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A Note On A Recent Metrization Theorem

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In this paper, I am going to give a generalization of a recent metrization theorem of Stares [5] and show that it can be obtained from the main theorem of [1].

0. Notations and Terminology

0.1 Definitions. On a T_1-space X, a *pair-network* is a collection $\mathcal{A}$ of pairs $\{A, A'\}$ of sets, A' being *open* and *nonvoid* and contained in A, such that, for any neighbourhood U of any $x \in X$, there is such a $\{B, B'\} \in \mathcal{A}$ that $x \in B' \subset B \subset U$. On $\mathcal{A}$, we define a partial order $\prec$ so that, given $\{A, A'\}$ and $\{B, B'\}$ of $\mathcal{A}$, $\{A, A'\} \prec \{B, B'\} \Leftrightarrow A' \supsetneq B$. A *nest* $\mathcal{B}$ is an *infinite* subcollection of $\mathcal{A}$, well-ordered by $\prec$. If $\bigcap\{A' : A, A'\} \in \mathcal{B}\}$ is nonvoid (and contains ξ) we say $\mathcal{B}$ is *fixed* (at ξ). Given a nest $\mathcal{B}$, its *first* element $\{I, I'\}$; if there is, for every $\{B, B'\} \in \mathcal{B}$, one $\{\hat{B}, \hat{B}'\} \in \mathcal{A}$ such that $\hat{B}' \cap B' \neq \emptyset$ and $\hat{B}' \backslash I \neq \emptyset$, we say $\mathcal{C} \equiv \{\{\hat{B}, \hat{B}'\} : \{B, B'\} \in \mathcal{B}\}$ is a *companion* of $\mathcal{B}$. We call a companion $\mathcal{C}$ of a $\mathcal{B}$ fixed at ξ *close* if $\mathcal{C}$ consists of but one pair $\{C, C'\}$ and is such that $\xi \in C'$. We also say that $\mathcal{C}$ *captures* $\mathcal{B}$ if $\bigcup\{C : \{C, C'\} \in \mathcal{C}\} \supset B$ for some $\{B, B'\} \in \mathcal{B}$ and that it *closes in on* $\mathcal{B}$ fixed at ξ if $\xi \in C$ for some $\{C, C'\} \in \mathcal{C}$.

0.2 Theorem [1]. *A T_1-space is metrizable if (and only if) there is on it such a pair-network that every* fixed nest *in it is*

(i) *(order) isomorphic to ω, and*

(ii) captured *by each of its* companions.

Remark: Implicit in the proof of the Theorem above in [1], ii) can be replaced by

ii)' closed in on *by each of its* companions *and* captured *by each of its* close companions [3].

1. Main Theorem

1.1 Theorem. *A (necessary and) sufficient condition for the metrizability of a T_1-space X is: For any open neighbourhood V of any $y \in X$, there is an open neighbourhood $V_y \subset V$, with which a natural number $n(y, V)$ is associated in such a manner that*

(†) $V_y \cap U_x \neq \emptyset$ and $V_y \backslash U \neq \emptyset \Rightarrow n(y, V) \leq n(x, U)$, the inequality being strict if in addition $U \subset V_y$.

Proof: Clearly, the family $\mathcal{A} \equiv \{\{U, U_x\} : x \in U, U$ is open$\}$ is a *pair-network*. Given any fixed *nest* $\mathcal{B} \subset \mathcal{A}$. If $\mathcal{B}$ is not (order) isomorphic to ω, there is $\{V, V_y\} \in \mathcal{B}$ that has infinitely many predecessors, associated with which is a *strictly* increasing sequence of natural numbers, bounded above by $n(y, V)$, which is of course impossible. $\mathcal{B}$ is therefore (order) isomorphic to ω and i) of Theorem 0.2 is satisfied. On the other hand let $\{U, U_x\}$, $\{V, V_y\}$ be two elements of $\mathcal{B}$ so that $\{U, U_x\} \prec \{V, V_y\}$. Let $\mathcal{C}$ be a *companion* of $\mathcal{B}$ and $\{W, W_z\} \in \mathcal{C}$ be such that $W_z \cap V_y \neq \emptyset$ and $W_z \backslash U \neq \emptyset$. Clearly, $n(z, W) \leq n(x, U) < n(y, V)$ and we have $V_y \subset W$ and the *capture* of $\mathcal{B}$ by $\mathcal{C}$. For, otherwise, $V_y \backslash W \neq \emptyset$ and $n(y, V) \leq n(z, W)$ which is a contradiction. $\mathcal{B}$ is therefore captured by $\mathcal{C}$ and ii) of Theorem 0.2 is satisfied and metrizability follows. □

Remarks Stares' metrizing condition, c) of Theorem 2.9 of [5], amounts to the insistence on a strict inequality in (†) on all occasions, which implies that 1) if $V_y \cap U_x \neq \emptyset$, then *either* $V_y \backslash U \neq \emptyset$ *or* $U_x \backslash V \neq \emptyset$ but *never* both, and 2) given a natural number ν and an $x \in X$, writing (x, ν) for the collection $\{U : x \in U, n(x, U) = \nu\}$, we have $\bigcup\{U_x : U \in (x, \nu)\} \subset \bigcap(x, \nu)$, enabling us to define $W_\nu(x)$, unless x is isolated, to be $\bigcup\{U_x : n(x, U) = \mu\}$ where μ is the smallest natural member greater than or equal to ν so that $(x, \mu) \neq \emptyset$. Clearly, the $W_\nu(x)$'s so defined constitute a local base at x. Indeed, we have in $\{W_\nu(p) : \nu \in \omega, p \in X\}$ the *classical metrizing structure* of Frink as described in the Corollary to Theorem VI.2 of [4]. For, the $m(n, p)$ required of every n and p can, for example, unless p is isolated, be defined to be $n(p, W)$ for some (definite) choice of U among the summands of $W_n(p)$, some (definite) choice of $V \subsetneq U_p$, and some (definite) choice of $W \subsetneq V_p$. That is, Stares' metrizing condition begets Frink's metrizing structure. See also Example 4 of [1] for the converse.

Condition b) of Theorem 2.9 of [5] can be seen to be a very special case of the Theorem in [2] if one sees $\Xi(x, n)$ in $\{x\} \cup$ Int $W(n, x)$.

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