Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Vol. 21, 1996

Rimcompact Spaces As Remainders Of Compactifications*

James P. Hatzenbuhler and Don A. Mattson

Abstract

The remainder of a compactification αX of a space X is the space $\alpha X - X$. The residue of X is the set of all points in X which do not possess compact neighborhoods. It is shown that the following conditions are equivalent: X is rimcompact; X is the residue of a space having a strongly 0-dimensional remainder; X is a remainder of a strongly 0-dimensional space. Similar characterizations are given for almost rimcompact spaces.

The rimcompact residue $RCR(X)$ of X is the set of points which do not have a base of neighborhoods with compact boundaries. Conditions on $RCR(X)$ are provided which preclude any remainder of X from being 0-dimensional.

1 Introduction

The *remainder* of a compactification αX of a space X is the space $\alpha X - X$. A major problem in the theory of compactifications is to determine when, for each X in a certain class

*This research was partially supported by a grant from Moorhead State University

of spaces, there is a member of another class of spaces which can serve as a remainder of X. (See [1], [2], [6], [8], [9], and [13], for example.) Herein all spaces are completely regular and Hausdorff and all compactifications are Hausdorff. Recall that a space X is *0-dimensional* if it has a base of clopen sets and X is *strongly 0-dimensional* whenever its Stone-Čech compactification βX is totally disconnected, or equivalently, when disjoint zero sets in X can be separated by a clopen set. A space X is a *0-space* if it possesses a compactification with 0-dimensional remainder. Any 0-space has a compactification ϕX which is maximal with respect to the property that the remainder is 0-dimensional (cf.[3], for example).

An open set O in a space X is called *π-open* if its boundary $Fr_X O$ is compact and a space is *rimcompact* if it has a base of π-open sets. Every rimcompact space is a 0-space, but not conversely. For rimcompact spaces ϕX is called the Freudenthal compactification of X.

In this paper we show that the class of rimcompact spaces is precisely the class of spaces which can serve as remainders of strongly 0-dimensional spaces.

A space X is *almost rimcompact* if X admits a compactification αX such that each point of $\alpha X - X$ has a base of αX-neighborhoods with boundaries in X. Internal characterizations of almost rimcompact spaces and development of their properties may be found in [3] and [4]. We show that almost rimcompact spaces are exactly the class of spaces which can serve as remainders of 0-dimensional spaces.

The *residue* $R(X)$ of X is the set of points in X which do not possess compact neighborhoods. We also prove that X is rimcompact if and only if X is the residue of a space having a strongly 0-dimensional remainder and that X is almost rimcompact whenever it is the residue of a 0-space. It follows that if a 0-space Y has a nonrimcompact residue, then Y has no strongly 0-dimensional remainder.

Since rimcompact spaces are always 0-spaces, it is clear that it is the presence of a non-empty set $RCR(X)$ of points of X which do not possess a base of π-open neighborhoods which may cause X to fail to be a 0-space. In section 3 we provide conditions on $RCR(X)$ which preclude X from being 0-space. However, it is also shown that when X is a 0-space every open set in X contains a non-trivial π-open set.

2 The main results

We begin by listing some definitions, results and notation. Let O be an open set in X and αX any compactification of X. The *extension* of O to αX is the set $Ex_\alpha O = \alpha X - (\mathrm{Cl}_\alpha(X - O))$ which is the largest open subset of αX whose trace on X is O. See [4] for properties of $Ex_\alpha O$.

A compactification αX of X is *perfect* whenever $f^{-1}(z)$ is connected, for all $z \in \alpha X$, where f is the natural mapping of βX onto αX. If αX is perfect and O is open in X, Theorems 1 and 2 of [12] show that $\mathrm{Cl}_{\alpha X}(Fr_X O) = Fr_{\alpha X}(Ex_\alpha O)$. Thus, for a π-open set O in X, $Ex_\alpha O \cap (\alpha X - X)$ and $Ex_\alpha(X - \mathrm{Cl}_X O) \cap (\alpha X - X)$ partition $\alpha X - X$. Additional results concerning perfect compactifications are available in [9],[10], and [12].

We denote the space of countable ordinals by W, and $\beta W = W^*$, where $W^* = W \cup \{\omega_1\}$ and ω_1 is the first uncountable ordinal.

2.1 Theorem *For any space X, the following are equivalent:*

(A) X is almost rimcompact.

(B) X is a remainder of a 0-dimensional space.

(C) X is the residue of some 0-space.

Proof: (A) implies (B). Suppose X is almost rimcompact. For each $x \in \phi X$, let N_x be a countable discrete space and let $Z = \phi X \cup [\cup\{N_x | x \in \phi X\}]$. A base for the topology on Z consists of the following: points of each N_x are open in Z and for $y \in \phi X$ a base for the open neighborhoods of y consists of the sets $O_y \cup [\cup\{N_x | x \in O_y\} - S]$, where O_y is an open ϕX-neighborhood of y and S is any finite subset of $\cup\{N_x | x \in O_y\}$. It is clear that equipped with this topology Z is a compact Hausdorff space and $Z - X$ is dense in Z. We next show $Z - X$ is 0-dimensional. Obviously, points of $Z - \phi X$ are clopen in Z. Let $y \in \phi X - X$ and M_y be any neighborhood of y in $Z - X$ and let $\hat{M}_y$ be a Z-neighborhood of y for which $\hat{M}_y \cap (Z - X) = M_y$. If U_y is a basic open Z-neighborhood of y satisfying $U_y \subseteq \hat{M}_y$, then since X is almost rimcompact, there is a ϕX-open neighborhood O_y of y for which $O_y \subseteq U_y \cap \phi X$ and $Fr_\phi O_y \subseteq X$.

Set $A = O_y$, $B = \phi X -$ $\mathrm{Cl}_\phi O_y$ and $C = Fr_\phi O_y$. Thus, $\phi X = A \cup B \cup C$, where C is compact and A and B are open in ϕX. Take $\hat{A} = [A \cup (\cup\{N_x | x \in A\})] \cap U_y$. Since U_y is a basic open Z-neighborhood of y, it follows that $\hat{A}$ is a basic open Z-neighborhood of y. Thus $\hat{A} \cap (Z - X)$ is open in $Z - X$ and satisfies $\hat{A} \cap (Z - X) \subseteq U_y \cap (Z - X) \subseteq \hat{M}_y \cap (Z - X) \subseteq M_y$. Also, $(Z - X) - \hat{A} = [(B \cup (\cup\{N_x | x \in B\})) \cap (Z - X)] \cup [\cup\{N_x | x \in C\}] \cup [\cup\{N_x | x \in A\} - U_y]$. Thus, $(Z - X) - \hat{A}$ is also open in $Z - X$ so that $\hat{A} \cap (Z - X)$ is clopen in $Z - X$, as desired. Hence $Z - X$ is 0-dimensional and Z is a compatification of $Z - X$ having X as its remainder.

(B) implies (C). Suppose Y is 0-dimensional and $\alpha Y - Y = X$, for some compactification αY of Y. Take $S = W^* \times \alpha Y - \{\omega_1\} \times Y$. Evidently S is a 0-space and $R(S) = X$.

(C) implies (A). Suppose Y is a 0-space and $R(Y) = X$. Let $K = \mathrm{Cl}_{\phi Y} X$. We show that each point z of $K - X$ has a base of K-open neighborhoods having boundaries which lie in X. Let M_z be any K-neighborhood of z and set $T = (\phi Y - Y) \cup X$. Let

$\hat{M}_z$ be any T-neighborhood of z for which $\hat{M}_z \cap K = M_z$. Let N_z be a T-neighborhood of z satisfying $\text{Cl}_T N_z \subseteq \hat{M}_z$. Since $\phi Y - Y$ is 0-dimensional, there is a $\phi Y - Y$ clopen neighborhood O_z of z such that $O_z \subseteq N_z$. Now $Ex_T O_z \subseteq \hat{M}_z$ and since O_z and $((\phi Y - Y) - O_z$ are disjoint, the sets $Ex_T O_z$ and $Ex_T((\phi Y - Y) - O_z)$ are disjoint in T and cover $\phi Y - Y$. Thus, $Ex_T O_z \cap K$ and $Ex_T((\phi Y - Y) - O_z) \cap K$ are disjoint open sets in K which cover $K - X$. Now $[\text{Cl}_K(Ex_T O_z \cap K)] \cap [Ex_T((\phi Y - Y) - O_z)] = \emptyset$, so that $Fr_K[Ex_T O_z \cap K] \subseteq X$. Clearly $Ex_T O_z \cap K \subseteq M_z$, so X is almost rimcompact and the proof is complete.

We define X to be a *strong 0-space* (S.O.S.) iff X has a strongly 0-dimensional remainder.

2.2 Theorem *For any space X, the following are equivalent:*

(A) X is rimcompact.

(B) X is a remainder of a strongly 0-dimensional space.

(C) X is a residue of a strong 0-space.

Proof: (A) implies (B). Since X is rimcompact, ϕX exists. Let Z be the space defined in the proof of (A) implies (B) of Theorem 2.1 and let $Y = Z - X$. Now Z is a compactification of Y, and, accordingly, let t be the canonical mapping of βY into Z which is the identity on Y. Suppose K is a component of $\beta Y - Y$. Then $t(K) \subset X$. If $t(p) \neq t(q)$ in $t(K)$, separate $t(p)$ and $t(q)$ by a π-open set $N_{pq} \subseteq X$, where $t(p) \in N_{pq}$ and $t(q) \notin \text{Cl}_X N_{pq}$. Take $\hat{A} = Ex_{\phi X} N_{pq}$ and $\hat{B} = Ex_{\phi X}(X - \text{Cl}_X N_{pq})$. Set $A = \hat{A} \cup [\bigcup\{N_x | x \in \hat{A}\}]$ and $B = \hat{B} \cup [\bigcup\{N_x | x \in \hat{B}\}] \cup [\bigcup\{N_y | y \in Fr_X N_{pq}\}]$. Since ϕX is perfect, $\hat{A}$ and $\hat{B}$ determine a partition of $\phi X - X$ (cf.[10], for example). It follows that $C = A \cap Y$ and $D = B \cap Y$ are clopen sets which partition Y. But $p \in \mathrm{Cl}_\beta C$ and $q \in \mathrm{Cl}_\beta D$, where $\mathrm{Cl}_\beta C$ and $\mathrm{Cl}_\beta D$ partition βY into clopen sets, contradicting the fact that K is connected. Hence $t(K)$ is a singleton for each component K of $\beta Y - Y$.

Since Y is 0-dimensional, so is ϕY. Recall that ϕY is obtained from βY by identifying components of $\beta Y - Y$ to points (cf. [3], for example). Let g be the natural projection of βY onto ϕY. Thus t is single-valued on fibres of g, so there is a continuous mapping f of ϕY onto Z which is identity on Y and carries $\phi Y - Y$ onto X. We note that f is one-one on $R(Y) = \phi X - X$.

Next, let $S = W^* \times \phi Y - \{\omega_1\} \times (\phi Y - Y)$. Then $\beta S = W^* \times \phi Y$ and βS is 0-dimensional since W^* and ϕY are, hence S is strongly 0-dimensional. But f induces a continuous mapping of $\mathrm{Cl}_{\beta S}(\beta S - S)$ onto ϕX which is one-one on $R(S) = \{\omega_1\} \times R(Y) = \{\omega_1\} \times (\phi X - X)$. Hence, according to Theorem 1.1 of [11], X is a remainder of the strongly 0-dimensional space S.

(B) implies (C). This is similar to (B) implies (C) of Theorem 2.1.

(C) implies (A). If X is a residue of a S.O.S., it is then a remainder of a strongly 0-dimensional space Y also. Thus, let $\alpha Y - Y = X$. There is a continuous mapping f of $\phi Y = \beta Y$ onto αY which is identity on Y. Let $x \in X$ and let $\hat{U}$ be an open X-neighborhood of x. Choose αY-open U such that $U \cap X = \hat{U}$. Now $K = f^{-1}(x)$ is compact and $f^{-1}(U)$ is a ϕY-open neighborhood of K. Since ϕY is 0-dimensional, K can be covered by a ϕY-clopen set V such that $V \subseteq f^{-1}(U)$. Let $\hat{V} = V \cap Y$. Then $\hat{V}$ is clopen in Y, so $Fr_\alpha Ex_\alpha \hat{V} \subseteq X$. Hence $Ex_\alpha \hat{V} \cap X$ is π-open in X.

Next, we show that $Ex_\alpha \hat{V} \cap X$ is an X-neighborhood of x contained in $\hat{U}$. Note that $f(V)$ is compact and $\hat{V} \subseteq f(V)$. Thus, $Ex_\alpha \hat{V} \subseteq \mathrm{Cl}_\alpha \hat{V} \subseteq f(V) \subseteq U$. Also, $K \cap \mathrm{Cl}_{\phi Y}(Y - \hat{V}) = \emptyset$, hence $x \notin f(\mathrm{Cl}_{\phi Y}(Y - \hat{V})) = \mathrm{Cl}_\alpha f(Y - \hat{V}) = \mathrm{Cl}_\alpha (Y - \hat{V})$. Thus,

$x \in Ex_\alpha \hat{V} \subseteq U$ and $Ex_\alpha \hat{V} \cap X$ is π-open in X so that X is rimcompact and the proof is complete.

It follows from 2.1 that if X is almost rimcompact but not rimcompact, then X is the residue of some 0-space Y. Thus Y is not rimcompact and by 2.2 Y cannot have any strongly 0-dimensional remainder. Also, 5.3 of [8] affords an example of a non-rimcompact S.O.S.

3 The rimcompact residue of X

Recall that every rimcompact X is a 0-space and that when X is metric then the two conditions are equivalent (cf. [9], for example). We define the *rimcompact residue* of a space to be the set $RCR(X)$ of points which do not possess a base of π-open sets. While $RCR(X)$ is contained in $R(X)$, we note that, unlike $R(X), RCR(X)$ need not be closed. Clearly, it is the presence of a non-empty $RCR(X)$ which may cause X to fail to be a 0-space. In this section we provide conditions on $RCR(X)$ which preclude X from being a 0-space. If αX is any compactification of X, following [3], for $p \in \alpha X$ we set $G(\alpha X, p) = \cap\{\mathrm{Cl}_{\alpha X} U | U$ is π-open in X and $p \in Ex_{\alpha X} U\}$. In case $\alpha X = \beta X$ we denote $G(\beta X, p)$ by G_p. Lemma 2.2 of [3] shows that any $G(\alpha X, p)$ is connected and obviously it is compact. From the definitions and the proof of 2.5 of [5] the following remark is easily established.

3.1 Remark *For $p \in X$ and any perfect $\alpha X, G(\alpha X, p) = \{p\}$ if and only if $p \notin RCR(X)$.*

3.2 Theorem *For a non-rimcompact X, if $RCR(X)$ is totally disconnected and locally compact, then X has no compactification with totally disconnected remainder.*

Proof: Suppose X has a compactification with totally disconnected remainder. Then there is a compactification αX of X

which is maximal with respect to this property, hence is perfect (cf. [3] and [12]). Take $x \in RCR(X)$. Then $G(\alpha X, x)$ is not a singleton. Now $H(x) = G(\alpha X, x) \cap X$ is a locally compact and totally disconnected subset of $RCR(X)$. Thus, x has a compact neighborhood N_x in $H(x)$. But $H(x)$ is dense in $G(\alpha X, x)$ so that N_x is a $G(\alpha X, x)$-neighborhood of x. Since N_x is compact and 0-dimensional, this disconnects $G(\alpha X, x)$, a contradiction.

This completes the proof.

3.3 Corollary *If $RCR(X)$ contains an $RCR(X)$-isolated point, then no remainder of X is totally disconnected.*

Proof: Let p be an isolated point of $RCR(X)$ and F an X-closed neighborhood of p such that $F \cap RCR(X) = \{p\}$. Now $RCR(F) = \{p\}$ and by 3.2 F cannot have a compactification with a totally disconnected remainder, hence neither can X.

This completes the proof.

The next result show that when $R(X)$ is totally disconnected, then the properties of rimcompactness and almost rimcompactness are equivalent. From the proof it follows that if $RCR(X)$ is non-empty and totally disconnected, X cannot be almost rimcompact.

3.4 Theorem *If X is almost rimcompact and $R(X)$ is totally disconnected, then X is rimcompact.*

Proof: Suppose $RCR(X) \neq \emptyset$. If $p \in RCR(X)$ and X is almost rimcompact, then $G(\phi(X), p) \subseteq RCR(X)$. Since $RCR(X)$ is totally disconnected and ϕX is a perfect compactification, it follows that $G(\phi X, p) = \{p\}$, in contradiction to 3.1.

This completes the proof.

Next we show that, in the presence of almost rimcompactness, if each point of X has a base of neighborhoods with locally compact boundaries, then X is rimcompact.

3.5 Theorem *Let X be almost rimcompact. Then X is rimcompact if and only if each point of X has a base of neighborhoods having locally compact boundaries.*

Proof: Only sufficiency requires proof. Let $p \in X$ and let N_p be any X-open neighborhood of p. Choose an X-open neighborhood M_p of p for which $M_p \subseteq N_p$ and $Fr_X M_p$ is locally compact. Then $D = \mathrm{Cl}_{\phi X} Fr_X M_p - Fr_X M_p$ is compact. Since X is almost rimcompact D can be covered by a collection of ϕX-open sets $M_1, \ldots, M_k$ such that $p \notin \mathrm{Cl}_\phi M_i$ and $Fr_{\phi X} M_i \subseteq X$, for $i = 1, \ldots, k$. Set $O_i = \phi X - \mathrm{Cl}_\phi M_i$, $i = 1, \ldots, k$, and take $O = M_p \cap O_1 \cap \ldots \cap O_k$. Clearly, O is X-open and $O \subseteq N_p$.

Let $x \in Fr_X O$. If $x \notin O_i$, for some i, then $x \in Fr_{\phi X} M_i$. If $x \notin M_p$, then $x \in Fr_X M_p$ and $x \notin M_i, i = 1, \ldots, k$. Thus $x \in (Fr_X M_p - \cup\{M_i | i = 1, \ldots, k\}) \cup \{Fr_{\phi X} M_i | i = 1, \ldots, k\}$, a compact set. Hence $Fr_X O$ is compact and the proof is complete.

Not every 0-space is rimcompact, but the following result shows that some amount of "rimcompactness" is present in every 0-space.

3.6 Theorem *If X is a 0-space, then every non-empty open subset of X contains a non-empty π-open set.*

Proof: It suffices to consider non-empty X-open sets O such that $O \subseteq R(X)$. Then $Ex_\phi O$ is open in ϕX and meets $\phi X - X$ since $(\phi X - X) \cup R(X)$ is a compactification of $\phi X - X$. Choose a non-empty $\phi X - X$ clopen set $U \subseteq (\phi X - X) \cap Ex_\phi O$. Let V be a ϕX-open set such that $V \cap (\phi X - X) = U$ and $V \subseteq Ex_\phi O$. Since U is clopen in $\phi X - X$, it follows that $Fr_X(V \cap X) \subseteq X$ is compact. This completes the proof.

From 2.2 it is clear that if X is a S.O.S, then either X contains a point having a compact neighborhood or X is rimcompact. Also, in view of 3.6, it is natural to ask whether some rimcompactness condition at points of X is necessary in order that X be a 0-space. It can be shown that either $X - RCR(X)$ is dense in any 0-space X or there is a 0-space which is nowhere rimcompact. We also note that if a 0-space $X = RCR(X)$ exists, then X must be almost rimcompact yet by the proof of 3.5 it follows that no point of X can have a base of open neighborhoods with locally compact boundaries.

In view of the above we state the open question: Can $X = RCR(X)$ be a 0-space?

References

[1] G. L. Cain, *Countable compactifications*, Gen. Top. and its Relations to Modern Analysis and Algebra VI (Proc. Sixth Prague Topological Symposium, 1986), Heldermann Verlag, Berlin, 1988.

[2] R. E. Chandler, *Hausdorff Compactifications*, Lecture notes in pure and applied mathematics, No 23, Marcel Dekker, Inc., New York and Basel, 1976.

[3] B. Diamond, *Almost rimcompact spaces*, Topology Appl., 25(1987), 81-91.

[4] ——, *Some properties of almost rimcompact spaces*, Pacific J. Math., **118** (1985), 63-77.

[5] ——, J. Hatzenbuhler and D. Mattson, *On when a 0-space is rimcompact*, Topology Proceedings, **13** (1988), 189-201.

[6] W. Fleissner, J. Kulesza and R. Levy, *Remainders of normal spaces*, Topology Appl., **49** (1993), 167-174.

[7] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, New York, 1960.

[8] J. Hatzenbuhler and D. Mattson, *Compactifications with discrete remainders*, Proc. Amer. Math. Soc., **123** (1995), 2927-2934.

[9] J. R. Isbell, *Uniform Spaces*, Amer. Math. Soc. Math. Surveys, No 12, 1962.

[10] J. R. McCartney. *Maximum zero-dimensional compactifications*, Proc. Camb. Phil. Soc., **68** (1970), 653-661.

[11] M. Rayburn, *On Hausdorff compactifications*, Pac. J. Math., **44** (1973), 707-714.

[12] E. G. Sklyarenko, *Some questions in the theory of bicompactifications.* Amer. Math. Soc. Tran., **58** (1966), 216-244.

[13] T. Terada, *On countable discrete compactifications*, Topology Appl., **7** (1977), 321-327.

[14] J. Terasawa, *Spaces* $N \cup \mathcal{R}$ *and their dimensions*, Topology Appl., **11** (1980), 93-102.

Moorhead State University
Moorhead, MN 56563