Topology Proceedings

Productive $[\lambda, \mu]$-Compactness And Regular Ultrafilters

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Abstract

Though full compactness is a productive property, weaker versions are not necessarily productive. A characterization of those spaces all whose powers are countably compact (or κ-initially compact) has been found using ultrafilters. A similar characterization has been worked out for $[\lambda, \mu]$-compactness, a more general notion encompassing both κ-initial and κ-final compactness. We show that the use of ultrafilters cannot be avoided: given λ, μ infinite cardinals, the following are equivalent:

(a) Every productively $[\lambda, \lambda]$-compact topological space is (productively) $[\mu, \mu]$-compact.

(b) Every (λ, λ)-regular ultrafilter is (μ, μ)-regular.

It is well known that, for most pairs of cardinals λ, μ, property (b) cannot be decided in ZFC.

We also prove a multi-cardinal version of the above result, and analyse some of its consequences.

Since by Tychonoff's theorem every product of compact spaces is compact, it was a surprise when in the 50's two countably compact spaces were discovered whose product is

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not countably compact. The counterexample has been sharpened in many ways, finding even a countably compact topological group whose square is not countably compact ([HM], under a version of Martin's Axiom; see also [Va]).

Whence the problem arose of characterizing productively countably compact spaces (spaces all whose powers are countably compact), or, more generally, productively countably compact families of spaces. Some characterizations have been found in [ScSt], and in [GS] using ultrafilters [Be]; similar characterizations were discovered for productively initially κ-compact spaces and families [Sa]; see [Va] and [St] for more references.

Both initial and final κ-compactness are particular cases of the notion of $[\lambda, \mu]$-compactness, already introduced in 1929 [AU] (see also [Va1]). Recently [Ca] generalized to productive $[\lambda, \mu]$-compactness the mentioned results and methods; actually, his treatment provides also some conceptual simplifications and furnishes a clearer understanding of the connections between ultrafilters and productive compactness.

In [Ca] the notion of a (λ, μ)-regular ultrafilter is used in an essential way: we show that its use cannot be avoided:

Theorem 1 *For every infinite cardinals λ, μ, κ, the following are equivalent:*

    (i) *Every productively $[\lambda, \mu]$-compact topological space is $[\kappa, \kappa]$-compact.*

    (ii) *Every productively $[\lambda, \mu]$-compact family of topological spaces is productively $[\kappa, \kappa]$-compact.*

    (iii) *Every (λ, μ)-regular ultrafilter is (κ, κ)-regular.*

        If κ is a regular cardinal, then the preceding conditions are also equivalent to

    (iv) *Every productively $[\lambda, \mu]$-compact Hausdorff normal topological space with a base of clopen sets in $[\kappa, \kappa]$-compact.*

We now recall the basic notions. By a *space* we mean any topological space; if not explicitly mentioned, no separation axiom is assumed. A space X is $[\lambda, \mu]$-*compact* iff every open cover of X of cardinality $\leq \mu$ has a subcover of cardinality $< \lambda$. *Initial κ-compactness* is $[\omega, \kappa]$-compactness, and *final κ-compactness* is $[\kappa, \mu]$-compactness for all $\mu \geq \kappa$. *Countable compactness* is the same as initial ω-compactness. A space X is *productively $[\lambda, \mu]$-compact* iff every power of X is $[\lambda, \mu]$-compact. A family F of spaces is *productively $[\lambda, \mu]$-compact* iff every product of members of F is $[\lambda, \mu]$-compact.

A *filter D* over a set I is a non-empty collection of subsets of I closed under finite intersections and such that $A \in D$ and $A \subseteq B$ imply $B \in D$. A filter D is *proper* iff $\emptyset \notin D$. D is an *ultrafilter* iff it is a maximal proper filter. D is *uniform* over I iff every member of D has cardinality $|I|$. The ultrafilter D is (λ, μ)-*regular* iff there exists a family of μ-many members of D the intersection of any λ-many of which is empty. We shall need the fact that if λ is a regular cardinal then D is (λ, λ)-regular iff there is a decreasing sequence $(A_\alpha)_{\alpha \in \lambda}$ of members of D whose intersection is empty (this is called λ-*descending incompleteness* in the literature).

D is κ-*complete* iff whenever I is partitioned into $< \kappa$ sets, there is a member of the partition belonging to D. A cardinal $\kappa > \omega$ is *measurable* iff there exists a uniform κ-complete ultrafilter over κ. $\kappa > \omega$ is λ-*strongly compact* iff there exists a κ-complete (κ, λ)-regular ultrafilter; $\kappa > \omega$ is *strongly compact* iff it is λ-strongly compact for all $\lambda \geq \kappa$.

If D is an ultrafilter over I and X is a space, a sequence $(x_i)_{i \in I}$ of elements of X D-*converges* to x iff $\{i \in I | x_i \in V\} \in D$, for any neighbourhood V of x. X is D-*compact* iff every I-family of elements of X D-converges to some element of X. Notice that, in particular, D is κ-complete iff every discrete space of cardinality $< \kappa$ is D-compact. See [CN] or [KV] for other notions used only occasionally.

The following has been proved in [Ca, Theorem 2.9], both

generalizing and simplifying the methods used in the proof of [St, Theorem 5.14]; compare also [Sa, §2].

Theorem 2 [Ca] *For every family F of spaces, the following are equivalent:*

    (i) F is productively [λ, μ]-compact.

    (ii) There exists a (λ, μ)-regular ultrafilter D such that all members of F are D-compact.

See [Ca] for further results along the lines of Theorem 2.

If κ is a regular cardinal, we endow it with the order topology (a base is given by the set of open intervals).

Proposition 1 *For every ultrafilter D and every regular cardinal κ, the space κ is D-compact iff D is not (κ, κ)-regular.*

Proof: If κ is not D-compact there exists a sequence α_i which does not D-converge. For $\alpha \in \kappa$, let $A_\alpha = \{i \in I | \alpha < \alpha_i\}$. Each A_α belongs to D, since otherwise if α is the least ordinal such that A_α is not in D then α_i D-converges to α. Clearly $\bigcap_{\alpha \in \kappa} A_\alpha = \emptyset$, so that D is κ-descendingly incomplete, whence (κ, κ)-regular.

On the other side, let A_α witness that D is κ-descendingly incomplete, and for $i \in I$ let $\alpha_i = \inf \{\alpha | i \notin A_\alpha\}$. Then the sequence α_i does not D converge.□

Corollary 1 *If κ is regular, the space κ is productively [λ, μ]-compact iff there exists a (λ, μ)-regular non-(κ, κ)-regular ultrafilter.*

If κ is a singular cardinal, we need a variation on Proposition 1. For $\lambda \leq \mu$, let $S_\lambda(\mu)$ be the set of subsets of μ of cardinality $< \lambda$, endowed with the smallest topology with open sets $\{X \in S_\lambda(\mu) | Y \not\subseteq X\}$, for Y varying among all finite subsets of μ.

Proposition 2 *For every ultrafilter D and cardinals λ, the space $S_\lambda(\lambda)$ is D-compact iff D is not (λ, λ)-regular.*

Corollary 2 *The space $S_\lambda(\lambda)$ is productively $[\lambda', \mu']$-compact iff there exists a (λ', μ')-regular non$-(\lambda, \lambda)$-regular ultrafilter.*

Proof of Theorem 1: (iii) $\Rightarrow$ (ii) is immediate from Theorem 2.

(ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv) are trivial.

(i) $\Rightarrow$ (iii) By Corollary 2, it is enough to observe that the space $S_\kappa(\kappa)$ is not $[\kappa, \kappa]$-compact (actually, no $S_\lambda(\mu)$ is $[\lambda, \lambda]$-compact).

(iv) $\Rightarrow$ (iii) Use Corollary 1 and the fact that, for κ a regular cardinal, the space κ is not $[\kappa, \kappa]$-compact. $\square$

Notice that the spaces $S_\lambda(\mu)$ are T_0 but not even T_1. We do not know whether (iv) $\Rightarrow$ (iii) holds for κ singular, even just for Hausdorff spaces. We just point out that in the parallel case for logics (see the Remark at the end) the situation is quite messy, to say the least [Lp, §4].

There are many results in Ultrafilter Theory asserting that condition (iii) in Theorem 1 holds, for given cardinals (the most notable example being that every (λ^+, λ^+)-regular ultrafilter is (λ, λ)-regular). By Theorem 1 (iii) $\Rightarrow$ (ii), any one of these "transfer" results for regularity of ultrafilters provides an analogous transfer result for productivity of $[\lambda, \lambda]$-compactness, as noted in [Ca] for many particular cases. Since the main new point given by the present paper is the implication (i) $\Rightarrow$ (iii), we get counterexamples to transfer of compactness.

For example:

Corollary 3   *(i) If $\mathrm{cf}\mu < \mathrm{cf}\lambda$ then there is a productively $[\mu, \mu]$-compact space which is not $[\lambda, \lambda]$-compact.*

(ii) $\mu > \omega$ is measurable iff there is a productively $[\mu, \mu]$-compact space which is not $[\lambda, \lambda]$-compact, for any $\lambda < \mu$.

(iii) $\mu > \omega$ *is κ-strongly compact iff there is a productively $[\mu, \kappa]$-compact space which is not $[\lambda, \lambda]$-compact, for any $\lambda < \mu$.*

(iv) $\mu > \omega$ *is strongly compact iff there is a productively finally μ-compact space which is not $[\lambda, \lambda]$-compact, for any $\lambda < \mu$.*

(v) *If the existence of a measurable cardinal is consistent then also the following are consistent:*

    (a) *there exist a weakly inaccessible cardinal $\kappa \leq 2^\omega$ and a productively $[\kappa, \kappa]$-compact space which is not $[\lambda, \lambda]$-compact, for every λ, with $\omega < \mathrm{cf}\,\lambda < \kappa$.*

    (b) *there exist a strongly inaccessible not weakly compact cardinal and a productively $[\kappa, \kappa]$-compact space which is not $[\lambda, \lambda]$-compact, for every λ, with $\omega < \mathrm{cf}\,\lambda < \kappa$.*

(vi) *If the existence of a κ^+-strongly compact cardinal κ is consistent, then it is consistent that there is a productively $[\omega_{\omega+1}, \omega_{\omega+1}]$-compact space which for no $0 < n < \omega$ is $[\omega_n, \omega_n]$-compact.*

Proof: From the results on ultrafilters listed below, and Theorem 1 (if we want $[\lambda, \lambda]$-incompactness for a single λ) or Theorem 3 below if we want the full result (incompactness for many cardinals at a time).

For (i) it is enough to use a uniform ultrafilter on $\mathrm{cf}\,\mu$, which must be (μ, μ)-regular, but cannot be (λ, λ)-regular by, say, [Lp, Corollary 1.4].

(ii)-(iv): D is v-complete iff it is not (v', v')-regular, for any $v' < v$. In order to prove (ii)-(iv) it is enough to use discrete spaces, rather than the spaces κ of Proposition 1.

See [Sh] and [AH] for the results on ultrafilters giving (v) and (vi).

The reader should be warned that results on regular ultrafilters are often stated using a rather different terminology. For κ regular, an ultrafilter is (κ, κ)-regular iff it is κ-descendingly incomplete, iff it is κ-*decomposable*, iff (in the *RK order*) it is larger than a *uniform* ultrafilter over κ (see [CN]). Usually, in the literature, the property "every (λ, λ)-regular ultrafilter is (κ, κ)-regular" is stated in some equivalent form, e.g.: "every uniform ultrafilter over λ is κ-decomposable" (we are assuming both λ and κ regular cardinals, otherwise the equivalence does not necessarily hold). $\square$

Notice that, contrary to Corollary 3, there are many hypotheses of Set Theory implying that every productively $[\kappa, \kappa]$-compact space is produtively $[\lambda, \lambda]$-compact, or even productively initially κ-compact. We shall mention only one result in this direction:

Proposition 3 *Suppose that there is no inner model with a measurable cardinal. Then for every μ and λ there is a (μ, μ)-regular non-(λ, λ)-regular ultrafilter iff* cfμ $<$cfλ. *Whence, every productively* $[\lambda, \lambda]$-*compact space is productively* cfλ-*initially compact.*

Proof: From [Do, Theorem 4.5] and [Lp, Corollary 1.4]. The last conclusion follows from the classical result [AU] that a space is $[\lambda, \mu]$-compact iff it is $[\kappa, \kappa]$-compact for every κ with $\lambda \leq \kappa \leq \mu$. (By passing, let us warn the reader that the situation for ultrafilters is different: there is, in some model of ZFC, a non-(λ, λ^+)-regular ultrafilters which is both (λ, λ)- and (λ^+, λ^+)-regular. Notice that this shows that in Theorem 1 we cannot replace $[\kappa, \kappa], (\kappa, \kappa)$ by $[\kappa, \kappa'], (\kappa, \kappa')$. However, the equivalence (i) $\Leftrightarrow$ (ii) still holds when we perform the replacement). $\square$

Now for the refinement of Theorem 1 which is needed in order to prove Corollary 3 in its full generality:

Theorem 3 *For cardinals λ, μ, and for a family $(\kappa_i)_{i \in I}$ of cardinals, the following are equivalent:*

(i) *Every productively $[\lambda, \mu]$-compact space is $[\kappa_i, \kappa_i]$-compact for some $i \in I$.*

(ii) *Every productively $[\lambda, \mu]$-compact family of topological spaces is productively $[\kappa_i, \kappa_i]$-compact for some $i \in I$.*

(iii) *Every (λ, μ)-regular ultrafilter is (κ_i, κ_i)-regular for some $i \in I$.*

If each κ_i is a regular cardinal, then the preceding conditions are also equivalent to

(iv) *Every productively $[\lambda, \mu]$-compact Hausdorff regular space is $[\kappa_i, \kappa_i]$-compact for some $i \in I$.*

Proof: (i) $\Rightarrow$ (iii). Suppose that D is (λ, μ)-regular and let $X = \prod_{i \in I} S_{\kappa_i}(\kappa_i)$. Since X is not $[\kappa_i, \kappa_i]$-compact, for $i \in I$, by (i) X is not productively $[\lambda, \mu]$-compact, and by Theorem 2 there is $i \in I$ such that $S_{\kappa_i}(\kappa_i)$ is not D-compact. Then by Propostion 2 D is (κ_i, κ_i)-regular.

The rest is like the proof of Theorem 1. $\square$

Problem   Provide a characterization of the following variant of condition (i) in Theorem 1:

(*) *For every space X, if the v-th power of X is $[\lambda, \mu]$-compact then the v'-th power of X is $[\kappa, \kappa]$-compact.*

The problem is open only for small values of v and v', since [Ca] (generalizing [ScSt], [GS]. [Sa]) proved that for every λ, μ there is a v such that any space X is productively $[\lambda, \mu]$-compact iff the v-th power of X is $[\lambda, \mu]$-compact. Notice that, for such a v, (*) becomes independent from v', by Theorem 1(i) $\Leftrightarrow$ (ii). See [Va] and [St] for the problem of estimating v, in the case of initial μ-compactness.

Remark   A result parallel to Theorem 1 has been proved a long time ago in a completely different setting (compactness properties of extensions of first-order logic). One direction is due to J. A. Makowsky and S. Shelah, and the other to the present author. Later H. Mannila realized the topological content of some arguments, but only X. Caicedo fully exploited their topological nature. See [Ca], [BF], [Lp] for references. Actually, for κ regular, we get a (more involved) proof of Theorems 1 and 3 just by putting together [Ca, Theorem 2.9, Corollary 3.6] and (a suitable generalization of) [Lp, Theorem 4.1].

We are indebted with X. Caicedo for providing us with a copy of [Ca] (in August 1995), without which we could not even begin writing the present work.

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