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A NOTE ON COMPACTIFICATION THEOREM FOR TRDIM

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ABSTRACT. P. Borst introduced a transfinite extension of covering dimension. In this paper we prove that every S-w.i.d. metacompact normal space X has a compactification αX such that trdim αX = trdim X and $w(\alpha X) = w(X)$.

1. INTRODUCTION

In this paper we assume that all spaces are normal unless otherwise stated. We refer the readers to [E1] and [E2] for dimension theory.

A space X is called *weakly-infinite-dimensional in the sense of Smirnov*, abbreviated S-w.i.d., when for every sequence $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X there exist a non-negative integer $n < \omega$ and a partition T_i in X between A_i and B_i for each $i \leq n$, such that $\bigcap\{T_i : i \leq n\} = \emptyset$.

P. Borst [B1] introduced a transfinite extension of covering dimension. In this paper we denote by trdim Borst's transfinite dimension. The values of Borst's transfinite dimension, trdim, are ordinals. Borst's transfinite dimension coincides with covering dimension if covering dimension is finite. Borst proved that a space X is S-w.i.d. if and only if trdim $X \leq \alpha$ for some ordinal α. Hence Borst's transfinite dimension classifies the class of all S-w.i.d. spaces.

In [K1] the author proved that every space X has a compactification αX such that trdim $\alpha X \leq$ trdim X and $w(\alpha X) = w(X)$, where $w(X)$ is the weight of X. Chatyrko [C1] and

Yokoi[Y] proved factorization theorem for trdim and obtained the above compactification theorem. However, Borst [B3] proved that the subspace theorem for trdim does not hold. Thus the equality trdim $\alpha X =$ trdim X need not hold even if trdim $\alpha X \leq$ trdim X.

It is well-known that every space X has a compactification αX such that $d(\alpha X) = d(X)$ and $w(\alpha X) = w(X)$ in the case when $d =$ dim, Ind or trInd (see [E1], [E2], [P]). In the case when $d =$ ind or trind not all spaces X have a compactification αX such that $d(\alpha X) \leq d(X)$ (see [L], [vMP], [K2]). In this paper we shall prove that every S-w.i.d. metacompact space X has a compactification αX such that trdim $\alpha X =$ trdim X and $w(\alpha X) = w(X)$.

2. Definitions and preliminaries

We begin with basic symbols.

For a set X, $[X]^{<\omega}$ denotes the collection of all finite subsets of X and $|X|$ denotes the cardinality of X. For a collection $\mathcal{A}$ of subsets of a space we write $\cap\mathcal{A}$ for $\cap\{A : A \in \mathcal{A}\}, \cup\mathcal{A}$ for $\cup\{A : A \in \mathcal{A}\}, \wedge \cdot \mathcal{A}$ for $\{\cap\mathcal{A}' : \mathcal{A}' \in [\mathcal{A}]^{<\omega}\}$ and $\vee \cdot \mathcal{A}$ for $\{\cup\mathcal{A}' : \mathcal{A}' \in [\mathcal{A}]^{<\omega}\}$. For a collection $\sigma = \{(A_i, B_i) : i \leq n\}$ of pairs of subsets of a space we write $\sigma^{\#}$ for $\{A_i : i \leq n\} \cup \{B_i : i \leq n\}$. For a pari $a = (A, B)$ of subsets of a space we write $a^{\#}$ for $\{A, B\}$.

We need some preparation for the definition of Borst's transfinite dimension.

2.1. Definition. Let L be a set. We denote by Fin L the collection of all non-empty finite subsets of L (i.e. Fin $L = [L]^{<\omega} - \{\emptyset\}$). For a subset M of Fin L and an element $\sigma \in [L]^{<\omega}$ we put

$$M^{\sigma} = \{\tau \in \text{ Fin } L : \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

We abbreviate $M^{\{a\}}$ to M^a for each $a \in L$.

2.2. Definition. Let L and M be as in Definition 2.1. We define the ordinal number, Ord M, inductively, as follows. Ord $M = 0$ if $M = \emptyset$. For an ordinal α, Ord $M \leq \alpha$ if Ord $M^a < \alpha$ for every $a \in L$. We put Ord $M = \alpha$ if Ord $M \leq \alpha$ and Ord $M \not< \alpha$. If there is no ordinal α for which Ord $M \leq \alpha$, then we put Ord $M = \infty$.

2.3. Definition. Let X be a space. We set

$$L(X) = \{(A, B) : A \text{ and } B \text{ are disjoint closed in } X\}$$

A collection $\sigma = \{(A_i, B_i) : i \leq n\} \in [L(X)]^{<\omega}$ is called *inessential* if there is a partition T_i in X between A_i and B_i for each $i \leq n$ such that $\cap\{T_i : i \leq n\} = \emptyset$. Otherwise σ is called *essential.* Let us set

$$M_L = \{\sigma \in \text{ Fin } L : \sigma \text{ is essential }\}$$

for each $L \subset L(X)$.

We now come to the definition of Borst's transfinite dimension.

2.4. Definition. For a space X we define

$$\text{trdim } X = \text{ Ord } M_{L(X)}.$$

2.5. Remark. Borst [B1, 3.1.1] proved that the above dimension function, trdim, coincides with covering dimension if covering dimension is finite. He [B1, 3.1.3] also proved that a space X is S-w.i.d. if and only if trdim $X \leq \alpha$ for some ordinal α.

To prove the compactification theorem we need some information and facts about Wallman compactifications.

2.6. Definition. Let $\mathcal{F}$ be a base for the closed sets of a space X. Then $\mathcal{F}$ is called a *normal base* for X provided that $\mathcal{F}$ satisfies the following conditions (1) and (3).

(1) $\wedge \cdot \vee \cdot \mathcal{F} = \mathcal{F}$,

(2) for every closed subset E of X and for any point $x \in X - E$ there is $F \in \mathcal{F}$ such that $x \in F$ and $E \cap F = \emptyset$,

(3) for $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap F_2 = \emptyset$ there exist $E_1, E_2 \in \mathcal{F}$ such that $E_1 \cap F_2 = \emptyset = E_2 \cap F_1$ and $E_1 \cup E_2 = X$.

For every normal base $\mathcal{F}$ for a space X we can construct the Wallman compactification $w(X, \mathcal{F})$ of X with respect to $\mathcal{F}$. The underlying set of $w(X, \mathcal{F})$ is the set of all ultrafilters in $\mathcal{F}$ and the topology of $w(X, \mathcal{F})$ is induced by $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$ as a base for the closed sets of $w(X, \mathcal{F})$, where $F^* = \{\mathcal{A} \in w(X, \mathcal{F}) : F \in \mathcal{A}\}$ (see [F]). Then we have $F^* = \mathrm{Cl}_{w(X,\mathcal{F})} F$ for every $F \in \mathcal{F}$.

In this paper we use the following fact

2.7. Fact. *Let $\mathcal{F}$ be a normal base for a space X. Then we have*

(a) $\wedge \cdot \vee \cdot \mathcal{F}^* = \mathcal{F}^*$,
(b) $Cl_{w(X,\mathcal{F})}(F_1 \cap F_2) = Cl_{w(X,\mathcal{F})} F_1 \cap Cl_{x(X,\mathcal{F})} F_2$ *for* $F_1, F_2 \in \mathcal{F}$.

A subset N of an ordinal α is *cofinal* in α if for every $\beta < \alpha$ there exists $\gamma \in N$ such that $\beta \leq \gamma$.

The following lemma is used in the proof of Lemma 3.3.

2.8. Lemma. *Let N be a cofinal subset of an ordinal α. If $N \cap \beta$ is cofinal in β for every $\beta \in N$, then the equality $|N| = |\alpha|$ holds.*

Proof: Suppose that this lemma has been proved for any ordianl β with $\beta < \alpha$, and we shall prove it for α. For every $\beta \in N$ we set $N_\beta = N \cap \beta$. Then N_β is cofinal in β. Since $B_\beta \cap \gamma = N \cap \gamma$ for every $\gamma \in N_\beta, N_\beta \cap \gamma$ is cofinal in γ. By the induction hypothesis, we have $|N_\beta| = |\beta|$. On the other hand, since N is cofinal in α, we have $\alpha = \cup\{\beta : \beta \in N\}$. Hence $|\alpha| = |\cup\{\beta : \beta \in N\}| \leq |\oplus\{\beta : \beta \in N\}| = |\oplus\{N \cap \beta : \beta \in N\}| = |N|$. This completes the proof of Lemma 2.8.

3. Compactification theorem

The following lemma is essentially due to Borst [B1. 2.1.6], so we omit the proof.

3.1. Lemma. *Let L and L' be sets, $M \subset [L]^{<\omega}, M' \subset [L']^{<\omega}$ and $\varphi : L \to L'$ be a one-to-one mapping satisfying the following condition (*); (*) for every $\sigma \in [L]^{<\omega}$ with $\varphi(\sigma) \notin M'$ we have $\sigma \notin M$. Then we have* Ord $M \leq$ Ord M'.

3.2. Lemma [B1, 3.3.5]. *Let X be a space and $l \subset L(X)$. Furthermore assume that for every $(E, F) \in L(X)$ there exists $(G, H) \in L$ such that $E \subset G$ and $F \subset H$, then we have* $\mathrm{Ord} M_L = \mathrm{Ord} M_{L(X)}$.

3.3. Lemma. *Let X be a compact S-w.i.d. space. Then we have trdim $X < w(X)^+$, where $w(X)^+$ is the smallest cardinal number larger than $w(X)$.*

Proof: Suppose that $\alpha =$ trdim X. Take a base $\mathcal{B}$ for X such that $|\mathcal{B}| = w(X)$ and $\wedge \cdot \mathcal{B} = \mathcal{B}$. Let us set

$$L = \{(\mathrm{Cl}_X B,\ \mathrm{CL}_X B') : B, B' \in \mathcal{B} \text{ with } \mathrm{Cl}_X B \cap \mathrm{Cl}_X B' = \emptyset\}.$$

Then, obviously, we have $|L| \leq |\mathcal{B}| = w(X)$. Since X is compact and since $\vee \cdot \mathcal{B} = \mathcal{B}$, by Lemma 3.2, we have Ord $M_L =$ Ord $M_{L(X)}$. Let $\varphi : M_L \to \alpha$ be the mapping defined by $\varphi(\sigma) =$ Ord M_L^σ for every $\sigma \in M_L$. We shall show that $\varphi(M_L)$ is cofinal in α. For every $\beta < \alpha$ we can take $a \in L$ such that Ord $M_L^a \geq \beta$, because Ord $M_L = \alpha$. Put $\sigma = \{a\}$. Then we have $\beta \leq \varphi(\sigma) \in \varphi(M_L)$. Hence $\varphi(M_L)$ is cofinal in α. Next, we shall show that $\varphi(M_L) \cap \beta$ is cofinal in β for every $\beta \in \varphi(M_L)$. Let $\beta \in \varphi(M_L)$. Take $\sigma \in M_L$ with $\varphi(\sigma) = \beta$. Then for every $\gamma < \beta$ we can take $a \in L$ such that $\mathrm{Ord}(M_L^\sigma)^a \geq \gamma$, because Ord $M_L^\sigma = \beta$. Put $\tau = \sigma \cup \{a\}$. Then we have $\gamma \leq \varphi(\tau) \in \varphi(M_L) \cap \beta$. Hence $\varphi(M_L) \cap \beta$ is cofinal in β. By Lemma 2.8, we have $|\varphi(M_L)| = |\alpha|$. On the other hand, since $|L| \leq w(X)$, we have $|M_L| \leq w(X)$. Thus we have $|\alpha| = |\varphi(M_L)| \leq |M_L| \leq w(X)$. This implies that trdim $X = \alpha < w(X)^+$.

For a space X we set

$$G_n(X) = \cup\{U : U \text{ is open in } X \text{ such that Ind } \mathrm{Cl}_X U \leq n\}$$

for every $n < \omega$, and

$$S(X) = X - \cup\{G_n(X) : n < \omega\}$$

3.4. Lemma [S] *Let $G_n(X)$ and $S(X)$ be as above. If a space X is S-w.i.d., then*

(a) *$S(X)$ is compact, and*
(b) *every closed subset F in X with $F \cap S(X) = \emptyset$ is contained in $G_n(X)$ for some $n < \omega$.*

We now generalize Lemma 3.3.

3.5. Theorem. *Let X be a S-w.i.d. metacompact space. Then we have* tridem $X < w(X)^+$.

Proof: Since X is metacompact, by Lemma 3.4(b) and the point-finite sum theorem (see [E1, 3.1.14]), for every closed subset F in X with $F \cap S(X) = \emptyset$ we have dim $F \leq n$ for some $n < \omega$. Thus, by Hattori's result [Ha], we have trdim $X \leq \omega+$ trdim $S(X)$. By Lemmas 3.3 and 3.4(a), trdim $S(X) < w(S(X))^+ \leq w(X)^+$. This implies that trdim $X < w(X)^+$. □

We now come to the main result in this paper.

3.6. Theorem. *Let X be a S-w.i.d. space with trdim $X < w(X)^+$. Then X has a compactification αX such that trdim αX= trdim X and $w(\alpha X) = w(X)$.*

Proof: We may assume that trdim $X = \alpha \geq \omega$. Put $M_0 = \{\emptyset\}$. By induction on $i, 0 < i < \omega$, we shall construct a subset M_i of $M_L(X)$. Suppose that M_i has been constructed. For every $\sigma \in M_i$ we shall construct a subset $N(\sigma)$ of $M_{L(X)}$. Let $\sigma \in M_i$. We distinguish three cases.

Case 1. $\mathrm{Ord}(M_{L(X)})^\sigma = \beta + 1$.

Take $a \in L(X)$ such that $\mathrm{Ord}((M_{L(X)})^\sigma)^a = \beta$. Put $N(\sigma) = \{\sigma \cup \{a\}\}$.

Case 2. $\mathrm{Ord}(M_{L(X)})^\sigma = \gamma \neq 0$, where γ is limit.

For every $\beta < \gamma$ take $a(\beta) \in L(X)$ such that $\mathrm{Ord}((M_{L(X)})^\sigma)^{a(\beta)} > \beta$. Put $N(\sigma) = \{\sigma \cup \{a(\beta)\} : \beta < \gamma\}$.

Case 3. $\mathrm{Ord}(M_{L(X)})^\sigma = 0$.

Put $N(\sigma) = \emptyset$.

Let us set

$M_{i+1} = \bigcup\{N(\sigma) : \sigma \in M_i\}$.

By the construction of M_i we have $|M_i| \leq |\ \mathrm{trdim} X|$. Since $\mathrm{trdim}\ X < w(X)^+$, we have $|M_i| \leq w(X)$. Take a base $\mathcal{B}$ for the open sets of X such that $|\mathcal{B}| = w(X)$. We set $\mathcal{B}' = \mathcal{B} \cup \{X - \mathrm{Cl}\ B : B \in \mathcal{B}\}$. By induction on $m < \omega$ we shall construct a collection $\mathcal{F}_m$ of closed subsets of X. Let us set

$$\mathcal{F}_0 = \wedge \cdot \vee \cdot (\{\ \mathrm{Cl}\ B : B \in \mathcal{B}'\} \cup \bigcup\{\sigma^\# : \sigma \in M_i \text{ and } i < \omega\}).$$

Then we have $|\mathcal{F}_0| = w(X)$. Suppose that $\mathcal{F}_m$ has been constructed. We shall construct $\mathcal{F}_{m+1}$. Let us set

$$\mathcal{G}_m = \{(A, B) : A, B \in \mathcal{F}_m \text{ and } A \cap B = \emptyset\}$$

If $\sigma = \{(A_i, B_i) : i \leq n\} \in [\mathcal{G}_m]^{<\omega}$ is inessential, then there exists a collection $\sigma' = \{(E_i, F_i) : i \leq n\}$ of pairs of closed subsets of X such that

$E_i \cap B_i = \emptyset = F_i \cap A_i$ for every $i \leq n$,

$E_i \cup F_i = X$ for every $i \leq n$, and

$\bigcap\{E_i \cap F_i : i \leq n\} = \emptyset$.

For every $a = (A, B) \in \mathcal{G}_m$ there exists a pair $a' = (E, F)$ of closed subsets of X such that $E \cap B = \emptyset = F \cap A$ and $E \cup F = X$. Let us set

$\mathcal{F}_{m+1} = \wedge \cdot \vee \cdot (\mathcal{F}_m \cup \bigcup\{\sigma'^\# : \sigma \in [\mathcal{G}_m]^{<\omega}$ such that σ is inessential $\} \cup \bigcup\{a'^\# : a \in \mathcal{G}_m\})$, and $\mathcal{F} = \bigcup\{\mathcal{F}_m : m < \omega\}$. Then it is easy to see that $\mathcal{F}$ is a normal base for X and $|\mathcal{F}| = w(X)$. Let αX be the Wallman compactification $w(X, \mathcal{F})$ of X with respect to $\mathcal{F}$. Because $\mathcal{F}^*$ is a base for the closed sets of αX, we have $w(\alpha X) = w(X)$.

Claim 1. $\mathrm{trdim}\ \alpha X \leq \mathrm{trdim}\ X$.

Let us set

$$L = \{(A^*, B^*) : A, B \in \mathcal{F} \text{ and } A \cap B = \emptyset\}.$$

Since $\mathcal{F}^*$ is a base for the closed sets of αX and since αX is compact, by Fact 2.7(a), for every $(E, F) \in L(\alpha X)$ there is $(A^*, B^*) \in L$ such that $E \subset A^*$ and $F \subset B^*$. By Lemma 3.2, we have Ord $M_{L(\alpha X)}$ = Ord M_L, therefore trdim αX = Ord M_L. Let $\varphi : L \to L(X)$ be the mapping defined by $\varphi((A^*, B^*)) = (A, B)$ for every $(A^*, B^*) \in L$. Then for every $\sigma = \{(A_i^*, B_i^*) : i \leq n\} \in [L]^{<\omega}$ with $\varphi(\sigma) \notin M_{L(X)}$ there is $m < \omega$ such that $A_i, B_i \in \mathcal{F}_m$ for each $i \leq n$. Since $\varphi(\sigma)$ is inessential, by the construction of $\mathcal{F}_{m+1}$, there exists $E_i, F_i \in \mathcal{F}_{m+1}$ for each $i \leq n$ such that $E_i \cap B_i = \emptyset = F_i \cap A_i$, $E_i \cup F_i = X$ and $\cap\{E_i \cap F_i : i \leq n\} = \emptyset$. Put $T_i = \mathrm{Cl}_{\alpha X}(E_i \cap F_i)$ for each $i \leq n$. Then T_i is a partiotion in αX between A_i^* and B_i^* for each $i \leq n$, and, by Fact 2.7(b), $\cap\{T_i : i \leq n\} = \emptyset$. Thus σ is inessential. This implies that $\sigma \notin M_L$. By Lemma 3.1, we have Ord $M_L \leq$ Ord $M_{L(X)}$. Hence we have trdim $\alpha X \leq$ trdim X.

For every $\sigma = \{(E_i, F_i) : i \leq n\} \in \cup\{M_j : j < \omega\}$ let us set $\sigma^* = \{(E_i^*, F_i^*) : i \leq n\}$. Since $\sigma^* \subset \mathcal{F}_0 \subset \mathcal{F}$ and since σ is essential in X, by Fact 2.7(b), σ^* is essential in αX.

Claim 2. trdim $\alpha X \geq$ trdim X.

Assume that trdim $\alpha X <$ trdim X. We shall construct $n < \omega$ and $\sigma_i \in M_i$ for every $i \leq n$ satisfying the following conditions;

(i) $\sigma_{i+1} \in N(\sigma_i)$ for every $i \leq n-1$,
(ii) $\mathrm{Ord}(M_{L(\alpha X)})^{\sigma_i^*} < \mathrm{Ord}(M_{L(X)})^{\sigma_i}$ for every $i \leq n$,
(iii) Ord $(M_{L(X))})^{\sigma_i} \geq 1$ for every $i \leq n$, and
(iv) $\mathrm{Ord}(M_{L(X)})^{\sigma_n} = 1$.

Put $\sigma_0 = \emptyset$. Since we assume that trdim $\alpha X <$ trdim $X \geq \omega$, σ_0 satisfies the conditions (ii) and (iii). Suppose that $\sigma_i \in M_i$ has been constructed. If $\mathrm{Ord}(M_{L(X)})^{\sigma_i} = 1$, then we set $n = i$. Suppose that $\mathrm{Ord}(M_{L(X)})^{\sigma_i} > 1$.

Case 1. $\mathrm{Ord}(M_{L(X)})^{\sigma_i} = \beta + 1$.

Take $\sigma \in N(\sigma_i)$ and put $\sigma_{i+1} = \sigma$. Then we have $\text{Ord}(M_{L(\alpha X)})^{\sigma_{i+1}^*} < \text{Ord}(M_{L(\alpha X)})^{\sigma_i^*} < \text{Ord}(M_{L(X)})^{\sigma_i} = \beta + 1$. By the construction of $N(\sigma_i)$, we have $\text{Ord}(M_{L(X)})^{\sigma_{i+1}} = \beta$. This implies that $\text{Ord}(M_{L(\alpha X)})^{\sigma_{i+1}^*} < \text{Ord}(M_{L(X)})^{\sigma_{i+1}}$, therefore $\sigma_i + 1$ is as required.

Case 2. $\text{Ord}(M_{L(X)})^{\sigma_i} = \gamma$, where γ is limit.

We shall show that there exists $\lambda < \gamma$ such that $\text{Ord}(M_{L(\alpha X)})^{(\sigma_i \cup \{a(\beta)\})^*} < \lambda$ for every $\beta < \gamma$. Assume the contrary. Then we have $\text{Ord}(M_{L(\alpha X)})^{\sigma_i^*} \geq \gamma$. This contradicts that Ord $(M_{L(\alpha X)})^{\sigma_i^*} < \text{Ord}(M_{L(X)})^{\sigma_i} = \gamma$. Take β with $\lambda < \beta < \gamma$ and put $\sigma_{i+1} = \sigma_i \cup \{a(\beta)\}$. Then $\text{Ord}(M_{L(X)})^{\sigma_{i+1}} = \text{Ord}(M_{L(X)})^{\sigma_i)^{a(\beta)}} > \beta > \lambda\ \text{Ord}(M_{L(\alpha X)})^{\sigma_{i+1}^*}$. Obviously, $\text{Ord}(M_{L(X)})^{\sigma_{i+1}} > \beta \geq 1$. Thus σ_{i+1} is as required.

Since σ_i is a proper subset of σ_{i+1} for every $i \leq n-1$, we have $\text{Ord}(M_{L(X)})^{\sigma_{i+1}} < \text{Ord}(M_{L(X)})^{\sigma_i}$. Thus $\text{Ord}(M_{L(X)})^{\sigma_n} = 1$ for some $n < \omega$. This completes the construction of n and $\sigma_i \in M_i$.

Take $\sigma \in N(\sigma_n)$. Since $\text{Ord}(M_L(X))^{\sigma_n} = 1$ we have $\text{Ord}(M_{L(X)})^{\sigma} = 0$. This implies that σ is essential in X. On the other hand, since $\text{Ord}(M_{L(X)})^{\sigma_n^*} < \text{Ord}(M_{L(X)})^{\sigma_n} = 1$, we have $\text{Ord}(M_{L(X)})^{\sigma_n^*} = 0$. This implies that σ^* is inessential in αX. This is a contradiction. Hence we have trdim $\alpha X \geq$ trdim X. This completes the proof of Theorem 3.6. □

We now present the following consequence of Theorems 3.5 and 3.6.

3.7. Corollary. *Every S-w.i.d. metacompact space X has a compactification αX such that* trdim αX = trdim$X$ *and* $w(\alpha X) = w(X)$.

3.8. Corollary. *Every S-w.i.d. separable metrizable space X has a metrizable compactification αX such that* trdim αX = trdim X.

3.9. Corollary. *Let X be a S-w.i.d. space with* trdim $X < \omega_1$. *Then X has a compactification αX such that* trdim αX = trdim X *and* $w(\alpha X) = w(X)$.

Proof: Since $w(X) \geq \omega$, we have trdim $X < \omega_1 \leq w(X)^+$. Apply Theorem 3.6.

4. Embeddings into the Hilbert cube

In this section we assume that all spaces are separable and metrizable. Luxemburg [L] proved that if a space X has trInd, then the following sets are residual in $C(X, I^\omega)$, where $C(X, I^\omega)$ is the space of all continuous mappings from X into the Hilbert cube I^ω with the topology of uniform convergence.

(1) $\{h \in C(X, I^\omega) : h$ is an embedding such that trind Cl $f(X) =$ trind $X\}$.
(2) $\{h \in C(X, I^\omega) : h$ is an embedding such that trInd Cl $f(X) =$ trInd $X\}$.

In this section we prove the following theorem that is similar to Luxemburg's results above.

4.1. Theorem. *For a space X the set of all embeddings $f : X \to I^\omega$ such that*
trdim Cl $f(X) =$ trdim X
is residual in $C(X, I^\omega)$.

4.2. Lemma. *For a space X the set of all continuous mappings $f : X \to I^\omega$ such that*
trdim Cl $f(X) \leq$ trdim X
is residual in $C(X, I^\omega)$.

Proof: Let τ be a finite collection of pairs of disjoint closed subsets of X and let $f : X \to I^\omega$ be continuous. Let us set

$$f(\tau) = \{(\text{Cl } f(A), \text{Cl } f(B) : (A, B) \in \tau\} \text{ and}$$

$$U(\tau) = \text{Int}\{g \in C(X, I^\omega) : g(\tau) \text{ is inessential in Cl } g(X)\}$$

By [K3], if τ is inessential in X, then $U(\tau)$ is open and dense in $C(X, I^\omega)$. Take a countable base $\mathcal{B}$ for I^ω with $\vee \cdot \mathcal{B} = \mathcal{B}$ and let $\mathcal{G} = \{\text{Cl } B : B \in \mathcal{B}\}$. Since the set $\mathcal{F} = \{\tau : \tau$ is a finite collection of pairs of disjoint sets from $\mathcal{G}\}$ is countable, enumerate $\mathcal{F}$ as $\mathcal{F} = \{\tau_i : i < \omega\}$. For every $f \in C(X, I^\omega)$ and for any $n < \omega$ we set

$$U(f,n) = \cap\{U(f^{-1}(\tau_i) : i \leq n \text{ and}$$

$$f^{-1}(\tau_i) \text{ is inessential in } X\},$$

where $f^{-1}(\tau_i) = \{(f^{-1}(A), f^{-1}(B)) : (A,B) \in \tau_i\}$. Then $U(f,n)$ is open and dense in $C(X, I^\omega)$. By induction on n, we shall construdct a pairwise disjoint collection $\mathcal{G}_n$ of open subsets of $C(X, I^\omega)$ and a continuous mapping $f_U \in U$ for every $U \in \mathcal{G}_n$ satisfying the following conditions:

(a) $\cup\mathcal{G}_n$ is dense in $C(X, I^\omega)$,
(b) mesh $\mathcal{G}_n \leq 1/n$,
(c) $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ and
(d) $\cup\{V \in \mathcal{G}_{n+1} : V \subset U\} \subset U(f_U, n)$ for every $U \in \mathcal{G}_n$.

Let $\mathcal{G}_0 = \{C(X, I^\omega)\}$and $f_{C(X,I^\omega)} = f$ for some $f \in C(X, I^\omega)$. Suppose that $\mathcal{G}_n$ has been constructed. For every $U \in \mathcal{G}_n$ let $f_U = f$ for some $f \in U$. Since $U(f_U, n)$ is open and dense in $C(X, I^\omega)$, we can take a pairwise disjoint collection $\mathcal{G}(U)$ of open subsets of $U \cap U(f_U, n)$ such that mesh $\mathcal{G}(U) \leq \frac{1}{n+1}$ and $\cup\mathcal{G}(U)$ is dense in U. Let us set

$$\mathcal{G}_{n+1} = \cup\{\mathcal{G}(U) : U \in \mathcal{G}_n\}.$$

Then, obviously, all the conditions are satisfied. Let us set

$$G_n = \cup\mathcal{G}_n \text{ and } G = \cap\{G_n : n < \omega\}.$$

Then G is residual in $C(X, I^\omega)$. Thus it suffices to show that

$$\text{trdim Cl } f(X) \leq \text{trdim } X$$

for every $f \in G$. Let $f \in G$. Take $U_n \in \mathcal{G}_n$ with $f \in U_n$. We set $f_n = f_{U_n}$ for every $n < \omega$. For $A, B \in \mathcal{G}$ with $A \cap B = \emptyset$ take $A^*, B^* \in \mathcal{G}$ such that $A \subset \text{Int } A^*, B \subset \text{Int } B^*$ and $A^* \cap B^* = \emptyset$. Since $\{f_n : n < \omega\}$ converges to f, there exists $N = N(A,B) > 0$ such that

$$f_n^{-1}(A^*) \supset f^{-1}(A) \text{ for every } n \geq N,$$
$$f_n^{-1}(B^*) \supset f^{-1}(B) \text{ for every } n \geq N \text{ and}$$
$$K(A) \cap K(B) = \emptyset,$$

where $K(A) = \text{Cl}\{f_n^{-1}(A^*) : n \geq N\}$ and $K(B) = \text{Cl}\{f_n^{-1}(B^*) : n \geq N\}$. For every $n < \omega$ let us set

$\tau_i^* = \{(A^*, B^*) : (A, B) \in \tau_i\}$,
$K(\tau_i) = \{(K(A), K(B)) : (A, B) \in \tau_i\}$ and
$\tau_i^\# = \{(\mathrm{Cl}((\mathrm{Int}A) \cap f(X), \mathrm{Cl}((\mathrm{Int}B) \cap f(X))) : (A, B) \in \tau_i\}$.

By Lemmas 3.1 and 3.2, to prove trdim Cl $f(X) \leq$ trdim X, if suffices to show that if $\tau_i^\#$ is essential in Cl$f(X)$ then so is $K(\tau_i)$ in X. Suppose that $\tau_i^\#$ is essential in Cl $f(X)$ and $\tau_i^\# = \tau_m$. Take $n < \omega$ such that $n \geq m$ and $n \geq N(A, B)$ for every $(A, B) \in \tau_i$. For every $(A^*, B^*) \in \tau_m$, we have

$\mathrm{Cl}\ ff_n^{-1}(A^*) \supset \mathrm{Cl}\ ((\mathrm{Int}A) \cap f(X))$ and
$\mathrm{Cl}\ ff_n^{-1}(B^*) \supset \mathrm{Cl}\ ((\mathrm{Int}B) \cap f(X))$.

Since $\tau_i^\#$ is essential in Cl $f(X)$, either $ff_n^{-1}(\tau_m)$ is essential in Cl $f(X)$ or Cl $ff_n^{-1}(A^*) \cap$ Cl $ff_n^{-1}(B^*) \neq \emptyset$ for some $(A^*, B^*) \in \tau_m$. Hence $ff_n^{-1}(\tau_m)$ is not inessential. Assume that $f_n^{-1}(\tau_m)$ is inessential in X. Since

$f \in U_{n+1} \subset U(f_n, n) \subset U(f_n^{-1}(\tau_m))$,

$ff_n^{-1}(\tau_m)$ is inessential. This is a contradiction. Thus $f_n^{-1}(\tau_m)$ is essential in X. Since $f_n^{-1}(A^*) \subset K(A)$ and $f_n^{-1}(B^*) \subset K(B)$, $K(\tau_i)$ is essential in X. This completes the proof of Lemma 4.2. □

4.3. Lemma. *For a space X the set of all continuous mappings $f : X \to I^\omega$ such that*
trdim Cl $f(X) \geq$ trdim X is residual in $C(X, I^\omega)$.

Proof: We distinguish two cases.

Case 1. trdim $X = \alpha$ for some ordinal α. Let M_i be as in the proof of Theorem 3.6 and let $M = \cup\{M_i : i \leq \omega\}$. Since $\cup M = \cup\{\sigma : \sigma \in M\}$ is countable, we enumerate $\cup M$ as $\cup M = \{(A_i, B_i) : i < \omega\}$. Then the set

$G = \{f \in C(X, I^\omega) : \mathrm{Cl}f(A_i) \cap \mathrm{Cl}\ f(B_i) = \emptyset$ for every $i < \omega\}$

is residual in $C(X, I^\omega)$. Similarly in the rproof of Theorem 3.6, we can prove that trdim Cl $f(X) \geq$ trdim X for every $f \in G$.

Case 2. trdim $X = \infty$.

In this case X is not S-w.i.d. Thus there exists a collection $\{(A_i, B_i) : i < \omega\}$ of pairs of disjoint closed subsets of X such that $\cap\{T_i : i \leq n\} \neq \emptyset$ for every partition T_i in X between A_i and B_i and for every $n < \omega$. Then the set

$G = \{f \in C(X, I^\omega) : \mathrm{Cl} f(A_i) \cap \mathrm{Cl}\ f(B_i) = \emptyset$ for every $i < \omega\}$

is residual in $C(X, I^\omega)$. It is easy to see that $\mathrm{Cl} f(X)$ is not S-w.i.d. for every $f \in G$. Hence we have trdim Cl $f(X) = \infty$. This completes the proof of Lemma 4.3. □

Since the set of all embeddings from X into the Hilbert cube is residual in $C(X, I^\omega)$, Theorem 4.1 follows from Lemmas 4.2 and 4.3. Applying Luxemburg's theorem, we obtain the following corollary.

4.4. Corollary. *If a space X has trInd , then X has a metrizable compactification αX such that*

trind αX = trind X, trInd αX = trInd X *and* trdim αX = trdim X.

5. Comments and questions

In [C2] Chatyrko proved that if trdim $X = \alpha < \omega_1$ and if X admits an essential mapping $f : X \to J^\alpha$, then X has a compactification αX such that trdim αX = trdim X and $w(\alpha X) = w(X)$, where J^α is Henderson's transfinite cube [He]. However, not all spaces X admit an essential mapping $f : X \to J^\alpha$, where α = trdim X, even if $\alpha < \omega_1$ (see [B2]). Thus Chatyrko's result above does not imply that Corollary 3.8 remains true.

In Theorem 3.5 we prove that trdim $X < w(X)^+$ for every S-w.i.d. metacompact space X. However, it is unknown whether there exists a S-w.i.d. space X such that trdim $X \geq w(X)^+$.

5.1. Question. *Does there exist a S-w.i.d. space X such that trdim $X \geq w(X)^+$?*

The negative answer to Question 5.1 implies that the condition of metacompactness can be dropped in Corollary 3.7.

Assume that there exists a S-w.i.d. space X such that trdim $X \geq w(X)^+$. By Lemma 3.3, for any compactification αX of X with $w(\alpha X) = w(X)$, we have trdim $\alpha X < w(\alpha X)^+ =$

$w(X)^+ \leq$ trdim X. Thus there exists no compactification αX such that trdim $\alpha X =$ trdim X and $w(\alpha X) = w(X)$. Hence the following statements are equivalent:

(1) every S-w.i.d. space X has a compactification αX such that trdim $\alpha X =$ trdim X and $w(\alpha X) = w(X)$,

(2) for every S-w.i.d. space X the inequality trdim $X < w(X)^+$ holds.

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