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Spaces Having σ-Compact-Finite k-Networks, and Related Matters

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Introduction

Let X be a space, and let $\mathcal{P}$ be a collection of subsets of X. We recall that $\mathcal{P}$ is *point-countable* (resp. *star-countable*) if every $x \in X$ (resp, $P \in \mathcal{P}$) meets at most countably many $Q \in \mathcal{P}$. Also, $\mathcal{P}$ is *compact-finite* (resp. *compact-countable*) if every compact subset of X meets at most finitely (resp. countably) many $P \in \mathcal{P}$. A collection $\cup\{\mathcal{P}_n : n \in N\}$ is *σ-compact-finite* if each $\mathcal{P}_n$ is compact-finite. Clearly, every σ-compact-finite collection is compact-countable, and thus, point-countable.

Let X be a space, and let $\mathcal{P}$ be a cover of X. Recall that $\mathcal{P}$ is a *k-network* if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\, \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. As is well-known, spaces with a countable (resp. σ-locally finite) k-network are called *$\aleph_0$-spaces* (resp. *$\aleph$-spaces*).

Every CW-complex, more generally, every space dominated by locally separable metric subspaces has a star-countable k-network. Also, every Lašnev space has a σ-hereditarily closure

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preserving (briefly, σ-HCP) k-network, and every space dominated by Lašnev subspaces has a σ-compact-finite k-network.

We recall that spaces with a star-countable k-network, and spaces with a σ-HCP k-network have σ-compact-finite k-networks.

Spaces with a star-countable k-network are investigated in [9], [18], [20], and [26]. Spaces with a σ-HCP k-network are investigated in [16], and so are spaces with a compact-countable k-network in [19], and [20].

In this paper, we shall investigate spaces with a σ- compact-finite k-network as well as related spaces, and their examples and applications.

We assume that all spaces are regular, T_1, and that all maps are continous and onto.

Results

Let X be a space, and let $\mathcal{C}$ be a cover of X. Then X *is determined by* $\mathcal{C}$ [7] ($= X$ has the weak topology with respect to $\mathcal{C}$ in the usual sense), if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathcal{C}$. Every space is determined by its open cover. We recall that a space X is a *k-space* (resp. *sequential space*) if it is determined by a cover of compact subsets (resp. compact metric subsets) of X. A space has *countable tightness* if X is determined by a cover of countable subsets; cf. [22]. For a cover $\mathcal{C}$ of a space X, X *is dominated by* $\mathcal{C}$ if the union of any subcollection $\mathcal{C}'$ of $\mathcal{C}$ is closed in X, and the union is determined by $\mathcal{C}'$. Every space is dominted by its σ-HCP closed cover. As is well-known, every CW-complex is dominated by a cover of compact metric subsets.

Lemma 1 *Let X have a point countable k-network. Then (1) and (2) below hold, here (1); (2) is due to* [2]; [7] *respectively.*

(1) If X is compact, then X is metric.

(2) If X is a k-space, then X is sequential, thus, of countable tightness.

Let $\mathcal{P}$ be a colletion of subsets of X. Then $\mathcal{P}$ is *cs-finite* [15] if any convergent sequence meets only finitely many $P \in \mathcal{P}$. Let $\mathcal{P}$ be a cover of X. Then $\mathcal{P}$ is a *cs-network* [8], if whenever $L = \{x_n; n \in N\}$ is a sequence converging to a point $x \in X$ such that $x \in U$ with U open in X, then there exists $P \in \mathcal{P}$ such that $x \in P$, $P \subset U$, and P contains L eventually (i.e., P contains $\{x_n : n \geqq m\}$ for some $m \in N$). If we replace "eventually" by "frequently (i.e., P contains a subsequence of L)", then $\mathcal{P}$ is a *cs*-network* [6]. Also, if we need not require "$x \in P$" in the definition of a cs*-network, then such a cover is a *wcs*-network* [14]. Every cs-network and every k-network of closed subsets are cs*-networks, and every cs*-network is a wcs*-network. Every quotient s-image of a metric space is characterized as a sequential space with a point-countable cs*-network [32].

In view of the following, we see that, among sequential spaces, the theory of spaces with a σ-compact-finite k-network can be done by means of " convergent sequences" instead of "compact subsets". Here, a space has *G_δ points* if every point is a G_δ-set. We note that not every compact space with a cs-finite and star-countable cs-network has a point-countable k-network (hence, not a σ-compact-finite k-network); indeed, the Stone-Čech compactification $\beta(X)$ of a non-compact space X is such a space by Lemma 1(1).

Proposition 2 *(1) For a cover $\mathcal{P}$ of a space X, the following are equivalent.*

(a) $\mathcal{P}$ is a σ-compact-finite k-network.

(b) $\mathcal{P}$ is a σ-cs-finite k-network.

(c) $\mathcal{P}$ is a σ-cs-finite wcs-network, and each compact subset of X is sequentially compact.*

(2) Let X be a sequential space, or a space with G_δ points. Then, a cover of X is a σ-compact-finite k-network if and only if it is a σ-cs-finite wcs-network.*

Proof: For (1), obviously, (a) implies (b). (b) implies (c), because each compact subset of X is metric by Lemma 1(1), and thus, sequentially compact. So, we show that (c) $\Rightarrow$ (a) holds. Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$ be a σ-cs-finite wcs*-network for X. Since $\mathcal{P}$ is a point-countable wcs*-network and any compact subset of X is sequentially compact, $\mathcal{P}$ is a k-network by [32; Proposition 1.2(1)]. To show that $\mathcal{P}$ is σ-compact-finite, suppose that some compact set K of X meets infinitely many $P \in \mathcal{P}_n$ for some $n \in N$. Then, there exist $\{x_n : n \in N\} \subset K$ and $\{P_n : n \in N\} \subset \mathcal{P}_n$ such that $x_n \in P_n$, and the x_n are distinct, and so are the P_n. Since K is sequentially compact, there exists a convergent subsequence C of $\{x_n : n \in N\}$. But the convergent sequence C meets infinitely many $P \in \mathcal{P}_n$. This is a contradiction. Thus, $\mathcal{P}$ is a σ-compact-finite k-network . For (2), note that if X is a sequential space, or a space with G_δ points, then each compact subset of X is sequentially compact. Thus, (2) follows from (1).

Now, let us consider the operations: (i) Subsets; (ii) Domination; (iii) Countable products; (iv) Closed maps with k-space domain; and (v) Perfect maps.

The property of having a star-countable k-network is preserved by the all operations; see [9], [19], etc. However, the property of having a σ-HCP k-network need not be preserved by (ii); nor (iii); see [34]; [11] respectively. But, the property of having a σ-compact-finite k-network is preserved by the all operations in view of Theorem 3 below (for (iii), cf. [7; Theorem 7.1]). We note that every closed image of a space with a compact-finite k-network of singletons need not have even a point-countable k-network; see [25]. Thus, the additional assumptions on X in case (d) of Theorem 3 are essential.

Theorem 3 *Each of the following (a) $\sim$ (d) implies that Y has a σ-compact-finite k-network.*

(a) Y has a star-countable k-network.

*(b) Y has a σ-*HCP k-*network.*

(c) Y is dominated by spaces with a σ-compact-finite k-*network.*

(d) Y is the closed image of a space X with a σ-compact-finite k-network, and one of the following properties holds.

(i) X is a k-*space.*

(ii) X is a space with G_δ points.

(iii) X is a normal space, and each countably compact closed subset is compact.

(iv) X is realcompact

(v) Each $\partial f^{-1}(y)$ is Lindelöf.

Proof: For case (a); (b); or (c), the result is due to [18]; [15]; or [19] respectively. So, we show the result for (d) holds. Let $f : X \to Y$ be a closed map, and let X be a space with a σ-compact-finite k-network $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$. For each $y \in Y$, choose $x_y \in f^{-1}(y)$, and let $A = \cup\{x_y : y \in Y\}$. For each $n \in N$, let $\mathcal{C}_n = \{f(A \cap P) : P \in \mathcal{P}_n\}$. Then $\mathcal{C} = \cup\{\mathcal{C}_n : n \in N\}$ is σ-point-finite. Let us consider the following conditions (C_1) and (C_2) with respect to the closed map f.

(C_1): For any infinite compact subset K of Y, and any sequence S in $f^{-1}(K)$ with $f(S)$ infinite, there exists a convergent subsequence of S.

(C_2): f is compact-covering (i.e., every compact subset of Y is the image of a compact subset of X), and for any sequence $\{y_n : n \in N\}$ in Y converging to $y \in Y$, and any points $x_n \in f^{-1}(y_n)$, a closed map $f|F$ is also compact-covering, where $F = \partial f^{-1}(y) \cup \{x_n : n \in N\}$ which is closed in X.

Then, (C_1) holds for (i) & (ii) by Lemma 1(2) and [14: Lemma 2]. Also, (C_2) holds for (iii) & (v); and (iv) by [14; Lemma 4]; and [4; Theorem 3.4] respectively. But, (C_1) or (C_2) implies that each compact subset of Y is sequentially compact by Lemma 1(1), besides, $\mathcal{C}$ is a wcs*-network. To show that $\mathcal{C}$ is σ-cs-finite, for some $n \in N$, suppose that an infinite convergent sequence K in Y meets infinitely many distinct members $f(A \cap P_k) \in \mathcal{C}_n$. Since $\mathcal{C}_n$ is point-finite, we can assume that each point of K is contained in some of these $f(A \cap P_k)$. Then, there exists a sequence $S = \{x_m : m \in N\}$ in $A \cap f^{-1}(K)$ such that $x_m \in P_{k(m)} \in \mathcal{P}_n$, and the x_m are distinct, and also so are the $P_{k(m)}$. But, (C_1) or (C_2) implies that there exists a convergent subsequence C of S. Then C meets infinitely many elements of $\mathcal{P}_n$. This is a contradiction. Thus, $\mathcal{C}$ is a σ-cs-finite wcs*-network for Y. Thus, by Proposition 2(1), $\mathcal{C}$ is a σ-compact-finite k-network for X.

For a space X, the *character* $\chi(X)$ of X is the smallest cardinal number of the form $|\mathcal{B}_x|$, here $\mathcal{B}_x$ is a local base at $x \in X$. A space X is *ω_1-compact* if every subset of cardinality ω_1 has an accumulation point in X.

Lemma 4 *Let X be a k-space, and let $\mathcal{C} = \cup\{\mathcal{C}_n : n \in N\}$ be a σ-compact-finite collection in X. Then (1) and (2) below hold.*

(1) If $\chi(X) \leq \omega_1$, then $\mathcal{C}$ is σ-locally countable.

(2) If X is locally ω_1-compact, then $\mathcal{C}$ is locally countable.

Proof: For (1), let $x \in X$, and let $\{V_\beta : \beta < \omega_1\}$ be a local base at x in X. Then, for each $n \in N$, there exists some V_β such that V_β meets only countably many $C \in \mathcal{C}_n$. Indeed, for some $n \in N$, suppose not. Then, by induction, there exist a subset $S = \{x_\beta : \beta < \omega_1\}$ of X and a subcollection $\{C_\beta : \beta < \omega_1\}$ of $\mathcal{C}_n$ such that $x_\beta \in V_\beta \cap C_\beta$, where $x_\beta \neq x$, and the C_β are distinct. But, S has an accumulation point in X, so it can be assumed to be not closed in X. Then, since X is a k-space, there exists a compact subset K of X which contains infinitely many points in S. This shows that the compact set K meets infinitely many elements of $\mathcal{C}_n$. This is a contradiction. Thus, for each $n \in N$, any point of X has a nbd V such that V meets only countably many $C \in \mathcal{C}_n$. Hence, $\mathcal{C}$ is σ-locally countable. For (2), let $x \in X$, and let V be a nbd of x which is ω_1-compact. Then V meets only countably many elements of $\mathcal{C}$ in view of the proof of (1). Thus, $\mathcal{C}$ is locally countable.

Remark 5 We note that not every k-space with a σ-compact-finite k-network has a σ-HCP k-network in view of [11] & [15], and not every space with a compact-finite and locally countable k-network consisting of singletons is a σ-space [25]. In [15], the first author shows that the following hold.

(1) Among Fréchet spaces, every σ-compact-finite k-network is σ-HCP. Thus, a space is Lašnev if and only if it is a Fréchet space with a σ-compact-finite k-network.

(2) Among k-spaces, every σ-compact-finite k-network of *closed subsets* is σ-locally finite. Thus, a k-space is an $\aleph$-space if and only if it has a σ-compact-finite k-network of closed subsets.

Theorem 6 (CH) *Let X be a k-space with a σ-compact-finite k-network. Then X is locally separable if and only if X is the topological sum of $\aleph_0$-spaces.*

Proof: For $x \in X$, let V be a nbd of x which is separable. Then V is sequential by Lemma 1(2), and it has a σ-compact-finite k-network. Also, V is separable, so $\chi(V) \leqq 2^\omega = \omega_1$ under (CH). Thus, V is a sequential space with a σ-locally countable k-network by Lemma 4(1). While, by [12; Proposition 1], every sequential space with a σ-locally countable k-network is meta-Lindelöf (i.e., every open cover has a point-countable refinement). Then the separable space V is meta-Lindelöf. But, every separable, meta-Lindelöf space is Lindelöf. Then V is Lindelöf, thus, ω_1-compact. This shows that X is locally ω_1-compact. Hence, X has a locally countable k-network by Lemma 4(2). Since X is a k-space, X is the topological sum of $\aleph_0$-spaces by [12; Theorem 1] (or [9]).

We note that not every separable, $\aleph$-space with a compact-finite, locally countable k-network need be an $\aleph_0$-space [9; Example 4.1]. And, not every separable, k-space with a point-countable closed cs-network need be an $\aleph_0$-space [7; Example 9.3]. Also, we note that not every cosmic, k-space with a point-countable closed k-network need be an $\aleph_0$-space [35; Example 1.6], where a space is *cosmic* if it has a countable network.

As for conditions for separable spaces to be $\aleph_0$-spaces, the following holds. In (1), case (c) gives an affirmative answer to [20: Question 3.1] under (CH).

Theorem 7 *(1) Let X be a separable space. Then each of the following implies that X is an $\aleph_0$-space.*

(a) X is a Fréchet space with a point-countable k-network.

(b) X is a k-space with a star-countable k-network.

(c) (CH). X is a k-space with a σ-compact-finite k-network. (When X is meta-Lindelöf, or $\chi(X) \leqq \omega_1$, (CH) *can be omitted).*

(2) Let X be a cosmic space. If X has a point-countable cs-network, then X is an $\aleph_0$-space.

Proof: In (1), for case (a); (b), the result is respectively due to [7]; [26]. For case (c), the result holds in view of the proof of Theorem 6. To see (2) holds, let $\mathcal{P}$ be a point-countable cs-network for X. Since X is cosmic, it is easy to show that X has a countable subset D such that, for any $x \in X$, there exists a sequence in D converging to x. Let $\mathcal{P}' = \{P \in \mathcal{P} : P \cap D \neq \emptyset\}$. Then, $\mathcal{P}'$ is countable. To see that $\mathcal{P}'$ is a cs-network for X, let $\{x_n : n \in N\}$ be a sequence converging to $x \in X$, U be a nbd of x. But, there exists a sequence $\{y_n; n \in N\}$ in D converging to the point x. Clearly, $L = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots\}$ converges to the point x. Since $\mathcal{P}$ is a cs-network, there exists $P \in \mathcal{P}$ such that P contains x, and L eventually, thus, $P \in \mathcal{P}'$. Then, $\mathcal{P}'$ is countable cs-network. Thus, X is an $\aleph_0$-space by [8; Theorem 1].

Lemma 8 *Let $\mathcal{P}$ be a point-countable cs*-network for a space X. Let $K = \{x_n : n \in N\} \cup \{x\}$ be a sequence with a limit point x, and let U be an open set with $U \supset K$. Then there exists a finite $\mathcal{P}' \subset \mathcal{P}$ such that, for some $i \in N, \{x_n : n \geqq i\} \cup \{x\} \subset \cup\mathcal{P}' \subset U$, and, for each $P \in \mathcal{P}'$, $P \cap K$ is closed in K (thus, if $P \cap K$ is infinite then P contains the point x).*

Proof: Let $\{P \in \mathcal{P} : P \subset U$, and $P \cap K$ is non-empty, closed in $K\} = \{P_n : n \in N\}$. Then, for some $i,\ j \in N, \{x_n : n \geqq i\} \cup \{x\} \subset \cup\{P_n : n \leqq j\}$. Indeed, suppose not. Then there exists a subsequence $L = \{x_{n(i)} : i \in N\}$ of K such that $x_{n(i)} \in X - \cup\{P_n : n \leqq i\}$. Since $L \cup \{x\} \subset U$, there exists $P_0 \in \mathcal{P}$ such that $P_0 \subset U$, and P_0 contains the point x and L frequently. Thus, $P_0 \cap K$ is non-empty, closed in K, so $P_0 = P_m$ for some $m \in N$, hence P_m contains L frequently. This is a contradiction

Theorem 9 *(1) Let X be a k-space with a σ-compact-finite k-network . Then X has a star-countable k-network if and only if every metric closed subset of X is locally ω_1-compact.*

(2) Let X be a sequential space with a σ-compact-finite (resp. compact-countable) cs-network. Then X is the topological sum of $\aleph_0$-spaces (resp. k_ω-and-$\aleph_0$-spaces) if and only*

if every metric closed subset of X is locally ω_1-compact (resp. locally compact). Here, a space is a k_ω-space [21] *if it is determined by a countable cover of compact subsets.*

Proof: The "only if" part of (1) holds, because every first countable space with a star-countable k-network is locally separable metric [9; Theorem 1.4]. For the "if" part of (1), let X have a σ-compact-finite k-network $\mathcal{P}$ which is closed under finite intersections, and let every first countable closed subset of X be locally ω_1-compact. Let K be a compact subset of X, and let $\mathcal{P}_K = \{P \in \mathcal{P}; P \cap K \neq \emptyset\}$, and let $\{\mathcal{P}_n : n \in N\}$ be the collection of all covers of K consisting of finite subcollection of $\mathcal{P}_K$. For each $n \in N$, let $\mathcal{C}_n = \{\cap\{P_i : P_i \in \mathcal{P}_i,\ i \leq n\}\}$, and $A_n = \cup\, \mathcal{C}_n$. Then $\{A_n : n \in N\}$ is a decreasing sequence such that A_n are finite unions of elements of $\mathcal{P}$, $A_n \supset K$, and any open subset containing K contains some $\mathrm{cl}A_m$. Suppose that any $\mathrm{cl}A_n$ is not ω_1-compact in X. Then any $\mathrm{cl}A_n$ contains a closed discrete subset D_n of X with cardinality ω_1. Let $F = K \cup (\cup\{D_n : n \in N\})$. Then F is a closed subset of X which is a σ-space, and an M-space, for X is the perfect pre-image of a metric space F/K. Then, as is well-known, F is metric. (We can also see that F is metric by Lemma 13 below, because F is a first countable space, and it has a σ-compact-finite k-network by Proposition 2(2)). But, F is not locally ω_1-compact. This is a contradiction. Then, for some $n \in N$, $\mathrm{cl}A_n$ is ω_1-compact. While, any open subset containing K contains some $\mathrm{cl}A_m$. This implies that, $\mathcal{P}^* = \{P \in \mathcal{P} : \mathrm{cl}P \text{ is } \omega_1\text{-compact in } X\}$ is a k-network for X. Also, since each closure of elements of $\mathcal{P}^*$ is ω_1-compact, $\mathcal{P}^*$ is star-countable in view of the proof of Lemma 4. Then, X has a star-countable k-network $\mathcal{P}^*$. For the "if" part of (2), let $K = \{x_n : n \in N\} \cup \{x\}$ be a sequence with a limit point x, and let $\mathcal{P}_K = \{P \in \mathcal{P} : P \cap K \text{ is non-empty, closed in } K\}$, and let $\{\mathcal{P}_n : n \in N\}$ be the collection of all finite subcollections of $\mathcal{P}_K$ such that any $\cup\mathcal{P}_n$ contains x, and K eventually. Then, replacing " k-network " by " cs*-network" in the above proof, $\mathcal{P}^*$ is a σ-compact-finite cs*-network by Lemma 8, and $\mathcal{P}^*$ is star-countable. Then, $\mathcal{P}^*$ is a star-countable k-network for X by Proposition 2(2). But, since X is sequential, X is determined by $\mathcal{P}^*$. Thus, X is the topological sum of $\aleph_0$-spaces by [9; Corollary 1.2]. For the parenthetic part, the "only if" holds, because, as is well-known, every first countable k_ω-space is locally compact (note that each point has a nbd which is contained in a finite union of compact subsets). For the "if" part, similarly, X is the topological sum of $\aleph_0$-spaces $X_\alpha (\alpha \in A)$. Since every metric closed subset of X is locally compact, similarly, each X_α has a countable k-network of compact subsets. Then X_α is a k_ω-space, for it is a k-space. Then, X is the topological sum of k_ω-and-$\aleph_0$-spaces.

Lemma 10 ([35]). *Suppose that X is determined by a point-countable cover $\mathcal{C}$, or X is dominated by cover $\mathcal{C}$. Let $\{A_n : n \in N\}$ be a collection of subsets of X such that if $x_n \in A_n$, then $\{x_n : n \in N\}$ has an accumulation point in X. Then, for some $m \in N$, A_m is contained in a finite union of elements of $\mathcal{C}$.*

Corollary 11 *Suppose that X is determined by a point-countable cover of locally ω_1-compact subsets, or dominated by a cover of locally ω_1-compact subsets. Then (1) and (2) below hold.*

(1) If X is k-space with a σ-compact-finite cs-network, then X has a star-countable k-network.*

(2) If X is a sequential space with a σ-compact-finite cs-network, then X is the topological sum of $\aleph_0$-spaces (hence, X is an $\aleph$-space).*

Under (CH), it is possible to replace "locally ω_1-compact" by "locally separable".

Proof: Let F be a metric closed subset of X. Suppose that X is determined by a point-countable cover $\{X_\alpha : \alpha \in A\}$ of locally ω_1-compact subsets. Since F is closed, F is determined by a point-countable cover $\{F \cap X_\alpha : \alpha \in A\}$. Since each $F \cap X_\alpha$ is locally separable metric, $F \cap X_\alpha$ is determined by a point-countable cover $\{X_{\alpha\beta} : \beta \in B_\alpha\}$ of separable metric subsets. Hence, F is determined by a point-countable cover $\{X_{\alpha\beta} : \alpha \in A,\ \beta \in B_\alpha\}$ of separable metric subsets. Next, suppose that X is dominated by a cover $\{X_\alpha : \alpha \in A\}$ of locally ω_1-compact subsets. Then, F is dominated by a cover $\{F \cap X_\alpha : \alpha \in A\}$ of locally separable metric subsets. Then, for any case, F is locally ω_1-compact by Lemma 10. Then, every metric closed subset is locally ω_1-compact. Then, (1) and (2) holds by Theorem 9. For the latter part holds by means of Theorem 6 and Corollary 7.

It is well-known that every quotient s-image of a locally compact metric space is precisely a space determined by a point-countable cover of compact metric subsets, and that every CW-complex is dominated by a cover of compact metric subsets. Also, recall that every space determined by a point-countable cover of metric subsets has a point-countable cs*-network ([32]), and that every space X dominated by metric subsets X_α has a σ-compact-finite k-network (Theorem 3), and, in particular, X has a star-countable k-network if the X_α are locally separable ([9]). However, every CW-complex need not have a point-countable cs*-network, also every CW-complex determined by a point-finite cover of compact metric subsets need not have a point-countable cs-network ([18]). But, for spaces determined by locally separable metric subsets, we have the following theorem.

Theorem 12 *(1) Suppose that X is determined by a point-countable cover of locally separable metric subsets. If X has a σ-compact-finite k-network, then X has a star-countable k-network.*

(2) Suppose that X is determined by a point-countable cover

of locally separable (resp. locally compact) metric subsets, or X is dominated by a cover of locally separable (resp. locally compact) metric subsets. If X has a σ-compact-finite (resp. compact-sountable) cs*-network, then X is the topological sum of $\aleph_0$-spaces (resp. k_ω-and-$\aleph_0$-spaces).

(3) (i) Suppose that X is determined by a point-countable closed cover of locally separable metric subsets (in particular, X is determined by a point-countable cover of locally compact metric subsets). If X has a point-countable cs-network, then X is a locally $\aleph_0$-spaces. When X is meta-Lindelöf, X is the topological sum of $\aleph_0$-spaces.

(ii) Suppose that X is dominated by a cover of locally separable metric subsets. If X has a point-countable cs-network, then X is the topological sum of $\aleph_0$-spaces [18].

Proof: Since X is sequential, (1) and (2) holds by Corollary 11. For the parenthetic part of (2), suppose that X is determined by a point-countable cover of locally compact metric subsets. Since any locally compact metric space is determined by a point-finite cover of compact metric, X is determined by a point-countable cover of compact metric subsets. Thus, X is the topological sum of k_ω-and-$\aleph_0$ spaces by means Lemma 10 and the parenthetic part of Theorem 9(2). For (3), let $\mathcal{P}$ be a point-countable cs-network for X which is closed under finite intersections . Let $K = \{x_n : n \in N\} \cup \{x\}$ be a sequence with a limit point x, and let $\mathcal{P}_K = \{P \in \mathcal{P} : P \ni x$, and P contains K eventually$\} = \{P_n : n \in N\}$. Let $A_n = \cap(P_i : i \leqq n\}$ for each $n \in N$. Then $\{A_n : n \in N\}$ is a decreasing sequence such that $A_n \in \mathcal{P}$, $A_n \ni x$, A_n contains K eventually, and any nbd of x contains some A_n. For (i), since any sequence $\{x_n : n \in N\}$ with $x_n \in A_n$ has an accumulation point in X, by Lemma 10, for some $i \in N$, A_i is contained in a locally separable metric space. Since A_i contains x and K eventually, for some $j \in N$ with $j \geqq i$, A_j is separable metric. Then X is a sequential space with a point-countable cs-network of separable metric subsets. Thus, in view of the proof of Theorem 2.4 in [13], X has a locally $\aleph_0$-space. When X is meta-Lindelöf, X has a point-countable open cover of $\aleph_0$-spaces, thus, X is determined by this star-countable cover. Then, X is the topological sum of $\aleph_0$-spaces by means of [9; Lemma 1.2]. For (ii), similarly, X is a locally $\aleph_0$-space. But, as is well-known, every space dominated by metric subsets is paracompact, so X is paracompact. Thus, X is the topological sum of $\aleph_0$-spaces.

Let us consider a canonical space dominatede by metric subsets (not every piece is locally separable).

Example 13 Let M be a *metric* space. For each $x \in M$, let L_x be a sequence converging to the point x such that $L_x \cap M = \emptyset$, and the L_x are pairwise disjoint. Let $S_x = M \cup L_x$, and let $X_x = L_x \cup \{x\}$. Let S be the space determined by a point-finite cover $\{M, X_x : x \in M\}$ of metric subsets. Equivalently, S is dominated by a cover $\{S_x : x \in M\}$ of metric subsets. When M is an infinite convergent sequence with a limit point x, a subspace $(S - L_x)$ of S is called the *Arens' space* S_2.

M. Sakai [27] ask the following questions on the space S.

Questions (1) What are topological properties of S in terms of k-network?

(2) When does S have a point-countable cs-network? Also, if S has a point-countable cs-network, then is S an $\aleph$-space?

We shall give answers to (1) and (2), and give characterizations for S to have certain k-networks in terms of the metric space M. First, let us recall definitions. For a space X, let T_x be a collection of subsets of X such that any element of T_x contains x. The collection $T_X = \cup\{T_x : x \in X\}$ is a *weak base* [1] for X if it satisfies: The T_1, $T_2 \in T_x$, there exists $T_3 \in T_x$ with $T_3 \subset T_1 \cap T_2$; and, $U \subset X$ is open in X if and only if for each $x \in U$, there exists $T \in T_x$ with $T \subset U$. The T_x is a *local weak base* at x in X. Every weak base is a cs-network [14]. A space

X is *g-first countable* [28] (or X satisfies the *weak first axiom of countability* [1]) if X has a weak base $T_X = \cup\{T_x : x \in X\}$ such that each T_x is countable.

Properties of the space S: (A) S is a g-first countable, paracompact, and σ-space. Besides, S has a σ-compact-finite k-network, and a point-countable closed cs*-network.

(B) S is metric $\Leftrightarrow$ S is locally compact $\Leftrightarrow$ S is Fréchet $\Leftrightarrow$ M is discrete.

(C) S has a star-countable k-network $\Leftrightarrow$ M is locally separable.

(D) S has a locally countable k-network $\Leftrightarrow$ S has a star-countable closed k-network $\Leftrightarrow$ S is locally separable $\Leftrightarrow$ M is the topological sum of countable subsets. In particular, S is an $\aleph_0$-space $\Leftrightarrow$ S is separable $\Leftrightarrow$ M is countable.

(E) S has a star-countable (or locally countable) k-network of compact subsets $\Leftrightarrow$ S is a locally k_ω-space $\Leftrightarrow$ M is the topological sum of countable, compact subsets. In particular, S has a countable k-network of compact subsets $\Leftrightarrow$ S is a k_ω-space $\Leftrightarrow$ M is countable, locally compact.

(F) S has a point-countable k-network of separable (resp. compact) subsets $\Leftrightarrow$ M is locally separable (resp. locally compact).

(G) The following (a) $\sim$ (g) are equivalent, and (g) implies (h).

(a) S is an $\aleph$-space.

(b) S has a σ-locally countable k-network.

(c) S has a σ-HCP k-network.

(d) S has a σ-compact-finite cs*-network.

(e) S has a point-countable cs-network.

(f) M is the countable union of closed discrete subsets.

(g) M has a point-countable open cover $\mathcal{V}$ satisfying (*): Each $V \in \mathcal{V}$ contains a point $x(V)$ such that $\{x(V) : V \in \mathcal{V}\} = M$.

(h) For any subspace A of M, $|A| = \omega(A)$, here $\omega(A)$ is the weight of A.

Proof: (A): Since S is dominated by metric subsets, as is well-known, S is a paracompact σ-space (indeed, M_1-space; see [31], for example). Let $X_0 = M$, and let $M' = \{0\} \cup M$. For $p \in X_x$ ($x \in M'$), let $\{V_{xn}(p) : n \in N\}$ be a decreasing local base at p in X_x. For each $p \in S$, and $n \in N$, let $Q_n(p) = \cup\{V_{xn}(p) : p \in X_x,\ x \in M'\}$. Since S is determined by a point-finite cover $\mathcal{C} = \{X_x : x \in M'\}, \{Q_n(p) : n \in N\}$ is a weak nbd of p in S. Thus, S is g-first countable. Since any compact subset of S contained in a finite union of elements of the closed cover $\mathcal{C}$, it is routine to show that S has a point-countable closed cs*-network. Also, S has a σ-compact-finite k-network by Theorem 3.

(B): It suffices to show that if S is Fréchet, then M is discrete. Assume that M is not discrete. Then M has an infinite convergent sequence. Hence, S contains a copy of S_2. But, S_2 is not Fréchet. Then, S is not Fréchet.

(C): If S has a star-countable k-network, then so does M. Since M is first countable, M is locally separable by [7; Proposition 3.3]. Conversely, if M is locally separable, then, S has a star-countable k-network by (a) and Theorem 12(1).

(D): Suppose that S is locally separable. Since every separable subset of S meets only countably many of L_x's. S is a locally $\aleph_0$-space. But, S is paracompact by (A). Thus, each of the first three equivalence holds by [9; Theorem 1.4 and Proposition 1.5]. For the last equivalence, if M is locally separable, then M is the topological sum of separable subsets M_α. But, $\cup\{L_x : x \in M_\alpha\} \cup M_\alpha$ locally separable, then each M_α is countable. Thus, M is the topological sum of countable subsets.

(E): This is shown by a similar way as in (D), so we omit the proof.

(F): Let S have a point-countable k-network of separable subsets. Then so does M. Thus, since M is first countable, M is locally separable by [7; Proposition 3.2]. Conversely, let M be locally separable. Then M is determined by a point-countable cover of separable metric closed subsets. But, since S is determined by a point-finite closed cover $\{M, X_x : x \in M\}$, it is routinely shown that S has a point-countable k-network of separable metric closed k-network.

(G): First, we show that (b) $\Rightarrow$ (a) holds. Let $\mathcal{P} = \cup\mathcal{P}_n$ be a σ-locally countable closed k-network for S. For $n \in N$, and $x \in S$, let V_{xn} be a nbd of X meeting only countably many elements of $\mathcal{P}_n$. Since $\{V_{xn} : x \in S\}$ is an open cover of a paracompact space S, there exists a locally finite open refinement $\mathcal{U}_n$ of $\{V_{xn} : x \in S\}$. For each $U \in \mathcal{U}_n$, $\{U \cap P : P \in \mathcal{P}_n\} = \{P_{ni}(U) : i \in N\}$. Let $\mathcal{U}_{ni} = \{P_{ni}(U) : U \in \mathcal{U}_n\}$, and $\mathcal{W}_n = \cup\mathcal{U}_{ni}$ and let $\mathcal{W} = \cup\mathcal{W}_n$. Then $\mathcal{W}$ is σ-locally finite in S. We show that $\mathcal{W}$ is a k-network. Let V be open in S, and let $\{x_n : n \in N\}$ be a sequence converging to $x \in V$. Then, there exists $P \in \mathcal{P}_n$ for some $n \in N$ such that P contains a subsequence of $\{x_n : n \in N\}$. But, $\mathcal{U}_n$ is an open cover of S, there exists $U \in \mathcal{U}_n$ containing x. Then, $P \cap U \in \mathcal{W}_n$, $P \cap U \subset V$, and $P \cap U$ contains a subsequence of $\{x_n : n \in N\}$. While, every compact subset of S is sequentially compact. Thus, $\mathcal{W}$ is a k-network by [32: Proposition 1.2]. Then, S is an $\aleph$-space, thus, (a) holds. Next, we show that (d) $\Rightarrow$ (f), and (f) $\Rightarrow$ (a) hold. Let (d) hold, and let $\mathcal{P} = \cup\mathcal{P}_n$ be a σ-compact-finite cs*-network for S which is closed under finite intersections. Since S is dominated by metric subsets, in view of the proof of Theorem 9(2), using Lemma 10, we can assume that, for each $P \in \mathcal{P}$, cl$P$ is metric. For each $P \in \mathcal{P}$, let $D(P) = \{x \in M : x \in P$, and L_x is contained in P frequently$\}$. For each $P \in \mathcal{P}$, cl$P$ is metric, so it contains no copy of S_2. Thus, each $D(P)$ is closed discrete in M. For each $n \in N$, let $D_n = \cup\{D(P) : P \in \mathcal{P}_n\}$. Then since $\mathcal{P}$ is a cs*-network for S, M is the union of these D_n's. To show each D_n is closed discrete in M, suppose not. Then, there exists an infinite sequence K in D_n converging to a point $x \in M$. But, for each D(P), the compact set $C = K \cup \{x\}$ contains at most finitely many points in $D(P)$. Then, the compact set C meets infinitely many elements of $\mathcal{P}_n$, a contradiction. Thus, M is the countable union of closed discrete subsets D_n. Thus, (f) holds. Conversely, let (f) hold, and let M be the countable union of closed discrete subsets $E_n (n \in N)$. For each $n \in N$, let $C_n = \cup\{L_x : x \in E_n\} \cup M$. Then each C_n is a metric closed subset of S. But, each convergent sequence in S is contained in some C_n, then S is determined by a cover $\{C_n : n \in N\}$, for S is sequential. Thus, S is determined by a countable cover $\{C_n : n \in N\}$ of metric closed subsets. Thus, S is an $\aleph$-space by [31; Proposition 11]. Hence, (a) holds. For (c) $\Rightarrow$ (a), since S is g-first countable, if S has a σ-HCP k-network, S is an $\aleph$-space by means of [33; Theorem 6]. To show that (e) $\Leftrightarrow$ (g) holds, first, let (e) hold. But, S is a g-first countable by (A). Then, S has a point-countable weak base $T_s = \cup\{T_p : p \in S\}$ by [14; Lemma 7]. While, S is dominated by a cover $\{S_x : x \in M\}$ of metric subsets. Then, by Lemma 10, we can assume that, for each $p \in S$ and each $T \in T_p$, T is contained in a finite union of S_x's. Thus we can assume, for any $T \in T_s$, cl$T$ is metric. Since M is closed in $S, \{T \cap M : T \in T_s\}$ is a weak base for M. We recall that, for a space X and for a weak base $T_X = \cup\{T_x : x \in X\}$ for X, any sequence converging to a point $x \in X$ is contained eventually in any element of T_x. Then, since M is first countable, for any $x \in M$ and $T \in T_x$, $x \in \text{int}_M(T \cap M)$. Thus, $\mathcal{V} = \{\text{int}_M(T \cap M) : T \in T_s\}$ is a point-countable open cover of M. For each $V = \text{int}_M(T \cap M) \in \mathcal{V}$, let $D_V = \{x \in V : L_x$ is contained eventually in $T\}$. Then, $\cup\{X_x : x \in D_V\} \subset$ cl$T$. But, since cl$T$ is metric, cl$T$ contains no copy of S_2. Thus, D_V is a discrete closed subset of M with $D_V \subset V$. Also, $\{D_V : V \in \mathcal{V}\}$ is a cover of M, because, for any $x \in M$

and $T \in T_x$, $x \in \text{int}_M(T \cap M)$ and T contains L_x eventually. Since each $D_V = \{x_t : t \in D_V\}$ is closed discrete in M, there exists a discrete open collection $\{G_t : t \in D_V\}$ in M such that $x_t \in G_t \subset V$. Then, $\{G_t : t \in D_V,\ V \in \mathcal{V}\}$ is a point-countable open cover of M with $x_t \in G_t$, and $\{x_t : t \in D_V, V \in \mathcal{V}\} = M$. Then (g) holds. Conversely, let (g) hold. Then, since M has a point-countable base, it is easy to show that S has a point-countable cs-network, thus, (e) holds. We show that (f) $\Leftrightarrow$ (g) holds. Since M is metric, (f) $\Rightarrow$(g) holds. Let (g) hold, and let M have a point-countable open cover $\mathcal{W}$ satisfying (*). Let $\mathcal{B}$ be a σ-locally finite base for M. For each $W \in \mathcal{W}$, choose $B_W \in \mathcal{B}$, with $x(W) \in B_W \subset W$. since $\mathcal{W}$ is point-countable, for each $B_W \in \mathcal{B}, \{B_{W'} : B_W = B_{W'}\}$ is countable. This shows that $\{B_W : W \in \mathcal{W}\}$ is a σ-locally finite open cover of X satisfying (*). Then, M is the countable union of closed discrete subsets. Thus, (f) holds. For (g) $\Rightarrow$ (h), let $\mathcal{B}$ be a base for A. Then, A has a dense subset D with $|D| \leqq |\mathcal{B}|$. But, A has a point-countable open cover $\mathcal{V}$ satisfying (*). Thus, $|\mathcal{V}| \leqq |D|$. Hence, $|A| \leqq |\mathcal{B}|$, thus, $|A| \leqq \omega(A)$. But, since A is metric, $|A| \geqq \omega(A)$. Hence $|A| = \omega(A)$.

A space X is *strongly* Fréchet [28] (= countably bi-sequential in the sense of [22]), if whenever $\{A_n : n \in N\}$ is a decreasing sequence of subsets of X such that $\text{cl}A_n \ni x$ for each $n \in N$, there exists a sequence $\{x_n : n \in N\}$ coverging to the point x with $x_n \in A_n$. A space X is *an inner-closed A-space* [23] (or [24]). if whenever $\{A_n : n \in N\}$ is a decreasing seqence of subsets of X such that $\text{cl}(A_n - \{x\}) \ni x$ for each $n \in N$, there exist $B_n \subset A_n$ which are closed in X, but $\cup\{B_n : n \in N\}$ is not closed in X. Every first countable space is strongly Fréchet, and every strongly Fréchet space is Fréchet. Every strongly Fréchet space, more generally, every countable bi-quasi-k-space in the sense of [22] is inner-closed A.

We recall canonical quotient spaces S_ω, S_{ω_1}, and S_2. S_ω is called the *sequential fan*, and S_2 is the *Arens' space.* S_ω, S_{ω_1}

is respectively the space obtained from the topological sum of $\omega : \omega_1$ many convergent sequences by identifying all limit points to a single point ∞. For the space S_2, see Example 13. We note that neither S_ω nor S_2 is an inner-closed A-space.

In [10], it is proved that a space X with a σ-HCP k-network is an $\aleph$-space if and only if X contains no closed copy of S_{ω_1}. For a k-space with a σ-compact-finite k-network, the following holds.

Theorem 14 *Let X be a k-space with a σ-compact-finite k-network. Then the following are equivalent.*

(a) X contains no closed copy of S_{ω_1}.

(b) X has a point-countable cs-network.*

(c) X is the quotient s-image of a metric space.

Proof: Since X is sequential, the equivalence between (b) and (c) holds by [32; Theorem 2.3]. The implication (b) $\Rightarrow$ (a) holds, because S_{ω_1} has no point-countable cs*-networks by [32; Lemma 2.4]. For the implication (a) $\Rightarrow$ (b), let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$ be a σ-compact-finite k-network for X. Let $\mathcal{P}^* = \{S(P) : P \in \mathcal{P}\}$, where $S(P)$ is the set of all limit points of sequences in P. Then $\mathcal{P}^*$ is point-countable. Otherwise, since $\mathcal{P}$ is a σ-compact-finite cover, for some point $x \in X$, and some $\mathcal{P}_n$, $\mathcal{P}_n$ contains uncountable many elements P_α such that each P_α contains an infinite sequence L_α converging to the point x, here the sequences L_α are disjoint. Then, the space $S = \cup\{L_\alpha : \alpha\} \cup \{x\}$ is a closed copy of S_{ω_1}, because X is a k-space, and $\mathcal{P}_n$ is compact-finite. Thus, X contains a closed copy S of S_{ω_1}. This is a contradiction. Thus, $\mathcal{P}^*$ is point-countable. Next, to show $\mathcal{P}^*$ is a cs*-network, let L be a sequence converging to a point y, and let U be a nbd of y. Let V be a nbd of y with cl$V \subset U$. Since $\mathcal{P}$ is a k-network, there exists $P_0 \in \mathcal{P}$ such that $P_0 \subset V$, and P_0 contains L frequently. Hence, $S(P_0) \subset U$, and $S(P_0)$ contains the point y, and contains L frequently. This show that $\mathcal{P}^*$ is a cs*-network for X. Thus, $\mathcal{P}^*$ is a point-countable cs*-network for X.

Lemma 15 ([15]). *Every strongly Frécht space with a σ-compact-finite k-network is metric.*

Lemma 16 ([30]). *Let X be a sequential space with G_δ points. If X contains no closed copy of S_ω and no S_2 (resp. no S_2), then X is strongly Fréchet (resp. Frécht).*

Not every paracompact space with a σ-disjoint base is metric; see [3]. Thus, not every first countable space with a σ-point-finite k-network is metric. But, for spaces with a σ-compact-finite k-network, the following metrization theorem holds. In particular, under (CH), every k-space with a σ-compact-finite k-network is metric if it contains no closed copy of S_ω, and no S_2. This gives an affirmative answer to the parenthetic part of Question 3.2 in [20] under (CH).

Theorem 17 *Let X be a k-space with a σ-compact-finite k-network. Then the following are equivalent. When X is a space with G_δ points, a meta-Lindelöf space, or (CH) holds, it is possible to omit "$\chi(X) \leqq \omega_1$" in (b).*

(a) X is metric.

(b) $\chi(X) \leqq \omega_1$, and X contains no closed copy of S_ω, and no S_2.

(c) X is an inner closed A-space.

Proof: (a) $\Rightarrow$ (b) & (c) is obvious. For (b) $\Rightarrow$ (a) suppose (b) holds. Then, X has a σ-locally countable k-network by Theorem 6. Thus, each point of X is a G_δ-set in X. But, X is a sequential space which contains no closed copy of S_ω, and no S_2. Thus, X is strongly Fréchet by Lemma 16. Then X is metric by Lemma 15. For (c) $\Rightarrow$ (a), note that X is a k-space with a point-countable k-network. Then X is first countable by [20; Theorem 1.16]. Thus, X is metric. For the latter part of the theorem, let C be a countable subset of X, and let $D = \mathrm{cl}C$. We note that every separable meta-Lindelöf space is Lindelöf, thus ω_1-compact. Then, if X is meta-Lindelöf, or (CH) holds, D is an $\aleph_0$-space in view of Lemma 4 and Theorem 6. Thus, D is a space with G_δ points. But, D is a k-space, thus sequential, and D contains no closed copy of S_ω, and no S_2. Thus, D is strongly Fréchet by Lemma 16. Then, C is strongly Fréchet. Thus, any countable subset of X is strongly Fréchet. But, X has countable tightness by Lemma 1(2). Thus, X is strongly Fréchet by [22; Propositions 8.5 & 8.7]. Thus, X is metric by Lemma 15.

Corollary 18 (CH) *Let X and Y have σ-compact-finite k-networks. For $Z \subset X \times Y$, Z is metric if and only if Z is a k-space which contains no closed copy of S_ω, and no S_2. In particular, if X is a Lašnev space or a CW-compaex, and so is Y, then it is possible to omit (CH).*

Remark 19 For $Z = X \times Y$, where X and Y have σ-compact-finite k-network, let us consider the k-ness of X. In [20], the authors show that a necessary and sufficient condition for the product of two k-spaces with a compact-countable k-network to be a k-space is independent of the usual axiom of set theory. As an application of this, the following holds by [20; Theorem 2.4], Lemma 15, and the fact that the product of two k_ω-spaces is a k_ω-space [21].

(CH). Let X and Y be k-spaces with a σ-compact-finite k-network. For $Z = X \times Y$, Z is a k-space if and only if X or Y is a locally compact metric space; otherwise, Z is a metric space, or a locally k_ω (equivalently, topological sum of k_ω-and -$\aleph_0$-spaces in view of the proof of Theorem 6). If $X = Y$, it is possible to omit (CH).

In [17], it is proved that a k-space X with a σ-HCP k-network is g-first countable if and only if X contains no closed copy of S_ω. The authors don't know whether the result remains true if we replace "σ-HCP" by " σ-compact-finite". But,the following holds. For the definition of g-first countable spaces, see Example 13.

Theorem 20 *Let X be a k-space. If (a), (b), (c), or (d) holds, then, X is g-first countable (resp. Lašnev) if and only if X contains no closed copy of S_ω (resp. S_2).*

(a) X has a star-countable k-network.

(b) X has a σ-HCP k-network; more generally,

(c) X has a σ-compact-finite k-network, and each point is a G_δ-set in X.

(d) (CH) X has a σ-compact-finite k-network.

Proof: The "only if" part is obvious, so we prove the "if" part holds. For (a), let $\mathcal{P}$ be a star-countable k-network for X. For $x \in X$, let $\mathcal{P}_x = \{P \in \mathcal{P} : P$ contains a sequence converging to $x\}$. X is sequential, and $\mathcal{P}$ is a star-countable k-network, then X is a disjoint union of X_α's, where each X_α is a countable union of elements of $\mathcal{P}$, and, for each finite subset F_α of X_α, $\cup\{F_\alpha : \alpha\}$ is closed discrete in X ([20, 26]). But, X contains no closed copy of S_ω, then $\mathcal{P}_x$ is countable. Let P_x =cl($\cup\mathcal{P}_x$). Then, $\mathcal{P}_x$ is separable, so P_x is an $\aleph_0$-space by Theorem 7(1). Thus P_x is g-first countable by [17]. Take a local weak base T_x at x in P_x such that T_x is countable. Then, for any sequence L converging to $x \in X$, and any $T \in T_x$, L is contained in T eventually. Let $U \subset X$, and for each $x \in U$, let $x \in T \subset U$ for some $T \in T_x$. Then U is open in X, for X is sequential. Then $\cup\{T_x : x \in X\}$ is a weak base for X. Thus, X is g-first countable. For (c), since X contains no closed copy of S_ω, X has a point-countable cs*-network by Theorem 14. Thus, X is g-first countable in view of the proof of Theorem 1 in [17]. For (d), under (CH), every closed separable subset F

of X is an $\aleph_0$-space by Theorem 7(1), hence each point of F is a G_δ-set in F. Thus, X is also g-first countable in view of the proof of Theorem 1 in [17]. For the parenthetic part, the result for (c) holds by Lemma 16 and Remark 5(1). For (d), since X contains no closed copy of S_2, X is Fréchet by the same way as in the proof of Theorem 17, thus, X is Lašnev. For (a), the proof is similar to (d), but use Theorems 3 and 7(1).

In conclusion of this paper, we shall pose a question on spaces with a σ-compact-finite k-network.

Question 21 Let X be a separable k-space with a σ-compact-finite k-network $\mathcal{P}$. Then, is X an $\aleph_0$-space, a σ-space, or a space with G_δ points?

Remark 22 We shall give the following comments related to Question 21.

(1) The space X is an $\aleph_0$-space when (CH) holds (Theorem 6); $\mathcal{P}$ is star-countable ([25]); or $\mathcal{P}$ contsits of closed subsets (because, by Remark 5(2), X is an $\aleph$-space, so X is meta-Lindelöf [5], then X is Lindelöf, hence X is an $\aleph_0$-space).

(2) If Question 21 is afirmative, then Corollary 18 remains valid without (CH), for example.

(3) Every k-space Y is meta-Lindelöf if Y has a star-countable k-network ([20]), or a σ-compact-finite k-network of closed subsets (see(1)). (For case where Y has a star-countable k-network, Y is actually a paracompact σ-space ([26])). But, the k-ness of Y is essential. Indeed, in view of [25], every space with a star-countable, compact-finite, and locally countable closed k-network is not a σ-space, thus meta-Lindelöf by [9; Proposition 1.5]. Also, every space with a star-countable and compact-finite closed k-network is not a space with G_δ points [25]. In terms of these, we have the following (more general) questions related to Question 21.

Questions: (i) Is every k-space with a σ-compact-finite (or σ-HCP) k-network a meta-Lindelöf space?

(ii) Is every k-space with a σ-compact-finite (or star-countable) k-network a σ-space, or a space with G_δ points?

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