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THE BOURBAKI QUASI-UNIFORMITY

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ABSTRACT. We study properties of the Bourbaki quasi-uniformity $\mathcal{U}_*$ defined on the collection $\mathcal{P}_0(X)$ of all nonempty subsets of a given quasi-uniform space $(X,\mathcal{U})$.

We note that $\mathcal{U}_*$ is precompact (totally bounded, respectively) if and only if $\mathcal{U}$ is precompact (totally bounded, respectively). Examples are given that show that for the properties of compactness and hereditary precompactness the corresponding statement does not hold.

Furthermore we establish that for a quasi-uniform space $(X,\mathcal{U})$ the Bourbaki quasi-uniformity $\mathcal{U}_*$ on $\mathcal{P}_0(X)$ is right K-complete if and only if each stable filter on $(X,\mathcal{U})$ has a cluster point. This theorem generalizes the well-known Isbell-Burdick theorem for uniform spaces to the quasi-uniform setting. The paper ends with a related theorem characterizing bicompleteness of $(\mathcal{P}_0(X),\mathcal{U}_*)$ in terms of a property of $(X,\mathcal{U})$.

1. INTRODUCTION

Let $(X,\mathcal{U})$ be a quasi-uniform space. It has been observed by various authors that it is possible to define a quasi-uniformity on the collection $\mathcal{P}_0(X)$ of all nonempty subsets of X in the same way as the Hausdorff uniformity is defined in the theory of uniform spaces. In [1,5,15] basic properties of this construction in some special cases are derived. In particular the case is

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studied where $\mathcal{U}$ is the Pervin quasi-uniformity of a topological space.

In this paper we are starting a more systematic study of the quasi-uniform space $(\mathcal{P}_0(X),\mathcal{U}_*)$. In particular we are interested in the following important question: Suppose that $(X,\mathcal{U})$ is a quasi-uniform space possessing a certain property P. Does $(\mathcal{P}_0(X),\mathcal{U}_*)$ necessarily have property P, too? Of course, it is known from the theory of uniform spaces that in general the answer to this question is negative. For example, Isbell [8, p. 31] has given a simple example of a complete uniform space whose Hausdorff uniformity is not complete. Recently, based on former work of Isbell [8], Burdick [2, Corollary 2] has answered a question of Császár [3] in the affirmative by proving the following elegant characterization of the uniform spaces possessing a complete Hausdorff uniformity: The Hausdorff uniformity on $\mathcal{P}_0(X)$ of a uniform space $(X,\mathcal{U})$ is complete if and only if each stable filter on $(X,\mathcal{U})$ has a cluster point. In this paper we wish to present a surprising extension of this result to quasi-uniform spaces. Since the concept of a complete quasi-uniformity is highly controversial, it is at first not clear how to proceed. However it turns out that the appropriate concept of completeness has already been studied in the literature. In fact, for sequences in quasi-metric spaces it was considered long ago [9,18] under the name of right K-completeness; in quasi-uniform spaces the corresponding notion for filters and nets has been studied by Stoltenberg and Romaguera [20,22]. Using this simple concept of completeness, which for uniformities coincides with the usual one, Burdick's result can be generalized without any further formal change. Of course, the method of proof has to be adjusted to the nonsymmetric situation considered in this paper.

We also prove that the hyperspace $(\mathcal{P}_0(X),\mathcal{U}_*)$ of a quasi-uniform space $(X,\mathcal{U})$ is precompact (totally bounded, respectively) if and only if $(X,\mathcal{U})$ is precompact (totally bounded, respectively). Examples are given that show that the corresponding result does not obtain for the properties of hereditary precompactness and compactness. Finally we characterize bicompleteness of $(\mathcal{P}_0(X), \mathcal{U}_*)$ in terms of a property of the quasi-uniform space $(X, \mathcal{U})$.

For basic facts about (quasi-)uniform (hyper)spaces we refer the reader to [4,16,17]. A discussion of further characterizations of those uniform spaces that possess a complete Hausdorff uniformity can be found in [7].

2. Preliminary results

Let $(X, \mathcal{U})$ be a quasi-uniform space. We denote the associated uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ by $\mathcal{U}^s$. For any $U \in \mathcal{U}$ let $U_+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}$ and $U_- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}$. Furthermore set $U_* = U_- \cap (U_+)$ whenever $U \in \mathcal{U}$. Then $\{U_- : U \in \mathcal{U}\}$ is a base for *the lower quasi-uniformity* on $\mathcal{P}_0(X)$ and $\{U_+ : U \in \mathcal{U}\}$ is a base for *the upper quasi-uniformity* on $\mathcal{P}_0(X)$. Moreover $\mathcal{U}_* = \mathcal{U}_+ \vee \mathcal{U}_-$ is the so-called *Bourbaki quasi-uniformity* of X (see [1]).

For a given quasi-pseudometric space (X, d) we shall denote the open ball of radius 2^{-n} at $x \in X$ by $B_n(x)$.

Lemma 1. *(a) Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $x \mapsto \{x\}$ is an embedding of $(X, \mathcal{U})$ into $(\mathcal{P}_0(X), \mathcal{U}_*)$.*

(b) Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces and let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a quasi-uniformly continuous map. Then the map $f : (\mathcal{P}_0(X), \mathcal{U}_*) \to (\mathcal{P}_0(Y), \mathcal{V}_*)$ defined by $f(A) := \{f(a) : a \in A\}$ is quasi-uniformly continuous, too.

Proof. (a) The assertion is readily verified.

(b) If $(f \times f)(U) \subseteq V$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$, then $(f \times f)(U_*) \subseteq V_*$.

For a uniform space $(X, \mathcal{U})$, it is usually the T_0-quotient of $(\mathcal{P}_0(X), \mathcal{U}_*)$, rather than $(\mathcal{P}_0(X), \mathcal{U}_*)$ itself, that is studied. It is well known that this quotient can be described as follows: Each nonempty subset A of X is identified with its closure $\overline{A}$

and then the Hausdorff uniformity is restricted to the collection of nonempty closed subsets of X.

Similarly, our first result can be used to show that for a quasi-uniform space $(X,\mathcal{U})$ the T_0-quotient of $(\mathcal{P}_0(X),\mathcal{U}_*)$ can be described as follows: Identify an arbitrary nonempty subset A of X with $\mathrm{cl}_{\mathcal{T}(\mathcal{U})}A \cap \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}A$ and study the Bourbaki quasi-uniformity restricted to the collection of all subsets of X obtained in that way.

Lemma 2. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $A,B \in \mathcal{P}_0(X)$. We have $(A,B) \in \bigcap\mathcal{U}_*\cap(\bigcap\mathcal{U}_*)^{-1}$ iff $cl_{\mathcal{T}(\mathcal{U})}A = cl_{\mathcal{T}(\mathcal{U})}B$ and $cl_{\mathcal{T}(\mathcal{U}^{-1})}A = cl_{\mathcal{T}(\mathcal{U}^{-1})}B$. In particular for any $A \in \mathcal{P}_0(X)$, $(A,C) \in \bigcap\mathcal{U}_* \cap (\bigcap\mathcal{U}_*)^{-1}$ where $C = cl_{\mathcal{T}(\mathcal{U})}A\cap cl_{\mathcal{T}(\mathcal{U}^{-1})}A$.*

Proof: Note that $(A,B) \in (\cap\mathcal{U}_+)^{-1}$ implies that

$$A \subseteq \bigcap_{U\in\mathcal{U}} U(B) \subseteq \bigcap_{U\in\mathcal{U}} U(\mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}(B)) \subseteq \bigcap_{U\in\mathcal{U}} U^2(B),$$

and thus

$$\mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}A \subseteq \bigcap_{U\in\mathcal{U}} \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}U^2(B) \subseteq \bigcap_{U\in\mathcal{U}} U^3(B) = \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}B$$

and $\mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}A \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}B$. The other three inequalities follow similarly. On the other hand, suppose that $\mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}A \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}B$. Then $A \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}A \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}B \subseteq \bigcap_{U\in\mathcal{U}} U(B)$. Therefore $(A,B) \in (\bigcap\mathcal{U}_+)^{-1}$. Again, the three remaining inequalities follow similarly.

We have $A \subseteq C \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U})}A$. Thus $\mathrm{cl}_{\mathcal{T}(\mathcal{U})}A \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U})}C \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U})}A$. Analogously, we obtain $\mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}A = \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}C$. The second assertion follows.

We omit the proofs of the following remark, which are obvious. (A quasi-uniform space is called *transitive* provided that its quasi-uniformity has a base of transitive relations.)

Remark 1. *Let $(X,\mathcal{U})$ be a quasi-uniform space.*

(a) Then the space $(X,\mathcal{U})$ is transitive if and only if $\mathcal{U}_*$ is transitive.

(b) The quasi-uniformity $\mathcal{U}$ has a base of cardinality κ if and only if $\mathcal{U}_*$ has a base of cardinality κ.

(c) The space $(X,\mathcal{U})$ is uniform if and only if $(\mathcal{P}_0(X),\mathcal{U}_*)$ is uniform. (In fact for a uniformity $\mathcal{U}$, $\mathcal{U}_*$ is the well-known Hausdorff uniformity on $\mathcal{P}_0(X)$.)

Recall (see e.g. [11]) that a quasi-uniform space $(X,\mathcal{U})$ is *totally bounded* provided that for each $U \in \mathcal{U}$ there is a finite cover $\mathcal{A}$ of X such that $A \times A \subseteq U$ whenever $A \in \mathcal{A}$. It is said to be *precompact* if for each $U \in \mathcal{U}$ there is a finite subset F of X such that $U(F) = X$. A quasi-uniform space is called *hereditarily precompact* if each of its subspaces is precompact. Clearly "totally bounded" implies "hereditarily precompact" implies "precompact". It is well known that in the class of quasi-uniform spaces the converses do not obtain in general.

Proposition 1. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{P}_0(X),\mathcal{U}_*)$ is precompact if and only if $(X,\mathcal{U})$ is precompact.*

Proof: Let $(X,\mathcal{U})$ be precompact and let $V \in \mathcal{U}_*$. There is $U \in \mathcal{U}$ such that $U_* \subseteq V$. Since $\mathcal{U}$ is precompact, there exists a finite set $F \subseteq X$ such that $\bigcup_{f\in F} U(f) = X$. Set $\mathcal{M} = \mathcal{P}_0(F)$. We want to show that $\mathcal{P}_0(X) = \bigcup_{E\in\mathcal{M}} U_*(E)$: Consider an arbitrary $B \in \mathcal{P}_0(X)$. Set $F_B = \{f \in F : B \cap U(f) \neq \emptyset\}$. Thus $F_B \subseteq U^{-1}(B)$ and therefore $B \in U_-(F_B)$. Furthermore $B \in U_+(F_B)$, because $B \subseteq \bigcup_{f\in F_B} U(f)$. Hence $B \in U_*(F_B)$. We conclude that $(\mathcal{P}_0(X),\mathcal{U}_*)$ is precompact.

On the other hand, suppose that $(\mathcal{P}_0(X),\mathcal{U}_*)$ and thus $(\mathcal{P}_0(X),\mathcal{U}_-)$ is precompact. Let $V \in \mathcal{U}$. Then there is a finite subcollection $\mathcal{A}$ of $\mathcal{P}_0(X)$ such that for each $B \in \mathcal{P}_0(X)$ there is $A \in \mathcal{A}$ with $A \subseteq V^{-1}(B)$. Choose some $x_A \in A$ for each $A \in \mathcal{A}$. Then $B = X \setminus \bigcup_{A\in\mathcal{A}} V(x_A)$ is necessarily empty. Therefore $(X,\mathcal{U})$ is precompact.

Corollary 1. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{P}_0(X),\mathcal{U}_-)$ is precompact if and only if $(\mathcal{P}_0(X),\mathcal{U}_*)$ is precompact.*

Proof: The second part of the proof of the preceding proposition shows that $(X, \mathcal{U})$ is precompact provided that $(\mathcal{P}_0(X), \mathcal{U}_-)$ is precompact. Moreover the first part of the proof given above shows that $(\mathcal{P}_0(X), \mathcal{U}_*)$ is precompact if $(X, \mathcal{U})$ is precompact. Hence the assertion follows.

Lemma 3. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{U}_*)^s \subseteq (\mathcal{U}^s)_*$ on $\mathcal{P}_0(X)$.*

Proof: It suffices to show that for any $U \in \mathcal{U}$ we have $(U \cap U^{-1})_* \subseteq U_* \cap (U_*)^{-1}$: Let $(A, B) \in (U \cap U^{-1})_*$. Then $A \subseteq (U \cap U^{-1})(B)$ and $B \subseteq (U \cap U^{-1})(A)$. Thus $A \subseteq U(B) \cap U^{-1}(B)$ and $B \subseteq U(A) \cap U^{-1}(A)$. Therefore $A \in U_*(B)$ and $A \in (U_*)^{-1}(B)$.

Corollary 2. *A quasi-uniform space $(X, \mathcal{U})$ is totally bounded if and only if $(\mathcal{P}_0(X), \mathcal{U}_*)$ is totally bounded.*

Proof: Suppose that $(X, \mathcal{U})$ is totally bounded. Then $(X, \mathcal{U}^s)$ is precompact. Thus $(\mathcal{P}_0(X), (\mathcal{U}^s)_*)$ is precompact by the preceding proposition. Since $(\mathcal{U}^s)_*$ is a uniformity, it is totally bounded. By Lemma 3, $(\mathcal{U}_*)^s$ is coarser than $(\mathcal{U}^s)_*$; hence it is totally bounded, too. Therefore $\mathcal{U}_*$ is totally bounded. For the converse, suppose that $(\mathcal{P}_0(X), \mathcal{U}_*)$ is totally bounded. Since total boundedness is a hereditary property of quasi-uniform spaces, the assertion follows from Lemma 1(a).

Proposition 2. *A quasi-uniform space $(X, \mathcal{U})$ is compact provided that $(\mathcal{P}_0(X), \mathcal{U}_-)$ is compact.*

Proof: Let $\mathcal{F}$ be a filter on $(X, \mathcal{U})$. Since $(\mathcal{P}_0(X), \mathcal{U}_-)$ is compact, the net $(F)_{F \in (\mathcal{F}, \supseteq)}$ has a cluster point C in $\mathcal{P}_0(X)$. Suppose that $x \in C$. For any $U \in \mathcal{U}$ and $F_0 \in \mathcal{F}$, there is $F \in \mathcal{F}$ such that $F \subseteq F_0$ and $C \subseteq U^{-1}(F)$. Thus $U(x) \cap F_0 \neq \emptyset$. We conclude that x is a cluster point of $\mathcal{F}$. Hence $(X, \mathcal{U})$ is compact.

Corollary 3. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(X, \mathcal{U})$ is compact if $(\mathcal{P}_0(X), \mathcal{U}_*)$ is compact.*

Example 1. *If* $(X,\mathcal{U})$ *is compact,* $(\mathcal{P}_0(X),\mathcal{U}_*)$ *need not be:* Let $X = \{0\} \cup \{\frac{1}{n} : n \in \omega \setminus \{0\}\}$ be equipped with its usual metric uniformity $\mathcal{U}_d$. Set $q_0 = 0$, and $q_n = \frac{1}{n}$ if $n \in \omega \setminus \{0\}$. Let $T(q_n) = \{q_k : k \in \omega \text{ and } k \geq n\}$ whenever $n \in \omega$ and consider the compatible quasi-uniformity $\mathcal{U}$ on X that is generated by $\mathcal{U}_d \cup \{T\}$ where $T = \bigcup_{x\in X}(\{x\} \times T(x))$. We show that $(\mathcal{P}_0(X),\mathcal{U}_*)$ is not (countably) compact, although $(X,\mathcal{U})$ is compact: To this end consider the sequence $(F_n)_{n\in\omega}$ in $\mathcal{P}_0(X)$ where $F_n = \{q_k : k \leq n \text{ and } k \in \omega\}$ whenever $n \in \omega$. Suppose that C is a cluster point of $(F_n)_{n\in\omega}$ in $(\mathcal{P}_0(X),\mathcal{U}_*)$. There is $n \in \omega$ such that $C \subseteq T^{-1}(F_n)$. Since $T^{-1}(x)$ is finite whenever $x \in X$, we conclude that C is finite. Choose $n \in \omega$ such that $X \setminus B^d_n(C) \neq \emptyset$ and $m \in \omega$ such that $q_m \notin B^d_n(C)$. But then $F_k \not\subseteq B^d_n(C)$ whenever $k \in \omega$ and $k \geq m$— a contradiction. We have shown that $(F_n)_{n\in\omega}$ does not have a cluster point in $(\mathcal{P}_0(X),\mathcal{U}_*)$. Thus $(\mathcal{P}_0(X),\mathcal{U}_*)$ is not (countably) compact.

Lemma 4. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space. Then* $(\mathcal{P}_0(X),\mathcal{U}_+)$ *is hereditarily precompact if and only if* $(\mathcal{P}_0(X),\mathcal{U}_-)$ *is hereditarily precompact.*

Proof: Suppose that $(\mathcal{P}_0(X),\mathcal{U}_-)$ is hereditarily precompact, but that $(\mathcal{P}_0(X),\mathcal{U}_+)$ is not hereditarily precompact. Hence there is $U \in \mathcal{U}$ and a sequence $(A_n)_{n\in\omega}$ in $\mathcal{P}_0(X)$ such that $A_i \not\subseteq U(A_j)$ whenever $i,j \in \omega$ and $i > j$. Choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$ and fix $i,j \in \omega$ such that $i > j$. It follows that $V(A_i) \not\subseteq V^2(A_j)$. Therefore $X \setminus V^2(A_j) \not\subseteq X \setminus V(A_i)$. Since $V^{-1}(X \setminus V^2(A_i)) \subseteq X \setminus V(A_i)$, we have that $X \setminus V^2(A_j) \not\subseteq V^{-1}(X \setminus V^2(A_i))$. Set $B_s = X \setminus V^2(A_s)$ whenever $s \in \omega$. Then $\{B_s : s \in \omega\}$ is a nonprecompact subspace of $(\mathcal{P}_0(X),\mathcal{U}_-)$ — a contradiction. Thus $(\mathcal{P}_0(X),\mathcal{U}_+)$ is hereditarily precompact provided that $(\mathcal{P}_0(X),\mathcal{U}_-)$ is hereditarily precompact. The converse is shown analogously.

Corollary 4. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space. Then* $(\mathcal{P}_0(X),\mathcal{U}_*)$ *is hereditarily precompact if and only if* $(\mathcal{P}_0(X),\mathcal{U}_+)$ *is hereditarily precompact.*

Proof: It is well-known that the supremum of hereditarily precompact quasi-uniformities is hereditarily precompact [11, Corollary 8]. The nontrivial part of the assertion follows.

Example 2. *We construct a hereditarily precompact quasi-uniform space $(X,\mathcal{U})$ such that $(\mathcal{P}_0(X),\mathcal{U}_*)$ is not hereditarily precompact.* Let $X = \omega \times \omega$. For any $n \in \omega$ set $A_n = (n \times n) \cup (\{n\} \times \omega)$. Furthermore set $\mathcal{C} = \{X \setminus A_n : n \in \omega\} \cup \{X\}$. Finally, for each $x \in X$, set $T(x) = \bigcap\{C \in \mathcal{C} : x \in C\}$. Consider the quasi-uniformity $\mathcal{U} = \text{fil}\{T\}$ on X, where $T = \bigcup_{x \in X}(\{x\} \times T(x))$.

We first show that $(X,\mathcal{U})$ is hereditarily precompact. Note that for each $x \in X$, there is $j_0 \in \omega$ such that $x \in A_j$ for all $j \in \omega$ whenever $j \geq j_0$; thus $X \setminus T(x) = \bigcup\{A_n : x \notin A_n, n \in \omega\}$ is equal to the union of finitely many sets A_j. Suppose that $(X,\mathcal{U})$ is not hereditarily precompact. Then there is a sequence $(z_n)_{n \in \omega}$ of points of X such that $z_j \notin T(z_i)$ whenever $i, j \in \omega$ and $i < j$.

Since $\emptyset \neq X \setminus T(z_0)$ consists of the union of finitely many sets A_j and since the elements $(z_i)_{i \in \omega}$ are pairwise distinct, we conclude that there exists $n \in \omega$ such that $z_j \in \{n\} \times \omega$ for infinitely many j. Thus there are $m_1, m_2 \in \omega$ such that $m_1 < m_2$ and $(n, m_2) \notin T(n, m_1)$. Clearly this is impossible, since $(n, m_2) \in A_j \subseteq X \setminus T(n, m_1)$ for some $j \in \omega$ implies that $(n, m_1) \in A_j$. We conclude that $(X,\mathcal{U})$ is hereditarily precompact.

Finally we prove that $(\mathcal{P}_0(X),\mathcal{U}_-)$ is not hereditarily precompact. In fact we show that $(\mathcal{E},\mathcal{U}_-|(\mathcal{E} \times \mathcal{E}))$ is not precompact where $\mathcal{E} = \{A_n : n \in \omega\}$. Observe first that $T^{-1}(A_n) = A_n$ for each $n \in \omega$. If $(\mathcal{E},\mathcal{U}_-|(\mathcal{E} \times \mathcal{E}))$ is precompact, then there are $i, j \in \omega$ such that $i < j$ and $A_i \not\subseteq T^{-1}(A_j)$. But $A_i \subseteq A_j$ clearly does not hold, since $\{i\} \times \omega \not\subseteq A_j$. We have shown that $\mathcal{E}$ is not precompact in $(\mathcal{P}_0(X),\mathcal{U}_-)$.

Let us note that an example having better separation properties can be obtained by putting $\mathcal{V} = \sup\{\mathcal{P},\mathcal{U}\}$ where $\mathcal{P}$ is the Pervin quasi-uniformity for the discrete topology on X. Of course, the same argument as given above applies.

3. Right K-completeness of the Bourbaki quasi-uniformity

It is known (compare [19, Lemma] and [20, Proposition 2]) that in any quasi-pseudometric space (X, d) the following properties are equivalent:

(a) each left K-Cauchy sequence converges in (X, d).

(b) each left K-Cauchy filter on $(X, \mathcal{U}_d)$ converges.

(c) each $\mathcal{U}_d^{-1}$-stable filter has a cluster point in (X, d).

Moreover it is shown in [13, Proposition 1] that the well-monotone quasi-uniformity of any topological space is left K-complete. It will follow from the results presented in this section that the property of right K-completeness behaves differently. Let us recall the necessary definitions first.

A sequence $(x_n)_{n\in\omega}$ in a quasi-pseudometric space (X, d) is called *right K-Cauchy* [18, Definition 1] if for each $\epsilon > 0$ there is $k \in \omega$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \in \omega$ and $n \geq m \geq k$. A quasi-pseudometric space (X, d) is said to be *right K-sequentially complete* [18, Definition 3] if each right K-Cauchy sequence converges. A filter on a quasi-uniform space $(X, \mathcal{U})$ is called a *right K-Cauchy filter* [20, Definition 1] if for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U^{-1}(x) \in \mathcal{F}$ whenever $x \in F$. A quasi-uniform space is called *right K-complete* [20, Definition 2] provided that any right K-Cauchy filter converges. A net $(x_d)_{d\in D}$ in a quasi-uniform space $(X, \mathcal{U})$ is called a *right K-Cauchy net* [12] (compare [22, p. 229]) if for any $U \in \mathcal{U}$ there exists $d_U \in D$ such that for any $d_1, d_2 \in D$ satisfying $d_1 \geq d_2 \geq d_U$ we have that $(x_{d_1}, x_{d_2}) \in U$.

Obviously, the concepts of Cauchy filters and completeness discussed above agree all with the usual ones in uniform and metric spaces. In [12, Lemma 1] it is shown that a quasi-uniform space is right K-complete if and only if each right K-Cauchy net converges. Obviously a quasi-pseudometric space

(X,d) is right K-sequentially complete provided that the quasi-pseudometric quasi-uniformity $\mathcal{U}_d$ is right K-complete. It is known that the converse holds for regular spaces [21, Proposition 3], but not in general [22, Example 2.4].

Finally we recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called *stable* (compare [8, p. 29]) provided that for any $U \in \mathcal{U}$, $\bigcap_{F\in\mathcal{F}} U(F)$ belongs to $\mathcal{F}$.

The following result belongs to the folklore of the subject and is included here for completeness. In fact, according to [20, Proposition 1] an ultrafilter on a quasi-uniform space is right K-Cauchy if and only if it is stable.

Lemma 5. *On a quasi-uniform space $(X,\mathcal{U})$ each right K-Cauchy filter is stable.*

Proof: Let $\mathcal{F}$ be a right K-Cauchy filter on $(X,\mathcal{U})$ and let $U \in \mathcal{U}$. Then there is $M(U) \in \mathcal{F}$ such that for any $y \in M(U)$ we have $U^{-1}(y) \in \mathcal{F}$. Consider any $x \in M(U)$. Note that $U^{-1}(x) \cap F \neq \emptyset$ whenever $F \in \mathcal{F}$. Thus $x \in \bigcap_{F\in\mathcal{F}} U(F)$ and, consequently, $M(U) \subseteq \bigcap_{F\in\mathcal{F}} U(F)$. We conclude that $\bigcap_{F\in\mathcal{F}} U(F) \in \mathcal{F}$. Therefore, $\mathcal{F}$ is stable.

Corollary 5. *Let $(X,\mathcal{U})$ be a quasi-uniform space in which each stable filter has a cluster point. Then $(X,\mathcal{U})$ is right K-complete.*

Proof: We use the preceding lemma and the fact that each right K-Cauchy filter converges to any of its cluster points [20, Lemma 1].

Not much seems to be known about quasi-uniform spaces in which each stable filter has a cluster point. In the following we collect some pertinent results.

Proposition 3. *For any Lindelöf right K-sequentially complete quasi-pseudometric space (X,d) each stable filter on $(X,\mathcal{U}_d)$ has a cluster point.*

Proof: In order to obtain a contradiction, let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U}_d)$ such that $\bigcap_{F\in\mathcal{F}}\overline{F} = \emptyset$. Since X is a Lindelöf space, there is a countable subcollection $\{F_n : n \in \omega\}$ of $\mathcal{F}$ such that $\bigcap_{n\in\omega}\overline{F_n} = \emptyset$. Choose $x_0 \in \bigcap_{F\in\mathcal{F}} B_0(F) \cap F_0$. Define inductively a sequence $(x_n)_{n\in\omega}$ of points in X by choosing $x_n \in [\bigcap_{F\in\mathcal{F}} B_n(F) \cap \bigcap_{k=0}^{n} F_k \cap B_{n-1}^{-1}(x_{n-1})]$ for any $n \in \omega$ such that $n > 0$. Then clearly $(x_n)_{n\in\omega}$ is a right K-Cauchy sequence. It has a limit point x in X, because (X,d) is right K-sequentially complete. Therefore $x \in \bigcap_{k\in\omega}\overline{F_k}$ —a contradiction. We conclude that each stable filter on $(X,\mathcal{U}_d)$ has a cluster point.

Corollary 6. *The quasi-metric Sorgenfrey line $(\boldsymbol{R}, d_S)$ has the property that each stable filter on $(\boldsymbol{R},\mathcal{U}_{d_S})$ has a cluster point. (Recall that the quasi-metric d_S is defined as follows: For any $x,y \in \boldsymbol{R}$ set $d_S(x,y) = 1$ if $y < x$ and $d_S(x,y) = y - x$ if $x \leq y$.)*

Proof: It is well-known [21, Remark 1(a)] and easy to see that the quasi-metric Sorgenfrey line is right K-(sequentially) complete. Thus the assertion is a consequence of the preceding lemma, since the space has the Lindelöf property.

Next we present an example of a right K-complete quasi-metric space that possesses a stable filter without cluster point.

Example 3. *The quasi-metric Sorgenfrey plane $(\boldsymbol{R}\times\boldsymbol{R}, d_S \times d_S)$ where $(d_S \times d_S)((x_1,x_2),(y_1,y_2)) = \max\{d_S(x_1,y_1), d_S(x_2,y_2)\}$ ($x_1,x_2,y_1,y_2 \in \boldsymbol{R}$) is right K-complete, but it contains a stable filter without cluster point:* Indeed, since a product of right K-complete quasi-uniform spaces is right K-complete (see e.g. [20, Remark 3(c)]), the quasi-metric Sorgenfrey plane is right K-complete. Consider the filter fil$\{A_{\delta,F} : \delta > 0$ and F is a finite subset of $\boldsymbol{R}\}$ on $\boldsymbol{R}\times\boldsymbol{R}$ where $A_{\delta,F} = \{(x,y) \in \boldsymbol{R}\times\boldsymbol{R} : -x < y < -x+\delta\}\setminus\bigcup_{x\in F}[B_\delta^d(x)\times B_\delta^d(x)]$. Clearly that filter has no cluster point. But it is stable, since $A_{\delta,\emptyset} \subseteq \bigcap_{\epsilon,F} B_\delta(A_{\epsilon,F})$ for any $\delta > 0$.

Let X be a topological space. Consider a point-finite open cover $\mathcal{C}$ of X. For any $x \in X$, set $T_{\mathcal{C}}(x) = \bigcap\{C : x \in C \in \mathcal{C}\}$. Moreover set $T_{\mathcal{C}} = \bigcup_{x\in X}(\{x\} \times T_{\mathcal{C}}(x))$. Then $\{T_{\mathcal{C}} : \mathcal{C}$ is a point-finite open cover of $X\}$ generates a compatible quasi-uniformity on X. It is called the *point-finite covering quasi-uniformity of* X [4 p. 30].

Example 4. *The point-finite covering quasi-uniformity $\mathcal{PF}$ of a metacompact space X has the property that each stable filter has a cluster point.*

Proof: Suppose that $\mathcal{F}$ is a filter on X such that $\bigcap_{F\in\mathcal{F}} \overline{F} = \emptyset$. Since X is metacompact, the open cover $\{X \setminus \overline{F} : F \in \mathcal{F}\}$ of X has a point-finite open refinement $\mathcal{M}$. Thus $T_{\mathcal{M}} \in \mathcal{PF}$. Then for each $x \in X$, $T_{\mathcal{M}}^{-1}(x)$ is contained in the union of finitely many open sets belonging to $\mathcal{M}$. Consequently $\bigcap_{F\in\mathcal{F}} T_{\mathcal{M}}(F) = \emptyset$. We conclude that $\mathcal{F}$ is not stable. Hence each stable filter on $(X, \mathcal{PF})$ has a cluster point.

Example 5. *Equip ω_1 with the order topology. We want to show that the space X so defined does not admit any quasi-uniformity that is right K-complete.* Indeed, let $\mathcal{U}$ be a compatible quasi-uniformity on X and set $\mathcal{F} = \text{fil}\{F \subseteq X : F$ is closed and unbounded$\}$ on X. Then $\mathcal{F}$ has no cluster point. We prove that it is a right K-Cauchy filter on $(X,\mathcal{U})$. Suppose that $U \in \mathcal{U}$ and that $V \in \mathcal{U}$ such that $V^2 \subseteq U$. For any $x \in \omega_1 \setminus \{0\}$ there is $\beta_x \in \omega_1$ such that $\beta_x < x$ and $]\beta_x, x] \subseteq V(x)$. By the Pressing-Down Lemma (see e.g. [14, p. 153]), there are $\beta \in \omega_1$ and an uncountable subset S of ω_1 such that $\beta_x < \beta$ whenever $x \in S$. Consider an arbitrary $\alpha \in \omega_1$ such that $\alpha \geq \beta$. Then $S \cap [\alpha, \rightarrow[\subseteq V^{-1}(\alpha)$.

Thus $\overline{S \cap [\alpha, \rightarrow[} \subseteq V^{-2}(\alpha) \subseteq U^{-1}(\alpha)$. Since $\overline{S \cap [\alpha, \rightarrow[} \in \mathcal{F}$, $U^{-1}(\alpha) \in \mathcal{F}$. Because $[\beta, \rightarrow[\in \mathcal{F}$, we have shown that $\mathcal{F}$ is a right K-Cauchy filter on $(X,\mathcal{U})$.

It is well known that a uniform space in which each stable filter has a cluster point induces a paracompact topology

[8, Chapter VII, Theorem 41]. Our next result shows that a quasi-uniformity with the property that each stable filter has a cluster point need not induce a metacompact topology.

Example 6. *Let $(X,\mathcal{U})$ be a quasi-uniform space having an entourage $V \in \mathcal{U}$ such that for each $x \in X$, $V(x)$ or $V^{-1}(x)$ is a singleton. Then each stable filter on $(X,\mathcal{U})$ has a cluster point.* Indeed, let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U})$. Set $H = \bigcap_{F\in\mathcal{F}} V(F)$. Suppose that $\bigcap\mathcal{F} = \emptyset$. If $x \in H$ and $V^{-1}(x) = \{x\}$, then $x \in \bigcap\mathcal{F}$. Thus $H \subseteq \{x \in X : V(x) = \{x\}\}$. Consequently for any $F \in \mathcal{F}$ such that $F \subseteq H$, we have that $H \subseteq V(F) = F$. Thus $\emptyset = \cap\mathcal{F} = H \in \mathcal{F}$ —a contradiction. Therefore $\cap\mathcal{F} \neq \emptyset$. In particular $\mathcal{F}$ has a cluster point in $(X,\mathcal{U})$.

Corollary 7. *The space Ψ [6, 5I] equipped with its usual quasi-metric quasi-uniformity $\mathcal{U}_d$ has the property that each stable filter has a cluster point: We have $B_1(x) = \{x\}$ for an isolated point x and $B_1^{-1}(x) = \{x\}$ otherwise. It is well known (and easy to see) that Ψ is not metacompact.*

Proposition 4. (Compare [2, Corollary 5].) *Any quasi-uniform space $(X,\mathcal{U})$ that possesses an entourage V such that for each $x \in X$ the set $V^{-1}(x)$ is compact in $(X,\mathcal{U})$ has the property that each stable filter on $(X,\mathcal{U})$ has a cluster point.*

Proof: Let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U})$. Then there exists some $x \in \bigcap_{F\in\mathcal{F}} V(F)$. Thus $V^{-1}(x) \cap \overline{F} \neq \emptyset$ whenever $F \in \mathcal{F}$. Since $V^{-1}(x)$ is compact, we conclude that $\mathcal{F}$ has a cluster point in $V^{-1}(x)$.

Let d be a bounded quasi-pseudometric on X. We shall consider the Hausdorff quasi-pseudometric on $\mathcal{P}_0(X)$ defined by

$$d_*(A,B) = \max\{\sup_{y\in B} d(A,y), \sup_{a\in A} d(a,B)\}$$

whenever $A, B \in \mathcal{P}_0(X)$. Of course (see [1]) this quasi-pseudometric induces on $\mathcal{P}_0(X)$ the Bourbaki quasi-uniformity of the quasi-uniform space $(X,\mathcal{U}_d)$. The following proposition generalizes the well-known result that the Hausdorff metric of a (bounded) metric space (X, d) is complete if and only if (X, d) is complete (see e.g. [2, Corollary 6]). We include a proof of Proposition 5, because — although many ideas are related to the proof of the generalized Isbell-Burdick Theorem presented below — the use of sequences allows some simplifications.

Proposition 5. *Let (X, d) be a bounded quasi-pseudometric space. Then (X, d) is right K-sequentially complete if and only if $(\mathcal{P}_0(X), d_*)$ is right K-sequentially complete.*

Proof: Suppose that $(\mathcal{P}_0(X), d_*)$ is right K-sequentially complete. Let $(x_n)_{n\in\omega}$ be a right K-Cauchy sequence in (X, d). We first verify that $(\{x_n : n \in \omega \text{ and } n \geq k\})_{k\in\omega}$ is a right K-Cauchy sequence in $\mathcal{P}_0(X)$: Let $s \in \omega$. Then there is $k_s \in \omega$ such that for all $n, m \in \omega$ with $n \geq m \geq k_s$ we have that $d(x_n, x_m) < 2^{-s}$. Thus $\{x_n : n \in \omega \text{ and } n \geq k_2\} \subseteq B_s(\{x_n : n \in \omega \text{ and } n \geq k_1\})$ whenever $k_1, k_2 \in \omega$ and $k_1 \geq k_2 \geq k_s$. Furthermore $\{x_n : n \in \omega \text{ and } n \geq k_1\} \subseteq B_s^{-1}(\{x_n : n \in \omega \text{ and } n \geq k_2\})$. Since $(\mathcal{P}_0(X), d_*)$ is right K-sequentially complete, there is C in $\mathcal{P}_0(X)$ such that $(\{x_n : n \in \omega \text{ and } n \geq k\})_{k\in\omega} \to C$ in $(\mathcal{P}_0(X), d_*)$. Hence for any $n \in \omega$ there is $p_n \in \omega$ such that $C \subseteq B_{n+1}^{-1}(\{x_n : n \in \omega \text{ and } n \geq k\})$ for any $k \in \omega$ satisfying $k \geq p_n$. Fix $x \in C$. We conclude that x is a cluster point of $(x_n)_{n\in\omega}$. Hence $(x_n)_{n\in\omega}$ converges to x : Indeed, consider any $\epsilon > 0$. There is $n_0 \in \omega$ such that for $n, m \in \omega$ with $n \geq m \geq n_0$ we have that $d(x_n, x_m) < \epsilon/2$. Given $m \in \omega$ with $m \geq n_0$, choose $n \in \omega$ such that $n \geq m$ and $d(x, x_n) < \epsilon/2$. Hence $d(x, x_m) < \epsilon$ for all $m \in \omega$ such that $m \geq n_0$ and $(x_n)_{\in\omega}$ converges to x. We have shown that (X, d) is right K-sequentially complete.

In order to prove the converse, suppose that (X, d) is right K-sequentially complete. Let $(A_n)_{n\in\omega}$ be a right K-Cauchy sequence in $(\mathcal{P}_0(X), d_*)$. Thus for each $n \in \omega$ there is $m_n \in \omega$ such that for all $h_1, h_2 \in \omega$ satisfying $h_1 \geq h_2 \geq m_n$ we have that $d_*(A_{h_1}, A_{h_2}) < 2^{-n}$. Consequently, $A_{h_2} \subseteq B_n(A_{h_1})$ and $A_{h_1} \subseteq B_n^{-1}(A_{h_2})$ whenever $h_1, h_2 \in \omega$ such that $h_1 \geq h_2 \geq m_n$. Without loss of generality we can suppose that $(m_n)_{n\in\omega}$ is a strictly increasing sequence. Let $C = \{x \in X$: For each $n \in \omega$ we have $B_n(x) \cap A_m \neq \emptyset$ for infinitely many $m \in \omega\}$. We first show that $C \neq \emptyset$. In fact, we shall verify that whenever there are given some fixed $k \in \omega$, $e_k \in \omega$ such that $e_k \geq m_{k+1}$, and $a_k \in A_{e_k}$, then inductively we can find sequences $(e_n)_{n \geq k}$ and $(a_n)_{n \geq k}$ such that for all $n \geq k$,

(1) $e_n \in \omega$ and $e_n \geq m_{n+1}$,

(2) $a_n \in A_{e_n}$ and

(3) $d(a_{n+1}, a_n) < 2^{-(n+1)}$.

Let us first note that then $(a_n)_{n \geq k}$ is a right K-Cauchy sequence in (X, d). Indeed by the triangle inequality $d(a_s, a_n) < 2^{-n}$ whenever $s, n \in \omega$ such that $s \geq n \geq k$. Thus $(a_n)_{n \geq k}$ converges in (X, d) to some $x \in X$. Obviously $x \in C$. We observe that in particular we can find such an $x \in C$ by choosing first some $a_1 \in A_{e_1}$ where $e_1 \geq m_2$. Thus $C \neq \emptyset$.

We are going to verify now that the described induction can be completed:

Suppose that for some $q \in \omega$ such that $q > k$ we have chosen $(e_n)_{q > n \geq k}$ and $(a_n)_{q > n \geq k}$ as described above. Since $a_{q-1} \in A_{e_{q-1}} \subseteq B_q(A_h)$ whenever $h \in \omega$ and $h \geq e_{q-1}$, we can find $e_q \in \omega$ such that $e_q \geq \{m_{q+1}, e_{q-1}\}$ and choose $a_q \in A_{e_q}$ such that $d(a_q, a_{q-1}) < 2^{-q}$. This concludes the induction.

Next we prove that $(A_n)_{n \in \omega}$ converges in $(\mathcal{P}_0(X), d_*)$ to C. Suppose that there exists $n_0 \in \omega$ such that $A_n \not\subseteq B_{n_0}(C)$ for infinitely many $n \in \omega$. Then find some element in ω, say e_{n_0+1}, such that $e_{n_0+1} \geq m_{n_0+2}$ and $A_{e_{n_0+1}} \setminus B^4_{n_0+2}(C) \neq \emptyset$. Furthermore choose $a_{n_0+1} \in A_{e_{n_0+1}} \setminus B^4_{n_0+2}(C)$. Thus $B^{-1}_{n_0+1}(a_{n_0+1}) \cap B_{n_0+1}(C) = \emptyset$. Inductively we construct a right K-Cauchy sequence $(a_n)_{n \geq n_0+1}$ in the way as described above by starting with the point a_{n_0+1} in X. Then for any $n \in \omega$ such that $n \geq n_0 + 1$ we have that $d(a_n, a_{n_0+1}) < 2^{-(n_0+1)}$ and thus $a_n \notin B_{n_0+1}(C)$. Since (X, d) is right K-sequentially complete, there is $x_0 \in X$ such that $(a_n)_{n \geq n_0+1}$ converges to x_0 in (X, d). Hence $x_0 \in C \cap (X \setminus B_{n_0+1}(C))$ —a contradiction. We conclude that for any $n_0 \in \omega$ there exists $s \in \omega$ such that $A_n \subseteq B_{n_0}(C)$ whenever $n \in \omega$ and $n \geq s$.

Let $s \in \omega$. We finally show that $C \subseteq B_s^{-1}(A_n)$ whenever $n \in \omega$ such that $n \geq m_{s+1}$. Fix $n \in \omega$ such that $n \geq m_{s+1}$. Consider any $x \in C$. By the definition of C there is $t \in \omega$ such that $t \geq n$ and $B_{s+1}(x) \cap A_t \neq \emptyset$. Because $A_t \subseteq B_{s+1}^{-1}(A_n)$ we have $x \in B_{s+1}^{-1}(A_t) \subseteq B_{s+1}^{-2}(A_n) \subseteq B_s^{-1}(A_n)$. We have shown that $C \subseteq B_s^{-1}(A_n)$. Consequently $(A_n)_{n\in\omega}$ converges to C in $(\mathcal{P}_0(X), d_*)$. Hence $(\mathcal{P}_0(X), d_*)$ is right K-sequentially complete.

Next we are going to generalize the Isbell-Burdick theorem to quasi-uniform spaces. As we have mentioned above, it says that the Hausdorff uniformity on $\mathcal{P}_0(X)$ of a uniform space $(X,\mathcal{U})$ is complete if and only if each stable filter on $(X,\mathcal{U})$ has a cluster point. (A proof of this theorem based on nets instead of filters is presented in [2].)

The main ingredient of our result is the following lemma.

Lemma 6. *Suppose that $(X,\mathcal{U})$ is a quasi-uniform space in which each stable filter has a cluster point. Let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U})$ and let C be its set of cluster points in $(X,\mathcal{U})$. Then for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $F \subseteq U(C)$.*

Proof: Suppose the contrary. Hence there is $U_0 \in \mathcal{U}$ such that $E \setminus U_0^2(C) \neq \emptyset$ whenever $E \in \mathcal{F}$. For each $U \in \mathcal{U}$ and $E \in \mathcal{F}$ set $H_{UE} = \{a \in X$: There is $V \in \mathcal{U}$ such that $V^2 \subseteq U$, $V^{-2}(a) \cap U_0(C)$ is empty and $a \in \bigcap_{F\in\mathcal{F}} V(F) \cap E\}$. Observe that each such set $H_{UE} \neq \emptyset$: To this end choose $V \in \mathcal{U}$ such that $V^2 \subseteq U_0 \cap U$. Then any $a \in (\bigcap_{F\in\mathcal{F}} V(F) \cap E) \setminus U_0^2(C)$ belongs to H_{UE}.

Note also that for any $U_1, U_2 \in \mathcal{U}$ such that $U_1 \subseteq U_2$ and any $E_1, E_2 \in \mathcal{F}$ such that $E_1 \subseteq E_2$ we have that $H_{U_1E_1} \subseteq H_{U_2E_2}$.

Thus $\{H_{UE} : U \in \mathcal{U}, E \in \mathcal{F}\}$ is a base for a filter $\mathcal{H}$ on X. In order to show that $\mathcal{H}$ is stable on $(X,\mathcal{U})$, we verify that for any $U, V \in \mathcal{U}$ and $E \in \mathcal{F}$ we have $H_{UX} \subseteq U(H_{VE})$: Let $a \in H_{UX}$. Then there exists $W \in \mathcal{U}$ such that $W^2 \subseteq U$, $W^{-2}(a) \cap U_0(C) = \emptyset$ and $a \in \bigcap_{F\in\mathcal{F}} W(F)$. Choose $Z \in \mathcal{U}$ such that $Z^2 \subseteq V \cap W$. There is $y \in [E \cap \bigcap_{F\in\mathcal{F}} Z(F)] \cap W^{-1}(a)$, because $a \in \bigcap_{F\in\mathcal{F}} W(F)$ and $E \cap \bigcap_{F\in\mathcal{F}} Z(F) \in \mathcal{F}$.

Furthermore $Z^{-2}(y) \subseteq W^{-1}(y) \subseteq W^{-2}(a)$ and thus $Z^{-2}(y) \cap U_0(C) = \emptyset$. We conclude that $y \in H_{VE}$ and $a \in W(y) \subseteq U(y)$. Therefore $H_{UX} \subseteq U(H_{VE})$. We have shown that $\mathcal{H}$ is stable on $(X,\mathcal{U})$. Hence it has a cluster point $x \in X$. Since $H_{X \times XF} \subseteq F$ whenever $F \in \mathcal{F}$, it follows that $x \in C$. But $H_{X \times XX} \cap \text{int } U_0(C) = \emptyset$ and $x \in C \subseteq \text{int } U_0(C)$. We have obtained a contradiction and deduce that our initial assumption was wrong.

Proposition 6. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{P}_0(X),\mathcal{U}_*)$ is right K-complete if and only if each stable filter on $(X,\mathcal{U})$ has a cluster point.*

Proof: Suppose that $(\mathcal{P}_0(X),\mathcal{U}_*)$ is right K-complete. Let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U})$. Consider the net $(F)_{F \in (\mathcal{F},\supseteq)}$ on $\mathcal{P}_0(X)$. Let $U \in \mathcal{U}$. Since $\mathcal{F}$ is stable, there is $F_U \in \mathcal{F}$ such that $F_U \subseteq U(F)$ whenever $F \in \mathcal{F}$. Thus for any $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq F_2 \subseteq F_U$, we have that $F_2 \subseteq U(F_1)$ and $F_1 \subseteq U^{-1}(F_2)$. Therefore $(F)_{F \in \mathcal{F}}$ is a right K-Cauchy net in $(\mathcal{P}_0(X),\mathcal{U}_*)$. Since $(\mathcal{P}_0(X),\mathcal{U}_*)$ is right K-complete, $(F)_{F \in \mathcal{F}}$ converges to some C in $(\mathcal{P}_0(X),\mathcal{U}_*)$. Fix $x \in C$. As in the proof of Proposition 2 we conclude that x is a cluster point of $\mathcal{F}$ in $(X,\mathcal{U})$. Hence each stable filter on $(X,\mathcal{U})$ has a cluster point.

In order to prove the converse suppose that each stable filter on $(X,\mathcal{U})$ has a cluster point. Consider any right K-Cauchy net $(F_d)_{d \in D}$ on $(\mathcal{P}_0(X),\mathcal{U}_*)$. For each $U \in \mathcal{U}$ there is $d_U \in D$ such that for any $d_1, d_2 \in D$ satisfying $d_1 \geq d_2 \geq d_U$ we have that $F_{d_2} \subseteq U(F_{d_1})$ and $F_{d_1} \subseteq U^{-1}(F_{d_2})$. Consider the filter $\mathcal{F} = \text{fil}\{E_e : e \in D\}$ on X where $E_e = \bigcup_{d \in D, d \geq e} F_d$ whenever $e \in D$. We verify that for each $U \in \mathcal{U}$ we have $E_{d_U} \subseteq \bigcap_{d \in D} U(E_d)$: Let $x \in E_{d_U}$ and $d \in D$. Then $x \in F_{d_0}$ for some $d_0 \in D$ such that $d_0 \geq d_U$. Choose $h \in D$ such that $h \geq d_0, d$. Observe that $x \in F_{d_0} \subseteq U(F_h) \subseteq U(E_d)$. We conclude that $E_{d_U} \subseteq U(E_d)$ and that $\mathcal{F}$ is a stable filter on $(X,\mathcal{U})$.

Let C be the (nonempty) set of cluster points of $\mathcal{F}$ in $(X,\mathcal{U})$. Take any $U \in \mathcal{U}$. Choose $W \in \mathcal{U}$ such that $W^2 \subseteq U$. We wish to show that $C \subseteq U^{-1}(F_d)$ whenever $d \in D$ and $d \geq d_W$: Let $x \in C$ and $d \in D$ such that $d \geq d_W$. Then $W(x) \cap E_d \neq \emptyset$. Hence $W(x) \cap F_p \neq \emptyset$ for some $p \in D$ with $p \geq d$. Thus $x \in W^{-1}(F_p) \subseteq W^{-1}(W^{-1}(F_d))$. We conclude that $C \subseteq U^{-1}(F_d)$ whenever $d \in D$ and $d \geq d_W$, as we have stated above. By Lemma 6 for each $U \in \mathcal{U}$ there exists $e \in D$ such that $\bigcup_{d \in D, d \geq e} F_d \subseteq U(C)$. We conclude that $(F_d)_{d \in D}$ converges in $(\mathcal{P}_0(X), \mathcal{U}_*)$ to C. Consequently $(\mathcal{P}_0(X), \mathcal{U}_*)$ is right K-complete.

The following result is well known in the case of uniform spaces (see e.g. [2, Corollary 3]).

Proposition 7. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a quasi-uniformly continuous surjection that is perfect. If $\mathcal{V}_*$ is right K-complete on $\mathcal{P}_0(Y)$, then $\mathcal{U}_*$ is right K-complete on $\mathcal{P}_0(X)$.*

Proof: Let $\mathcal{F}$ be a stable filter on $(X,\mathcal{U})$. Consider any $V \in \mathcal{V}$. There is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$, because f is quasi-uniformly continuous. Since $\mathcal{F}$ is stable on $(X,\mathcal{U})$, there is $F_0 \in \mathcal{F}$ such that $F_0 \subseteq U(F)$ whenever $F \in \mathcal{F}$. Consequently $f(F_0) \subseteq V(f(F))$ whenever $F \in \mathcal{F}$. Since the stable filter $f\mathcal{F} := \{fF : F \in \mathcal{F}\}$ has a cluster point y_0 on Y by Proposition 6 and since f is perfect (i.e. the mapping f is closed and the fibers $f^{-1}\{y\}$ are compact whenever $y \in Y$), it follows that the filter $\mathcal{F}$ has a cluster point $x_0 \in f^{-1}\{y_0\}$. The assertion follows from Proposition 6.

Corollary 8. *Let $(X,\mathcal{V})$ be a quasi-uniform space such that $(\mathcal{P}_0(X),\mathcal{V}_*)$ is right K-complete. Then for any quasi-uniformity $\mathcal{U}$ finer than $\mathcal{V}$ on X and generating the topology $\mathcal{T}(\mathcal{V})$, the Bourbaki quasi-uniformity $\mathcal{U}_*$ is right K-complete.*

Proof: Consider the identity map $i : (X,\mathcal{U}) \to (X,\mathcal{V})$ and apply Proposition 7.

Remark 2. *Let us note that if $(X,\mathcal{U})$ is compact, then $(\mathcal{P}_0(X),\mathcal{U}_*)$ is right K-complete by Proposition 6 and precompact by Proposition 1, but in general not left K-complete as*

Example 1 together with the fact that each precompact left K-complete quasi-uniform space is compact (see [10, Proposition 13]*) shows.*

A quasi-uniform space $(X,\mathcal{U})$ is called *bicomplete* [4] if the uniformity $\mathcal{U}^s$ is complete.

A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be *doubly stable* provided that for any $U \in \mathcal{U}$, $\bigcap_{F\in\mathcal{F}}(U(F)\cap U^{-1}(F))$ belongs to $\mathcal{F}$. A point $x \in X$ is called *a double cluster point* of a filter $\mathcal{F}$ on $(X,\mathcal{U})$ provided that $x \in \bigcap_{F\in\mathcal{F}} \mathrm{cl}_{\mathcal{T}(\mathcal{U})}F \cap \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}F$. (In the following we shall denote the set of cluster points $\bigcap_{F\in\mathcal{F}} \mathrm{cl}_{\mathcal{T}(\mathcal{U})}F$ of $\mathcal{F}$ with respect to the topology $\mathcal{T}(\mathcal{U})$ by $\mathrm{adh}_{\mathcal{T}(\mathcal{U})}\mathcal{F}$.)

Proposition 8. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{P}_0(X),\mathcal{U}_*)$ is bicomplete if and only if for any doubly stable filter $\mathcal{F}$ on $(X,\mathcal{U})$ and any $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $F \subseteq U(C)\cap U^{-1}(C)$. (Here C denotes the set of double cluster points of $\mathcal{F}$.)*

Proof: It suffices to sketch the proof, since it is similar to the proof of Proposition 6. Suppose that $(\mathcal{P}_0(X),\mathcal{U}_*)$ is bicomplete. Let $\mathcal{F}$ be a doubly stable filter on $(X,\mathcal{U})$. The net $(F)_{F\in(\mathcal{F},\supseteq)}$ on $\mathcal{P}_0(X)$ is a Cauchy net in $(\mathcal{P}_0(X),(\mathcal{U}_*)^s)$. Since $(\mathcal{P}_0(X),(\mathcal{U}_*)^s)$ is complete, $(F)_{F\in\mathcal{F}}$ converges to some C in $(\mathcal{P}_0(X),(\mathcal{U}_*)^s)$. Without loss of generality, by Lemma 2, we can replace C by D where $D = \mathrm{cl}_{\mathcal{T}(\mathcal{U})}C\cap\mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}C$. Consider any $U \in \mathcal{U}$. There is $F_0 \in \mathcal{F}$ such that $F \subseteq U(C)$, $F \subseteq U^{-1}(C)$, $C \subseteq U(F)$ and $C \subseteq U^{-1}(F)$ whenever $F \subseteq F_0$ and $F \in \mathcal{F}$. Therefore $\mathrm{adh}_{\mathcal{T}(\mathcal{U}^{-1})}\mathcal{F} \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}C$, $\mathrm{adh}_{\mathcal{T}(\mathcal{U})}\mathcal{F} \subseteq \mathrm{cl}_{\mathcal{T}(\mathcal{U})}C$, $\mathrm{cl}_{\mathcal{T}(\mathcal{U}^{-1})}C \subseteq \mathrm{adh}_{\mathcal{T}(\mathcal{U}^{-1})}\mathcal{F}$ and $\mathrm{cl}_{\mathcal{T}(\mathcal{U})}C \subseteq \mathrm{adh}_{\mathcal{T}(\mathcal{U})}\mathcal{F}$. Thus $D = \mathrm{adh}_{\mathcal{T}(\mathcal{U})}\mathcal{F} \cap \mathrm{adh}_{\mathcal{T}(\mathcal{U}^{-1})}\mathcal{F}$. We have shown that D is the set of double cluster points of $\mathcal{F}$ and conclude that the stated condition is satisfied.

In order to prove the converse suppose that each doubly stable filter on $(X,\mathcal{U})$ satisfies the given condition. Consider any $(\mathcal{U}_*)^s$-Cauchy net $(F_d)_{d\in D}$ on $\mathcal{P}_0(X)$. Define the filter $\mathcal{F} = \mathrm{fil}\{E_e : e \in D\}$ on X where $E_e = \bigcup_{d\in D, d\geq e} F_d$ whenever $d \in D$.

Then $\mathcal{F}$ is a doubly stable filter on $(X,\mathcal{U})$. Let C be the set of its double cluster points. In particular $C \in \mathcal{P}_0(X)$.

Let $U \in \mathcal{U}$. Choose $W \in \mathcal{U}$ such that $W^2 \subseteq U$. Similarly as in the last part of the proof of Proposition 6, one verifies that for some fixed $d_W \in D$, $C \subseteq U^{-1}(F_d) \cap U(F_d)$ whenever $d \in D$ and $d \geq d_W$. This fact together with our assumption on doubly stable filters implies that $(F_d)_{d \in D}$ converges to C in $(\mathcal{P}_0(X), (\mathcal{U}_*)^s)$.

Remark 3. *Note that a quasi-uniform space $(X,\mathcal{U})$ is bicomplete if $(\mathcal{P}_0(X),\mathcal{U}_*)$ is bicomplete.* Indeed, let $\mathcal{F}$ be a Cauchy filter on $(X,\mathcal{U}^s)$. Then $\mathcal{F}$ is doubly stable on $(X,\mathcal{U})$. Thus it has a double cluster point $x \in X$ according to the preceding proposition. Since $\mathcal{F}$ is a Cauchy filter on $(X,\mathcal{U}^s)$, it is clear that $\mathcal{F}$ converges to x with respect to the topology $\mathcal{T}(\mathcal{U}^s)$. Hence $(X,\mathcal{U})$ is bicomplete.

Corollary 9. *A quasi-uniform space $(X,\mathcal{U})$ is totally bounded and bicomplete if and only if $(\mathcal{P}_0(X),\mathcal{U}_*)$ is totally bounded and bicomplete.*

Proof: Because of the preceding remark and Corollary 2 it remains only to verify the condition stated in Proposition 8 under the assumption that $(X,\mathcal{U})$ is totally bounded and bicomplete. But then the topology $\mathcal{T}(\mathcal{U}^s)$ is compact. Thus any filter on X has a $\mathcal{T}(\mathcal{U}^s)$-cluster point x. Clearly such an x is a double cluster point of the filter under consideration on $(X,\mathcal{U})$. Hence it is readily seen that the condition formulated in Proposition 8 is satisfied.

Our final example shows that on a bicomplete quasi-metric space a doubly stable filter need not have a (double) cluster point.

Example 7. *Equip the set $\boldsymbol{Q}$ of rationals with the Sorgenfrey quasi-metric $d := d_S|(\boldsymbol{Q}\times\boldsymbol{Q})$. Then obviously $(\boldsymbol{Q},\mathcal{U}_d)$ is bicomplete. However $(\mathcal{P}_0(\boldsymbol{Q}), (\mathcal{U}_d)_*)$ is not bicomplete.*

Indeed, let $\{q_n : n \in \omega\}$ be an enumeration of $\boldsymbol{Q}$. For each $n \in \omega$ we choose inductively an open (with respect to the usual Euclidean topology $\mathcal{Q}$ on $\boldsymbol{Q}$) interval I_n with irrational end points of length $< 2^{-n}$ containing q_n such that

(1) if $q_n \in I_k$ for some $k < n$, then $I_n \subseteq I_k$;

(2) if $q_n \notin I_k$ for some $k < n$, then I_n and I_k have positive distance from each other with respect to the usual metric on $\boldsymbol{Q}$. (Note that this is possible, because each interval I_n has irrational end points.)

For each $n \in \omega$ set $A_n = \boldsymbol{Q} \setminus \bigcup_{i=0}^{n} I_i$. Observe that $(A_n)_{n\in\omega}$ is decreasing. Let $m \in \omega$. We want to show that $A_{m+1} \subseteq B_m^{-1}(A_s)$ and $A_{m+1} \subseteq B_m(A_s)$ whenever $s \in \omega$. Fix $s \in \omega$. Suppose that $r \in \omega$ and $q_r \in A_{m+1}$.

If $q_r \in A_s$, then we are finished. If $q_r \notin A_s$, then $q_r \in I_i$ for some minimal i such that $s \geq i > m+1$. Observe that, by the construction of the sequence $(I_n)_{n\in\omega}$, if $n, k \in \omega$, $n > k$ and the distance from I_n to I_k is zero with respect to the usual metric on $\boldsymbol{Q}$, then $I_n \subseteq I_k$. Thus there are points $u, v \in A_s$ such that $\inf I_i - 2^{-(m+1)} < u < \inf I_i$ and $\sup I_i < v < \sup I_i + 2^{-(m+1)}$. Since the length of I_i is $< 2^{-(m+1)}$ and since $q_r \in I_i$, we conclude that $q_r \in B_m(A_s)$ and $q_r \in B_m^{-1}(A_s)$. Thus $\mathcal{F} := \text{fil}\{A_m : m \in \omega\}$ is doubly stable on $(\boldsymbol{Q}, \mathcal{U}_d)$. Clearly $\mathcal{F}$ has no (double) cluster point on $(\boldsymbol{Q}, d)$.

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