$ is a map that is essential with respect to k-th Čech homology, then either $p \circ f$ or $q \circ f$ is essential with respect to k-th Čech homology. If $k < n$, then $q \circ f$ is essential with respect to k-th Čech homology.

Proof: We write the homology class of a cycle z as $[z]$. Express X as a nested intersection of compact neighborhoods X_i in Z. The space μ_n is a nested intersection of the compact polyhedra M_n^i defined above. So the space Y is a nested intersection of the compact neighborhoods $Y_i = M_n^i \times X_i$ in $Z \times R^{2n+1}$. The Čech homology of Y may be thus computed as the inverse limit of the singular homology of the spaces Y_i. Let p_i and q_i be projections from Y_i onto M_n^i and X_i respectively.

The spaces M_n^i have trivial singular homology in dimensions 1 through $(k-1)$. So the Künneth and Eilenberg–Zilber theorems for singular homology (see [Mu]) imply that if $\alpha_i = [c]$ is a nontrivial element of the k-th singular homology of Y_i, either $p_{i*}(\alpha_i)$ or $q_{i*}(\alpha_i)$ is nontrivial.

To see this, note that since we are using rational coefficients, the Künneth and Eilenberg Zilber Theorems provide isomorphisms

$$\begin{array}{ccc} H_k(Y_i) & \xrightarrow{u} & H_k\left(S(M_n^i) \otimes S(X_i)\right) \\ & & \Big\uparrow \theta \\ & & (H_k(M_n^i) \otimes H_0(X_i)) \oplus (H_0(M_n^i) \otimes H_k(X_i)) \end{array}$$

Here

$$u([c]) = \left[\sum_j p_{i_\#}\left(cF_j\right) \otimes q_{i_\#}\left(cB_{k-j}\right)\right]$$

where F_j and B_{k-j} are the front j-face and back $k-j$ face operators. Also, $\theta\left([\alpha] \otimes [\beta]\right) = [\alpha \otimes \beta]$. An inverse isomorphism λ to θ can be constructed that takes elements of the form $[\alpha_j \otimes \beta_{n-j}]$ to $[\alpha_j] \otimes [\beta_{n-j}]$ in $H_j(M_n^i) \otimes H_{n-j}(X_i)$ when α_j and β_{n-j} are cycles, and takes $[\alpha_j \otimes \beta_{n-j}]$ to 0 otherwise. Since $H_j(M_n^i)$ is nonzero only for $j = 0$ or $j = k$, it follows that

$$\lambda \circ u([c]) = \left[p_{i_{\#}}(c)\right] \otimes \left[q_{i_{\#}}(cB_0)\right] + \left[p_{i_{\#}}(cF_0)\right] \otimes \left[q_{i_{\#}}(c)\right]$$

Since λ and u are isomorphisms, if $\left[p_{i_{\#}}(c)\right]$ and $\left[q_{i_{\#}}(c)\right]$ are both 0, then $[c] = 0$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$be a nontrivial element of the k-th Čech homology of Y that is given by the hypotheses. That is, each $\alpha_i = f_{i*}(\gamma)$ where γ is a generator of $H_k(S^k)$ and where f_i is the composition of f with inclusion from Y to Y_i. Then there is a positive integer k such that for all $i \geq k$, $\alpha_i \neq 0$ in $H_k(Y_i)$. By the previous paragraph, either $p_{i*}(\alpha_i)$ or $q_{i*}(\alpha_i)$ is nontrivial for all $i \geq k$. The result now follows by considering the definition of the induced homomorphism on Čech homology groups.

If in fact $k < n$, then each $p_{i*}(\alpha_i)$ is trivial, so that $p_*(\alpha)$ is trivial, and thus $q \circ f$ is essential with respect to k-th Čech homology. □

4. The Main Results

Definition 4.1. A space X is locally homologically Čech n-connected if for each point $p \in X$ and for each neighborhood U of p, there is a neighborhood V of p such that the inclusion induced homomorphism from the n-th Čech homology of V to the n-th Čech homology of U is trivial.

Theorem 4.2. *Let $X = \mu_n \times \prod_{i=1}^{k} Y_i$ where $n \geq 1$, and where each Y_i is homeomorphic to μ_j for some $j \geq n$. Then X is not 2-homogeneous.*

Proof: Fix distinct points x and y in μ_n. Choose points r and s in $\prod_{i=1}^{k} Y_i$ so that for each $i, 1 \leq i \leq k, q_i \neq r_i$. We will show that there is no homeomorphism $h : X \to X$ such that

$h(\{(x,r),(x,s)\} = \{(x,r),(y,s)\}$. Assume to the contrary that there is such a homeomorphism h.

Let p be projection of X onto μ_n, let q be projection of X onto $\prod_{i=1}^k Y_i$ and let q_i be projection of X onto Y_i. Choose a neighborhood U of x in μ_n such that $p(h(U \times \{r\})) \cap p(h(U \times \{s\})) = \emptyset$, and such that for each i, $q_i(h(U \times \{r\})) \cap q_i(h(U \times \{s\})) = \emptyset$. Choose an embedding $e : S^n \to U$. This is possible by Lemma 3.2. Let $f_1 : S^n \to X$ be the map given by $f_1(a) = (e(a), \{r\})$ and let $f_2 : S^n \to X$ be the map given by $f_2(a) = (e(a), \{s\})$. Since $\prod_{i=1}^k X_i$ is path connected, the maps f_1 and f_2 are homotopic. By Lemma 3.1, the map e is essential with respect to n-th Čech homology. It follows that the maps f_1 and f_2 are also essential with respect to n-th Čech homology, and so the maps $h \circ f_1$ and $h \circ f_2$ are essential with respect to n-th Čech homology.

Lemma 3.3 implies that for each i, either $p \circ h \circ f_i$ or $q \circ h \circ f_i$ is essential with respect to n-th Čech homology. Since f_1 and f_2 are homotopic, this implies that either both $p \circ h \circ f_1$ and $p \circ h \circ f_2$ are essential with respect to n-th Čech homology, or both $q \circ h \circ f_1$ and $q \circ h \circ f_2$ are essential with respect to n-th Čech homology. Since $p \circ h \circ f_1$ and $p \circ h \circ f_2$ are homotopic and have disjoint images, Lemma 3.2 implies that these maps are not essential with respect to n-th Čech homology.

So both $q \circ h \circ f_1$ and $q \circ h \circ f_2$ are essential with respect to n-th Čech homology. Repeated application of Lemma 3.2 shows that for each i, both $q_i \circ h \circ f_1$ and $q_i \circ h \circ f_2$ are not essential with respect to n-th Čech homology. This leads to a contradiction. So there is no homeomorphism h as assumed, and it follows that X is not 2–homogeneous. □

Corollary 4.3. *Any finite product of two or more Menger spaces, where at least one of the Menger spaces is not μ_0, is not 2–homogeneous. In particular, $\mu_m \times \mu_n$, where* $\max\{m,n\} \geq 1$ *is not 2–homogeneous.*

Proof: The case where each factor is not μ_0 follows directly from the previous theorem. Since finite products of Cantor sets yield another Cantor set, the only remaining case to consider is $X = \mu_0 \times \prod_{i=1}^{k} X_i$ where each X_i is homeomorphic to μ_n for some $n \geq 1$. The path components of X are of the form $\{p\} \times \prod_{i=1}^{k} X_i$ where $p \in \mu_0$. Since self homeomorphisms of X can not take two points in a single path component to points in distinct path components, it follows that X is not 2–homogeneous. □

The preceding corollary provides a partial negative answer to the question in [KKT] as to whether any finite or countable product of μ_1 with itself is 2–homogeneous. Also, $\mu_n \times \mu_n$ provides an example of a homogeneous space that is $(2n)$-dimensional, that is LC^{n-1}, and that is not 2–homogeneous. We now proceed to produce an example of a homogenous LC^{n-1} space of dimension $(n+2)$ that is not 2 –homogeneous.

Theorem 4.4. *Let $X = \mu_n \times Y$ where $n \geq 1$, Y is compact, path connected, homogenous and locally homologically Čech n-connected. Then X is not 2–homogeneous.*

Proof: We proceed as in the proof of Theorem 4.1. Fix distinct points x and y in μ_n. Choose distinct points r and s in Y . We will show that there is no homeomorphism $h : X \to X$ such that $h(\{(x,r),(x,s)\}) = \{(x,r),(y,s)\}$. Assume to the contrary that there is such a homeomorphism h.

Let p be projection of X onto μ_n,and let q be projection of X onto Y. Choose a neighborhood U of x in μ_n such that $p(h(U \times \{r\})) \cap p(h(U \times \{s\})) = \emptyset$, such that $q(h(U \times \{r\})) \cap q(h(U \times \{s\})) = \emptyset$, and such that the inclusion induced homomorphisms from the n-th Čech homology of $q(h(U \times \{r\}))$ and from the n-th Čech homology of $q(h(U \times \{s\}))$ to the n-th Čech homology of Y are trivial.

As in Theorem 4.1, choose an embedding $e : S^n \to U$ and let $f_1 : S^n \to X$ be the map given by $f_1(a) = (e(a), \{r\})$ and let $f_2 : S^n \to X$ be the map given by $f_2(a) = (e(a), \{s\})$. Since Y is path connected, the maps f_1 and f_2 are homotopic. As before, the map e is essential with respect to n-th Čech homology. It follows that the maps f_1 and f_2 are also essential with respect to n-th Čech homology, and so the maps $h \circ f_1$ and $h \circ f_2$ are essential with respect to n-th Čech homology.

Since $p \circ h \circ f_1$ and $p \circ h \circ f_2$ are homotopic and have disjoint images, Lemma 3.2 again implies that these maps are not essential with respect to n-th Čech homology.

So both $q \circ h \circ f_1$ and $q \circ h \circ f_2$ are essential with respect to n-th Čech homology. But this contradicts the fact that the homomorphisms from the n-th Čech homology of $q\,(h(U \times \{r\}))$ and from the n-th Čech homology of $q\,(h(U \times \{s\}))$ to the n-th Čech homology of Y are trivial. So there is no homeomorphism h as assumed, and it follows that X is not 2–homogeneous.

Corollary 4.5. *The product of $\mu_n, n \geq 1$ with any ANR or with any manifold is not 2–homogeneous.*

Corollary 4.6. *For each positive integer n, there is an $(n+1)$–dimensional homogeneous compact metric space that is $(n-1)$–connected and is not 2–homogeneous.*

Proof: That the space $\mu_n \times S^1$ satisfies the conditions in the corollary follows directly from the previous theorem. □

Note that the techniques used in proving Theorems 4.2 and 4.4 can be used to prove that any finite product of Menger spaces and manifolds, or any finite product of Menger spaces and spaces that are locally homologically Čech n-connected for sufficiently many values of n, are not 2–homogeneous. These techniques can also be used to prove that such products are not n -homogeneous for values of n greater than 2. One can also use the techniques in the above theorems to analyze the type of self homeomorphisms of finite products of Menger spaces as was done in [KKT] for the product $\mu_1 \times \mu_1$.

5. Questions

The following questions remain open.

Question 1. Is there a compact metric space of dimension less than $(n+2)$ that is homogeneous, locally n–connected, and not 2–homogeneous?

Question 2. If a homogeneous compact metric space is locally n–connected for all n, is the space necessarily 2–homogeneous?

References

[An1] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math., **68** (1958), 313–324.

[An2] ———, *1–dimensional continuous curves and a homogeneity theorem*, Ann. of Math., **68** (1958), 1-16.

[Be] M. Bestvina, *Characterizing k–dimensional universal Menger compacta*, Memoirs Amer. Math. Soc., **71** (1988), no.380.

[CF] M. L. Curtis and M. K. Fort, *The fundamental group of one–dimensional spaces*, Proc. Amer. Math. Soc., **10** (1959), 140–148.

[ES] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, New Jersey, 1952.

[Ga] D. J. Garity, *On multiple homogeneity of products of Menger spaces*, Proc. of the Ninth Annual Workshop in Geometric Topology (J. Henderson, F. Tinsley ed.), Colorado College, Colorado Springs, Colorado, 1992, 18–24.

[Ke1] J. Kennedy Phelps, *Homeomorphisms of products of universal curves*, Houston J. Math., **6** (1980), 127–143.

[Ke2] ———, *A condition under which 2–homogeneity and representability are the same in continua*, Fund. Math., **121** (1984), 89–98.

[Ku] K. Kuperberg, *On the bihomogeneity problem of Knaster*, Trans. Amer. Math. Soc., **321** (1990), 129–143.

[KKT] K. Kuperberg, W. Kuperberg, and W. R. R. Transue, *On the 2–homogeneity of Cartesian products*, Fund. Math. **CX** (1980), 131–134.

[KKT2] ———, *Homology separation and 2–homogeneity*, Continua with the Houston problem book (H. Cook, W. T. Ingram, K. T. Kuperberg, A. Lelek, P. Minc, ed.), vol. 170, Marcel Decker, New Your, 1995.

[Mg] K. Menger, *Kurventheorie*, Teubner, Berlin – Leipzig, 1932.

[Mu] J. Munkres, *Elements of Algebraic Topology*, Benjamin/Cummings, Menlo Park, California, 1984.

[Un] G. S. Ungar, *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc., **212** (1975), 393–400.

Oregon State University
Corvallis, OR 97331
e-mail: garity@math.orst.edu