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COVERINGS OF CONTINUA

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ABSTRACT. Given a continuum X, we consider certain closed subsets of $\mathcal{C}(X)$ (the hyperspace of nonempty subcontinua of X) whose union is X and obtain information about X knowing these closed subsets of $\mathcal{C}(X)$ and vice versa.

Introduction. Krasinkiewicz and Nadler ([K–N]) intro duced the concept of covering property by saying that a continuum X has the *covering property* (written $X \in CP$) if and only if given a Whitney map μ for $\mathcal{C}(X)$ we have that for every $t \in (0, \mu(X))$, no proper subcontinuum of $\mu^{-1}(t)$ covers X.

In this paper we relax some of the conditions of this definition by considering closed subsets $\mathcal{A}$ of $\mathcal{C}(X)$, consisting of nondegenerate proper subcontinua of X, covering X, and minimal in the sense that no proper closed subset of $\mathcal{A}$ covers X. Such an $\mathcal{A}$ is called a *minimal closed cover.* We show the following:

Theorem 3. *Let X be a continuum and let $\mathcal{A}$ be a collection of nondegenerate proper subcontinua of X such that $\mathcal{A}$ is closed in $\mathcal{C}(X)$ and $\bigcup_{A \in \mathcal{A}} A = X$. Then $\mathcal{A}$ contains a minimal closed cover $\mathcal{B}$ of X.*

Theorem 5. *A continuum X is a graph if and only if every minimal closed cover of X is finite.*

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Theorem 6. *If X is a continuum such that all of its minimal closed covers are countable then X is locally connected.*

Theorem 7. *If X is a continuum such that all of its minimal closed covers are countable, then X is hereditarily locally connected.*

Definitions. If $(W, <)$ is a partially ordered set, by a *chain* A of elements of W we mean that $(A, <)$ is totally ordered. If Z is a topological space and $A \subset Z$, then the closure of A in Z is denoted by $Cl_Z(A)$, or by $Cl(A)$ if there is no confusion, its interior by A°, and its boudary by $\partial_Z(A)$. We say that Z is *connected im kleinen* at a point z if for every open set U of Z containing z, there exists an open set V such that $z \in V \subset U$ and for every $y \in V$ there is a connected set C_y such that $y, z \in C_y \subset U$. If (Y, d) is a metric space, then given $A \subset Y$ and $\varepsilon > 0$, the open ball around A of radius ε is denoted by $\mathcal{V}_\varepsilon(A)$, we will write $\mathcal{V}_\varepsilon(y)$ for $\mathcal{V}_\varepsilon(\{y\})$. If A is a subset of Y, then $\text{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$. The set of positive integers is denoted by $\mathbb{N}$.

If (Y, d) is a metric space then Y is said to have *property S* provided that for each $\varepsilon > 0$ there are finitely many connected subsets $A_1, \ldots, A_n$ of Y such that $Y = \bigcup_{k=1}^{n} A_n$ and for each $k \in \{1, \ldots, n\}$, $\text{diam}(A_k) < \varepsilon$.

A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a continuum X is a continuum contained in X. A continuum X is *decomposable* if $X = A \cup B$, where A and B are proper subcontinua of X. X is *indecomposable* if it is not decomposable. X is *hereditarily locally connected* if all of its subcontinua are locally connected. By a *graph* we mean a continuum which can be written as the union of finitely many arcs any two of which are either disjoint, or intersect only in one or both of their end points.

Given a continuum X, the *hyperspace of nonempty subcon-*

tinua of X is:

$$\mathcal{C}(X) = \{A \subset X \mid A \text{ is a continuum}\},$$

It is known that $\mathcal{C}(X)$ is a metric space with the Hausdorff metric ([N1], (0.1)), and in fact $\mathcal{C}(X)$ is a continuum ([N1], (0.8) and (1.12)). On $\mathcal{C}(X)$ we can define a real-valued function

$$\mu \colon \mathcal{C}(X) \to [0, \infty)$$

called a *Whitney map* which satisfies the following:

(i) μ is continuous;
(ii) for every $x \in X$, $\mu(\{x\}) = 0$;
(iii) if $A \subset B$ and $A \neq B$ then $\mu(A) < \mu(B)$ (see [N1], (0.50)).

For every $t \in [0, \mu(X)]$, $\mu^{-1}(t)$ is called a *Whitney level.* A couple of facts about Whitney maps are:

(1) for each $t \in [0, \mu(X)]$, $\mu^{-1}(t)$ is a continuum ([N1], (14.2));
(2) for each $t \in [0, \mu(X)]$, $\bigcup \mu^{-1}(t) = X$ ([N1], (1.213.1)).

A *minimal closed cover* $\mathcal{A}$ of the continuum X is a closed subset of $\mathcal{C}(X)$, where each element of $\mathcal{A}$ is a nondegenerate proper subcontinuum of X, $\bigcup_{A \in \mathcal{A}} A = X$, and no proper closed subset of $\mathcal{A}$ covers X.

Let us see some examples.

(i) If X is a decomposable continuum and $X = A \cup B$, where A and B are nondegenerate proper subcontinua of X, then clearly $\mathcal{A} = \{A, B\}$ is a minimal closed cover of X.

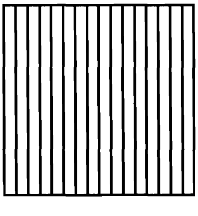

Figure 1

(*ii*) Let Y be the square $[0,1]\times[0,1]$ then if $\mathcal{A}=\{\{x\}\times[0,1]\mid x\in[0,1]\}$, then $\mathcal{A}$ is an uncountable minimal closed cover even though Y is locally connected.

(*iii*) Let $Z=Y\cup(\{0\}\times[1,2])$ then $\mathcal{B}=\mathcal{A}\cup\{\{0\}\times[0,2]\}$ is also an uncountable minimal closed cover. Observe that for every $y\in[0,1]$, $(0,y)$ belongs to both $\{0\}\times[0,1]$ and $\{0\}\times[0,2]$, and we cannot remove $\{0\}\times[0,1]$ because the covering must be closed.

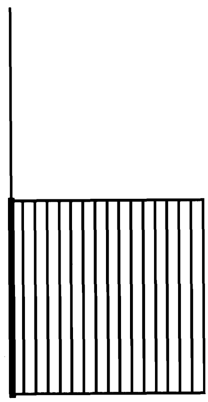

Figure 2

Main Theorems. *We begin with an easy technical lemma, whose proof can be found in* [E], *Corollary 3.1.5.*

Lemma 1. *Let Z be a compact Hausdorff space and let $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of closed subsets of Z with the finite intersection property. If U is an open subset of Z such that $\bigcap_{\alpha\in\Lambda}F_\alpha\subset U$ then there exist $\alpha_1,\dots,\alpha_n\in\Lambda$ such that $\bigcap_{k=1}^{n}F_{\alpha_k}\subset U$.*

Corollary 2. *Let Y be a compact metric space. If $\mathcal{A}$ is a chain of closed subsets of Y ordered by inclusion then there exists a nested sequence $\{A_n\}_{n=1}^{\infty}\subset\mathcal{A}$ such that $\bigcap_{n=1}^{\infty}A_n=\bigcap_{A\in\mathcal{A}}A$.*

Proof: Let $V_1 = \mathcal{V}_1\left(\bigcap_{A\in\mathcal{A}} A\right)$. By Lemma 1 we can find $A_{1,1},\dots,A_{1,n(1)} \in \mathcal{A}$ such that $\bigcap_{k=1}^{n(1)} A_{1,k} \subset V_1$. Without loss of generality we may assume that $A_{1,1} \supset A_{1,2} \supset \cdots \supset A_{1,n(1)}$. Thus $A_{1,n(1)} = \bigcap_{k=1}^{n(1)} A_{1,k}$ and we take $A_1 = A_{1,n(1)}$.

Let $m \geq 2$ such that $A_1 \not\subset \mathcal{V}_{\frac{1}{m}}\left(\bigcap_{A\in\mathcal{A}} A\right)$, and let $V_2 = \mathcal{V}_{\frac{1}{m}}\left(\bigcap_{A\in\mathcal{A}} A\right)$, by Lemma 1 we can find $A_{2,1},\dots,A_{2,n(2)} \in \mathcal{A}$ such that $\bigcap_{k=1}^{n(2)} A_{2,k} \subset V_2$. Without loss of generality we may assume that $A_{2,n(2)} = \bigcap_{k=1}^{n(2)} A_{2,k}$, and now we take $A_2 = A_{2,n(2)}$ and observe that $A_2 \subset A_1$.

If we continue with this process, we obtain a nested sequence $\{A_n\}_{n=1}^{\infty}$ of elements of $\mathcal{A}$. Clearly $\bigcap_{A\in\mathcal{A}} A \subset \bigcap_{n=1}^{\infty} A_n$, to see the other inclusion holds, let $x \in \bigcap_{n=1}^{\infty} A_n$ and let $s \in \mathbb{N}$, then $x \in A_s \subset \mathcal{V}_{\frac{1}{s}}\left(\bigcap_{A\in\mathcal{A}} A\right)$, which implies that $x \in Cl\left(\bigcap_{A\in\mathcal{A}} A\right) = \bigcap_{A\in\mathcal{A}} A$.

□

Theorem 3. *Let X be a continuum and let $\mathcal{A}$ be a collection of nondegenerate proper subcontinua of X such that $\mathcal{A}$ is closed in $\mathcal{C}(X)$ and $\bigcup_{A\in\mathcal{A}} A = X$. Then $\mathcal{A}$ contains a minimal closed cover $\mathcal{B}$ of X.*

Proof: We are going to use Zorn's Lemma to obtain $\mathcal{B}$. To this end let $\Im$ be the family of closed subcovers of $\mathcal{A}$ and note that $\mathcal{A} \in \Im$, hence $\Im$ is not empty.

We define an order "$>$" on $\Im$ as follows, given $\mathcal{B}_1, \mathcal{B}_2 \in \Im$. We say that $\mathcal{B}_1 > \mathcal{B}_2$ if $\mathcal{B}_1 \subset \mathcal{B}_2$. Let $\mathcal{H}$ be a chain of elements of $\Im$, we will see that $\mathcal{H}$ has an upper bound. By Corollary 2, there exists a nested sequence $\{\mathcal{C}_n\}_{n=1}^{\infty}$ of elements of $\mathcal{H}$ such that $\bigcap_{\mathcal{C} \in \mathcal{H}} \mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n = \mathcal{C}_0$. Clearly $\mathcal{C}_0$ is a closed subset of $\mathcal{C}(X)$ such that each element of $\mathcal{C}_0$ is a nondegenerate proper subcontinuum of X. Thus, to show that $\mathcal{C}_0$ is a minimal closed cover, we only need to show $\bigcup_{C \in \mathcal{C}_0} C = X$.

Let $x \in X$. Since each $\mathcal{C}_n \in \mathcal{H}$, there exists $D_n \in \mathcal{C}_n$ such that $x \in D_n$. By the compactness of $\mathcal{A}$, $\{D_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{D_{n_k}\}_{k=1}^{\infty}$ converging to an element D of $\mathcal{A}$. Observe that each D_{n_k} contains x. Hence D contains x.

Since $\{\mathcal{C}_{n_k}\}_{k=1}^{\infty}$ is a decreasing nested sequence, we have that for each $k \in \mathbb{N}$, $\{D_{n_\ell}\}_{\ell=k}^{\infty} \subset \mathcal{C}_{n_k}$, which implies that $D \in \mathcal{C}_{n_k}$, because $\mathcal{C}_{n_k}$ is closed. Therefore $D \in \bigcap_{k=1}^{\infty} \mathcal{C}_{n_k}$. It only remains to show that $\bigcap_{k=1}^{\infty} \mathcal{C}_{n_k} = \bigcap_{n=1}^{\infty} \mathcal{C}_n = \mathcal{C}_0$. Clearly $\bigcap_{n=1}^{\infty} \mathcal{C}_n \subset \bigcap_{k=1}^{\infty} \mathcal{C}_{n_k}$. Now take $D' \in \bigcap_{k=1}^{\infty} \mathcal{C}_{n_k}$, and let $n \in \mathbb{N}$. Since there exists a $k \in \mathbb{N}$ such that $n_k \geq n$, we have that $D' \in \mathcal{C}_{n_k} \subset \mathcal{C}_n$. Thus $D' \in \mathcal{C}_0$, and therefore $\mathcal{C}_0$ is a closed cover of X and it is an upper bound of $\mathcal{H}$. Zorn's Lemma implies that $\Im$ contains a maximal element.

Let $\mathcal{B}$ be a maximal element of $\Im$. Then $\mathcal{B}$ is a minimal closed cover of X. $\square$

The next theorem tells us which are the minimal closed covers of an arc.

Theorem 4. *Any minimal closed cover of $\mathcal{I} = [0,1]$ is finite.*

Proof: Let $\mathcal{A}$ be a closed cover of $\mathcal{I}$ formed by nondegenerate proper subcontinua, and suppose it is infinite. Since no member of $\mathcal{A}$ is degenerate, there exists an $\varepsilon > 0$ such that $\operatorname{diam}(A) > \varepsilon$ for every $A \in \mathcal{A}$.

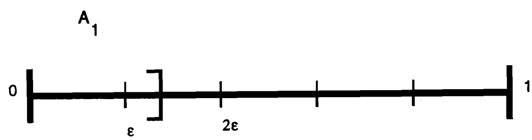

Figure 3

Let $A_1 \in \mathcal{A}$ such that $0 \in A_1$. Then since $\operatorname{diam}(A) > \varepsilon$, $[0, \varepsilon] \subset A_1$, and let

$$b_2 = \sup\{b \in \mathcal{I} \mid A = [a, b] \in \mathcal{A} \text{ and } \varepsilon \in A\}.$$

Since $\mathcal{A}$ is closed, there is an $A_2 \in \mathcal{A}$ such that $A_2 = [a_2, b_2]$. If $b_2 \geq 2\varepsilon$, then we have covered $[\varepsilon, 2\varepsilon]$ with only one element of $\mathcal{A}$. Thus suppose that $b_2 < 2\varepsilon$, and let

$$a_3 = \inf\{a \in \mathcal{I} \mid A = [a, b] \in \mathcal{A} \text{ and } 2\varepsilon \in A\}.$$

Then there exists an $A_3 \in \mathcal{A}$ such that $A_3 = [a_3, b_3]$. We claim that $a_3 \leq b_2$, otherwise if $a_3 > b_2$, then we take $x \in (b_2, a_3)$, and choose an $A \in \mathcal{A}$ such that $x \in A$. Thus either $\varepsilon \in A$ or $2\varepsilon \in A$, but this contradicts the election of either b_2 or a_3.

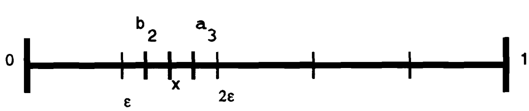

Figure 4

Hence, we have that $[\varepsilon, 2\varepsilon]$ can be covered with at most two elements of $\mathcal{A}$. In a similar way we see that the rest of the intervals $[\ell\varepsilon, (\ell+1)\varepsilon]$ can be covered with at most two elements of $\mathcal{A}$. $\square$

The following result gives us a characterization of graphs in terms of minimal closed covers.

Theorem 5. *A continuum X is a graph if and only if every minimal closed cover of X is finite.*

Proof: If X is a graph, it follows easily from Theorem 4 that all the minimal closed covers of X are finite.

Thus, let us suppose that X is a continuum for which all its minimal closed covers are finite, and let $\varepsilon > 0$ be given. For each $x \in X$, let A_x be a subcontinuum of X such that $x \in A_x$ and $\operatorname{diam}(A_x) = \varepsilon$. Let $\mathcal{A} = Cl_{\mathcal{C}(X)}\{A_x \mid x \in X\}$, then by Theorem 3, $\mathcal{A}$ has a minimal closed subcover $\mathcal{B}$, which is finite by hypothesis. On the other hand, for each $B \in \mathcal{B}$, $\operatorname{diam}(B) = \varepsilon$. Hence X has property S, and X is locally connected ([N2], Theorem 8.4).

Let A be any subcontinuum of X, we claim that $X \setminus A$ has a finite number of components. Suppose this is not true, so $X \setminus A$ has infinitely many components. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of components of $X \setminus A$, converging to K. Observe that, since X is locally connected, $K \subset A$.

$$\text{Let } T = \bigcup\{C \mid C \text{ is a component of } X \setminus A \text{ and, for each } n \in \mathbb{N}, C \neq K_n\},$$

and observe that since X is locally connected, $A \cup T$ is a continuum. Consider the following family of subcontinua of X:

$$\mathcal{A} = \{K_1 \cup A \cup T \cup K_n\}_{n=2}^{\infty},$$

then $\mathcal{A}$ is an infinite minimal closed cover of X, which contradicts our hypothesis. Therefore $X \setminus A$ has a finite number of components. By ([F], Theorem 4.7) we have that X is a graph. $\square$

Based on the previous theorem, one might ask the following question:

Is it true that a continuum X is hereditarily locally connected if and only if all its minimal closed covers are countable?

In one direction the answer to this question is no, because the Gehman dendrite

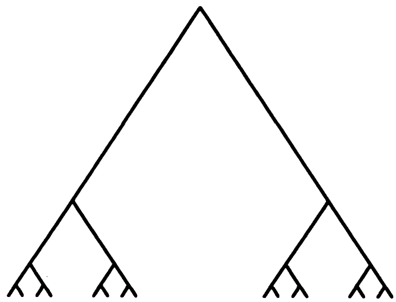

Figure 5

(see [G]) has an uncountable minimal closed cover, which consists of all the arcs from the vertex to the points of the Cantor set at the bottom.

In the other direction we have a positive answer. First we show the following:

Theorem 6. *If X is a continuum such that all its minimal closed covers are countable then X is locally connected.*

Proof: Suppose X is not connected im kleinen at the point p', then we can find an open set U' of X containing p', a continuum D that contains p', lying in $Cl_X(U')$ and meeting $Cl_X(U') \setminus U'$ and a sequenece of distinct components $\{D_n\}_{n=1}^{\infty}$ of U' converging to D, see ([H–Y], Theorem 3–12). Let p be a point in D such that, for some $\varepsilon > 0$, the closure of the open ball $U = \mathcal{V}_\varepsilon(p)$ is contained in U'.

Since $p \in D$ and the sequence $\{D_n\}_{n=1}^{\infty}$ converges to D, there is an $M \in \mathbb{N}$ such that if $n \geq M$ then $Cl_X(U) \cap D_n \neq \emptyset$. Hence, we can find a sequence $\{p_n\}_{n \geq M}$ of points of U converging to p such that p_n and p_m are in different components of $Cl_X(U)$

if and only if $n \neq m$. Let K_n be the component of $Cl_X(U)$ containing the point p_n.

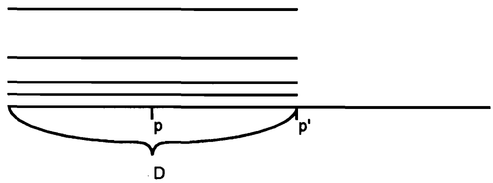

Figure 6

Since $Cl_X(U)$ is compact and metric, we have that its quasicomponents and components coincide, see ([H–Y], Theorem 2–14). It is easy to see that we can find a sequence $\{E_n\}_{n=1}^{\infty}$ of open sets of $Cl_X(U)$ such that each $K_n \subset E_n$, $n \in \mathbb{N}$, and $E_n \cap E_m = \emptyset$ if and only if $n \neq m$. By ([H–Y], Theorem 2–15), for every $n \in \mathbb{N}$, there is an open and closed subset H_n of $Cl_X(U)$ such that $K_n \subset H_n \subset E_n$, let $H = \bigcup\limits_{n=1}^{\infty} H_n$. By ([N2], 5.7), there is and $N \in \mathbb{N}$ such that $\alpha = \inf\limits_{n \geq N} \{\operatorname{diam}(H_n)\} > 0$.

Let μ be a Whitney map for $\mathcal{C}(X)$, and let $s \in (0, \mu(X))$ be such that for every $A \in \mu^{-1}(s)$, $\operatorname{diam}(A) < \frac{1}{16}\min\{\alpha, \varepsilon\}$. Now let us consider a closed interval $[r, s]$, where $0 < r < s$, and let $\{r_n\}_{n=1}^{\infty} = (r, s) \cap \mathbb{Q}$.

If $U \setminus Cl_X(H) \neq \emptyset$ then, for each $x \in U \setminus Cl_X(H)$, let us take $A_x \in \mathcal{C}(X)$ such that $x \in A_x$, $\mu(A_x) = r$, and call $\mathcal{A}_3$ this family of subcontinua of X; otherwise let $\mathcal{A}_3$ be empty.

Let

$$\mathcal{A}_1 = \{A \in \mu^{-1}(r_n) \mid A \subset H_n \text{ and } n \in \mathbb{N}\},$$

$$\mathcal{A}_2 = \{A \in \mathcal{C}(X) \mid \mu(A) = s \text{ and } A \cap (X \setminus U) \neq \emptyset\},$$

and

$$\mathcal{A} = Cl_{\mathcal{C}(X)}(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3),$$

then $\mathcal{A}$ is a closed subset of $\mathcal{C}(X)$. To see that $\mathcal{A}$ is a covering of X, let $x \in X$. If $x \in (X \setminus U)$, then there exists $A \in \mathcal{A}_2$ such that $x \in A$. Thus let us assume that $x \in U$. If $x \in U \setminus Cl_X(H)$, then there is an $A \in \mathcal{A}_3$ such that $x \in A$. Hence, suppose $x \in Cl_X(H)$. Since $\mathcal{A}$ is closed in $\mathcal{C}(X)$, it is enough to show that H is covered by the elements of $\mathcal{A}$. Thus, let us assume $x \in H$, then there is an $n \in \mathbb{N}$ such that $x \in H_n$. If there does not exist an element A of $\mu^{-1}(r_n)$ containing x such that $A \subset H_n$, then we have that for every $A \in \mu^{-1}(r_n)$ containing x, $A \cap (X \setminus U) \neq \emptyset$ (A cannot be contained in $U \subset Cl_X(U)$ because H_n is an open and closed subset of $Cl_X(U)$ and A is connected). Since $r_n < s$, by ([N1], (1.15)), there is a $B \in \mu^{-1}(s)$ such that $x \in A \subset B \in \mathcal{A}_1 \subset \mathcal{A}$. Therefore, $\mathcal{A}$ is a closed cover of X.

Observe that, clearly, $\mathcal{A}_2$ is closed in $\mathcal{C}(X)$ and that p is not contained in any element of $\mathcal{A}_2$. In fact, we claim there is a $\delta > 0$ such that for every $A \in \mathcal{A}_2$, $A \cap \mathcal{V}_\delta(p) = \emptyset$. If this is not true, for each $n \in \mathbb{N}$, there is an $A_n \in \mathcal{A}_2$ such that $A_n \cap \mathcal{V}_{\frac{1}{n}}(p) \neq \emptyset$. Let $x_n \in A_n \cap \mathcal{V}_{\frac{1}{n}}(p)$, since $\mathcal{A}_2$ is compact, without loss of generality we may assume that $\{A_n\}_{n=1}^{\infty}$ converges to an element A of $\mathcal{A}_2$. On the other hand, since $\{x_n\}_{n=1}^{\infty}$ converges to p, we have that $p \in A$, which is not possible. Therefore there is such a $\delta > 0$.

Since $\{p_n\}_{n=1}^{\infty}$ converges to p, there is an $N \in \mathbb{N}$ such that if $n \geq N$ then $p_n \in \mathcal{V}_\delta(p)$. Let us observe that if $n \geq N$ then no element of $Cl_{\mathcal{C}(X)}(\mathcal{A}_3)$ contains p_n. This is because, if for some $n \geq N$, there is an $A \in Cl_{\mathcal{C}(X)}(\mathcal{A}_3)$ such that $p_n \in A$, then there exists $B \in \mathcal{A}_3$ such that $B \cap H_n \cap \mathcal{V}_\delta(p) \neq \emptyset$. Since $B \cap U \setminus H_n \neq \emptyset$, we have that $B \cap X \setminus U \neq \emptyset$. Let $C \in \mu^{-1}(s)$ be such that $B \subset C$, then $C \in \mathcal{A}_2$ and $C \cap \mathcal{V}_\delta(p) \neq \emptyset$, this contradicts the choice of δ.

By Theorem 3, we can find a minimal closed cover $\mathcal{B}$ of $\mathcal{A}$. Now, let $t \in [r, s]$ and let $\{r_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{r_n\}_{n=1}^{\infty}$ converging to t. By the previous pragraph for each $n_k \geq N$, there is an $A_{n_k} \in \mathcal{A}_1 \cap \mathcal{B}$ such that $p_{n_k} \in A_{n_k} \subset H_{n_k}$. Then $\{A_{n_k}\}_{n_k \geq N}$ has a convergent subsequence $\{A_{n_{k_\ell}}\}_{\ell=1}^{\infty}$ converging to $A \in \mathcal{B}$. Since μ is continuous, we have that $\mu(A) = t$. Hence, for every $t \in [r, s]$, there is an $A \in \mathcal{B}$ such that $\mu(A) = t$. Thus, $\mathcal{B}$ is uncountable, a contradiction to the fact that all the minimal closed covers of X were countable. Therefore X is locally connected. $\square$

Theorem 7. *If X is a continuum such that all of its minimal closed covers are countable, then X is hereditarily locally connected.*

Proof: By Theorem 6, we may assume that X is locally connected. Suppose X is not hereditarily locally connected, then X contains a nondegenerate convergence continuum K, (see [N2], 10.4). Let U be an open connected subset of X containing K and let $\varepsilon > 0$ be such that $Cl_X(\mathcal{V}_\varepsilon(K)) \subset U$.

Since K is a convergence continuum, there is a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of X such that $K_n \cap K_m = \emptyset$ if and only if $n \neq m$, and for each $n \in \mathbb{N}$, $K_n \cap K = \emptyset$. Without loss of generality we may assume that for every $n \in \mathbb{N}$, $K_n \subset U$. Since we are working on locally connected metric spaces, we can find a sequence $\{U_n\}_{n=1}^{\infty}$ of open connected subsets of U such that for any $n \in \mathbb{N}$, $K_n \subset U_n \subset Cl_X(U_n) \subset U$, $Cl_X(U_n) \cap K = \emptyset$, and $Cl_X(U_n) \cap Cl_X(U_m) = \emptyset$ if and only if $n \neq m$. Let $H_n = Cl_X(U_n)$, $n \in \mathbb{N}$. We do not lose any generality if we suppose that the sequence $\{H_n\}_{n=1}^{\infty}$ converges to a continuum H containing K. Since K is nondegenerate and $\{K_n\}_{n=1}^{\infty}$ converges to K, there is an $N \in \mathbb{N}$ such that $\alpha = \inf_{n \geq N}\{\text{diam}(H_n)\} > 0$. Let $T = \bigcup_{n \geq N} H_n$, and observe that $Cl_X(T) = T \cup H$.

Let μ be a Whitney map for $\mathcal{C}(X)$, and let $s \in (0, \mu(X))$ be such that $s < \inf_{n \geq N}\{\mu(H_n)\}$, and for every $A \in \mu^{-1}(s)$, $\text{diam}(A) < \frac{1}{16}\min\{\alpha, \varepsilon\}$. Now let us consider a closed interval $[r, s]$, where $0 < r < s$, and let $\{r_n\}_{n \geq N} = (r, s) \cap \mathbb{Q}$.

Let $R = U \setminus (T \cup H)$, then note that R contains at most countably many components, see ([K], Theorem 1, p. 236). For each $n \geq N$, let $D_n = \cup\{Cl_X(F) \mid F$ is a component of R and $n = \min\{m \in \mathbb{N} \mid H_m \cap Cl_X(F) \neq \emptyset\}\}$, and $E_n = H_n \cup D_n$. Note that $Cl_X(E_n)$ is a continuum, for each $n \geq N$, and observe that if $Cl_X(E_n) \cap Cl_X(E_m) \neq \emptyset$, with $n \neq m$. Then by construction we have $Cl_X(E_n) \cap Cl_X(E_m) \subset \partial_X(Cl_X(E_n)) \cap \partial_X(Cl_X(E_m))$.

If there is a component F of R such that $Cl_X(F) \cap H \neq \emptyset$ and for every $n \geq N$, $Cl_X(F) \cap E_n = \emptyset$, then for each such component F and for any point $x \in F$, we take an $A_x \in \mathcal{C}(X)$ such that $x \in A_x \subset Cl_X(F) \cup H$ and $\mu(A_x) = r$. Let $\mathcal{A}_3$ be this family of continua; otherwise let $\mathcal{A}_3$ be empty.

Now we are ready to define a closed cover of X, let

$$\mathcal{A}_1 = \{A \in \mu^{-1}(r_n) \mid A \subset Cl_X(E_n) \text{ for some } n > N\},$$
$$\mathcal{A}_2 = \{A \in \mathcal{C}(X) \mid \mu(A) = s \text{ and } A \cap (X \setminus U) \neq \emptyset\},$$

and

$$\mathcal{A} = Cl_{\mathcal{C}(X)}(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3),$$

then $\mathcal{A}$ is a closed subset of $\mathcal{C}(X)$. We have to show that $\mathcal{A}$ is a covering of X. To this end, let $x \in X$. If $x \in X \setminus U$, then there is an $A \in \mathcal{A}_2$ such that $x \in A$, thus, let us suppose that $x \in U$. If $x \in U \setminus Cl_X\left(\bigcup_{n \geq N} Cl_X(E_n)\right)$, then there is a component F of R such that $Cl_X(F) \cap E_n = \emptyset$, for $n \geq N$ and $Cl_X(F) \cap H \neq \emptyset$. Thus we can find an $A \in \mathcal{A}_3$ such that $x \in A$. Hence suppose $x \in Cl_X\left(\bigcup_{n \geq N} Cl_X(E_n)\right)$. Since $\mathcal{A}$ is closed in $\mathcal{C}(X)$, it is enough to prove that $\bigcup_{n \geq N} Cl_X(E_n)$ is covered by $\mathcal{A}$. Thus, let us assume that $x \in \bigcup_{n \geq N} Cl_X(E_n)$. Then there is an $n \geq N$ such that $x \in Cl_X(E_n)$. Since $r_n < s < \mu(H_n) \leq \mu(Cl_X(E_n))$, by ([N1] (1.15)), there is an $A \in \mu^{-1}(r_n)$ such that $x \in A \subset Cl_X(E_n)$. Hence $A \in \mathcal{A}$ and therefore $\mathcal{A}$ is a closed cover of X.

Observe that $\mathcal{A}_2$ is closed in $\mathcal{C}(X)$ and no point p of K is contained in any element of $\mathcal{A}_2$. A similar argument to the one given in the proof of Theorem 6, shows that there is a $\delta > 0$ such that for any $A \in \mathcal{A}_2$, $A \cap \mathcal{V}_\delta(K) = \emptyset$. Since $\{K_n\}_{n \geq N}$

converges to K, there is an $M \geq N$ such that if $n \geq M$ then $K_n \subset \mathcal{V}_\delta(K)$.

Since $\mathcal{A}$ is a closed covering of X, by Theorem 3, there is a minimal closed cover $\mathcal{B}$ of X contained in $\mathcal{A}$. Now let $t \in [r, s]$ and let $\{r_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{r_n\}_{n \geq N}$ converging to t. Let $p \in K$ and for every $n_k \geq M$, let $p_{n_k} \in K_{n_k} \subset H_{n_k}^{\circ} \subset Cl_X(E_{n_k})^{\circ}$ such that the sequence $\{p_{n_k}\}_{n_k \geq M}$ converges to p. Let $L \geq M$ be such that if $n_k \geq L$, then $p_{n_k} \in \mathcal{V}_\delta(p)$, in particular we have that $K_{n_k} \subset \mathcal{V}_\delta(K)$ if $n_k \geq L$. For each $n_k \geq L$, there is $A_{n_k} \in Cl_{\mathcal{C}(X)}(\mathcal{A}_1) \cap \mathcal{B}$ such that $p_{n_k} \in A_{n_k}$ and $\mu(A_{n_k}) = r_{n_k}$, the existance of this A_{n_k} is guaranteed by the fact that $p_{n_k} \in Cl_X(E_{n_k})^{\circ}$. Then $\{A_{n_k}\}_{n_k \geq L}$ has a convergent subsequence $\{A_{n_{k_\ell}}\}_{\ell=1}^{\infty}$ converging to an element A of $\mathcal{B}$. Since μ is continuous, we have $\mu(A) = t$. Thus $\mathcal{B}$ is uncountable, which is contrary to our assumption. Therefore X is hereditarily locally connected. $\square$

It is known that if X is a hereditarily indecomposable continuum (a continuum for which all of its subcontinua are indecomposable) then $X \in CP$ (see [K–N]), which means that for each $t \in (0, \mu(X))$ the corresponding Whitney level is a minimal closed cover. Based on this we ask the following question: *Is it true that if X is a continuum such that for each $t \in (0, \mu(X))$, the corresponding Whitney level is a minimal closed cover, then X is hereditarily indecomposable?*

Or more general:

Is it true that if X is a continuum for which all of its minimal closed covers are connected, then X is hereditarily indecomposable?

In any case it is clear that such a continuum must be indecomposable. On the other hand we might reformulate our first question in the following way:

Is it true that a continuum X is hereditarily locally connected if and only if all its minimal closed covers are totally disconnected?

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References

[E] R. Engelking, *General Topology,* Vol.6, Sigma Series of Pure Mathematics, 1989.

[F] R. W. Fitzgerald, Connected Sets With a Finite Disconnection Property, Studies in Topology, N. M. Stavrakas and K. R. Allen, Editors (1975), 139–173.

[G] H. M. Gehman, *Concerning the subsets of a plane continuous curve,* Ann. of Math., **27** (1925), 29–46.

[H–Y] J. Hocking and G. Young, *Topology,* Dover, 1988.

[K–N] J. Krasinkiewicz and S. B. Nadler Jr., *Whitney properties,* Fund. Math., **98** (1978), 165–180.

[K] K. Kuratowski, *Topology,* Vol. 2, Academic Press, 1968.

[N1] S. B. Nadler Jr., *Hyperspaces of Sets,* Marcel Dekker, 1978.

[N2] S. B. Nadler Jr., *Continuum Theory,* Marcel Dekker, 1992.

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