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Hereditarily Monotone Mappings onto S^1

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Abstract

By modifying Knaster's continuum of $V's$ and $\Lambda's$ there is produced a colocally connected continuum that is not a simple closed curve but which admits an hereditarily monotone mapping onto a simple closed curve.

1 Introduction and Definitions

An *hereditarily weakly confluent mapping* (see [1]) between continua X and S is a continuous mapping $h : X \to S$ such that for each subcontinuum K of X and each subcontinuum M of $h(K)$ there exists a subcontinuum K_M of K with $h(K_M) = M$. In an unpublished paper of Davis and Nadler it was proved that every arcwise connected, semi-locally connected and cyclic continuum admitting an hereditarily weakly confluent mapping onto S^1 is a simple closed curve, i.e., is homeomorphic to S^1. The authors asked (a question posed by Nadler at the 1994 Joint Meeting of the AMS/MAA in Cincinatti) whether every semi-locally connected and cyclic continuum X admitting an hereditarily weakly confluent mapping onto S^1 is a simple closed curve. The purpose of this paper is to provide a negative answer to the question. The non-planar continuum and mapping constructed are derived from a technique briefly considered by Nadler and Davis; the technique was suggested to this author by Nadler. The given mapping onto S^1 is actually hereditarily monotone.

For most fundamental definitions the reader is referred to [2], [5] and [6]. A *compactum* is a nonempty compact metric space, and a *continuum* is a connected compactum. A continuum K contained in a continuum X is called a *subcontinuum* of X. A continuum X is said to be *colocally connected at* x ($x \in X$) if each open neighborhood of x contains an open neighborhood of x whose complement in X is connected . A continuum is *colocally connected* if it is colocally connected at each of its points. To use the terminology of [8], a continuum is colocally connected if and only if it is semi-locally connected and cyclic (see [3] and Lemma 4.14 in Chapter III of [8]). A continuum X is *arcwise connected* if each two of its points can be joined by an arc in X. A continuous mapping h from a continuum X into a continuum S is said to be *monotone* if $h^{-1}(s)$ is connected for each $s \in S$. A continuous mapping $h : X \to S$ is monotone if and only if $h^{-1}(M)$ is a subcontinuum of X whenever M is a subcontinuum of $h(X)$ (see Theorem 9 in Section 46, Chapter I, of [5]). When X is a continuum a continuous mapping h from a continuum X into a continuum S is said to be *hereditarily monotone* if the restriction $h\,|_K$ is monotone for each subcontinuum K of X. Thus, each hereditarily monotone mapping $h : X \to S$ is hereditarily weakly confluent, since we can choose K_M in the definition of hereditarily weakly confluent to be $(h\,|_K)^{-1}(M)$.

When X is any topological space we use $Cl_X(A)$, or just $Cl(A)$ when X is understood by context, to denote the closure of a set $A \subseteq X$. $Bd_X(A)$ and $Int_X(A)$ denote the interior and boundary of A in X, and the subscript X is again frequently omitted. If A and Z are nonempty *separated* subsets of X (i.e., $Cl(A) \cap Z$ and $A \cap Cl(Z)$ are empty), we write $A \cup Z = A\,|Z$. The cardinality of a set A is denoted by $|A|$. Given sets A and

Z, the symbol $A\backslash Z$ denotes the complement of $A \cap Z$ in A, i.e., $A\backslash Z$ is the set of all elements of A that are not elements of Z. The unit circle in the plane is denoted by S^1.

2 A Generalization of Knaster's Continuum

To arrive at the example mentioned above one first constructs generalized circular versions of Knaster's continuum of $V's$ and $\Lambda's$ (see Example 5 in Section 48, Chapter I, of [5]). Throughout we let C denote a Cantor set in S^1, and let $A_1, A_2, A_3, \ldots$ be an ordering of the components of $S^1\backslash C$. For each positive integer j let E_j be the set of endpoints of A_j. Thus $|E_j| = 2$ for each j. Define

$$\mathcal{E}_{-1} = \{E_{2m-1} : 1 \leq m < \infty\} \text{ and } \mathcal{E}_1 = \{E_{2m} : 1 \leq m < \infty\}.$$

$\bigcup \mathcal{E}_{-1}$ and $\bigcup \mathcal{E}_1$ are disjoint subsets of C whose union, which we denote by E, is dense in C,

$$(\bigcup \mathcal{E}_{-1}) \bigcap (\bigcup \mathcal{E}_1) = \emptyset \quad ; \quad E = \bigcup_{n=1}^{\infty} E_n\,.$$

Now let Z be an arbitrary continuum. A quotient continuum, $\mathcal{Z}_{\mathcal{F}} = \mathcal{Z}_{\mathcal{F}}(\mathcal{E}_{-1}, \mathcal{E}_1)$, will be defined by identifying certain pairs of points of $Z \times C$, the particular pairs identified depending upon a choice of compacta $F(-1)$ and $F(1)$ in Z. After the definition is given, we describe conditions under which $\mathcal{Z}_{\mathcal{F}}$ is colocally connected, define a natural map from $\mathcal{Z}_{\mathcal{F}}$ onto S^1, and cite a sufficient set of conditions for this map to be hereditarily monotone. The example described in the previous section is then constructed. The general method used to construct $\mathcal{Z}_{\mathcal{F}}$ is not new; see, e.g., Example 2.1 in [7].

Define a mapping π from C into $\{E \subseteq C : |E| = 1 \text{ or } 2\}$ by

$$\pi(c) = \begin{cases} E_n \text{ if } c \in E_n \text{ for some } n \in \{1, 2, 3, \ldots\} \\ \{c\} \text{ otherwise.} \end{cases}$$

Let $T(\pi(C)) = \{\mathcal{U} \subseteq \pi(C) : \bigcup \mathcal{U}$ is open in $C\}$. Then $T(\pi(C))$ is a quotient topology for the collection $\pi(C)$. When $\pi(C)$ carries this topology, which we assume henceforth, it is a simple closed curve, and π is continuous with respect to this topology.

Let $\mathcal{F} = \{F(-1), F(1)\}$ be a pair of compacta in Z. Define a mapping $\Pi_{\mathcal{F}}$ from $Z \times C$ into $\{S \subseteq Z \times C : |S| = 1 \text{ or } 2\}$ by

$$\Pi_{\mathcal{F}}(z,c) = \begin{cases} \{z\} \times E_n \text{ if } (z,c) \in F((-1)^n) \times E_n \text{ for some } n \\ \{(z,c)\} \text{ otherwise.} \end{cases}$$

$\Pi_{\mathcal{F}}$ is well-defined since the sets $E_1, E_2, E_3, \ldots$ are pairwise disjoint. Let $\mathcal{D} = \Pi_{\mathcal{F}}(Z \times C)$ and let $T(\mathcal{D}) = \{\mathcal{U} \subseteq \mathcal{D} : \bigcup \mathcal{U}$ is open in $Z \times C\}$. By Definition 3.1 and subsequent comments in [6], the space $(\mathcal{D}, T(\mathcal{D}))$ is a decomposition of $Z \times C$, and $\Pi_{\mathcal{F}}$ is a continuous map from $Z \times C$ onto $(\mathcal{D}, T(\mathcal{D}))$. Let

$$\mathcal{Z}_{\mathcal{F}} = \mathcal{Z}_{\mathcal{F}}[\mathcal{E}_{-1}, \mathcal{E}_1] = (\mathcal{D}, T(\mathcal{D})), \text{ or, in other words,}$$

$$\mathcal{Z}_{\mathcal{F}} = \Pi_{\mathcal{F}}(Z \times C).$$

Then $\mathcal{Z}_{\mathcal{F}}$ is compact, since $\Pi_{\mathcal{F}}$ is continuous. (If $Z = [0,1]$, $F(1) = \{1\}$ and $F(-1) = \{0\}$, $\mathcal{Z}_{\mathcal{F}}$ is a simple circular version of Knaster's continuum.)

Some collections of open arcs in $S^1$are now defined. For each $c \in C$, let

$$\mathcal{I}(c,1) = \{(ab) \subseteq S^1 : C \not\subseteq (ab) \supseteq \pi(c) \; ; \; a, b \in \textstyle\bigcup_{j=1}^{\infty} A_{2j}\}$$
$$\mathcal{I}(c,-1) = \{(ab) \subseteq S^1 : C \not\subseteq (ab) \supseteq \pi(c) \; ; \; a, b \in \textstyle\bigcup_{j=1}^{\infty} A_{2j-1}\}.$$

As in [8], we say that a set $S \subseteq Z \times C$ is an *inverse set* of $\Pi_{\mathcal{F}}$ provided $\Pi_{\mathcal{F}}^{-1}(\Pi_{\mathcal{F}}(S)) = S$. Equivalently, S is an inverse set of $\Pi_{\mathcal{F}}$ if and only if $\bigcup \Pi_{\mathcal{F}}(S) = S$. This is the same as saying that S is $\mathcal{D}$-saturated, as in [6]. Thus, because $\bigcup \Pi_{\mathcal{F}}(S) = S$ for each inverse set S of $\Pi_{\mathcal{F}}$, we see from the definition of the decomposition topology:

$$\text{If } S \text{ is an open inverse set of } \Pi_{\mathcal{F}} \text{ then } \Pi_{\mathcal{F}}(S) \text{ is open in } \mathcal{Z}_{\mathcal{F}}. \tag{1}$$

Observe also that, by the definition of $\Pi_{\mathcal{F}}$ and by the placement of the sets E_n in C,

$$\text{If } (ab) \text{ is an open arc in } S^1 \text{ with } a, b \in C \setminus E \text{ and } R \subseteq Z, \text{ then } R \times [(ab) \cap C] \text{ is an inverse set of } \Pi_{\mathcal{F}}. \tag{2}$$

We show the following:

$$\text{If } c_0 \in C,\ (ab) \in \mathcal{I}(c_0, 1), \text{ and } R \subseteq Z \text{ with } R \cap F(1) = \emptyset, \text{ then } R \times [(ab) \cap C] \text{ is an inverse set of } \Pi_{\mathcal{F}}. \tag{3}$$

Clearly, $\Pi_{\mathcal{F}}^{-1}(\Pi_{\mathcal{F}}(R \times [(ab) \cap C])) \supseteq R \times [(ab) \cap C]$. For the reverse containment, suppose $(z_1, c_1) \in \Pi_{\mathcal{F}}^{-1}(\Pi_{\mathcal{F}}(z, c))$ for some $(z, c) \in R \times [(ab) \cap C]$. We want to show that $(z_1, c_1) \in R \times [(ab) \cap C]$. Since $\Pi_{\mathcal{F}}(z, c) = \Pi_{\mathcal{F}}(z_1, c_1)$,

$$z = z_1 \in R.$$

Also, as $(ab) \in \mathcal{I}(c_0, 1)$, there exist *distinct* positive integers j and k so that $a \in A_{2j}$ and $b \in A_{2k}$. Since $(ab) \in \mathcal{I}(c_0, 1)$, we have $(ab) \cap C \neq \emptyset$. Then, since $a \in A_{2j}$, $(ab) \cap E_{2j} \neq \emptyset$. Similarly, $(ab) \cap E_{2k} \neq \emptyset$. Moreover, as $a \in A_{2j}$, $b \in A_{2k}$, and $j \neq k$, we have $|(ab) \cap E_{2j}| = 1 = |(ab) \cap E_{2k}|$. Without loss of generality, we can assume

$$(ab) \cap E_{2j} = \{e'_j\} \text{ and } (ab) \cap E_{2k} = \{e'_k\}, \tag{4}$$

where $E_{2j} = \{e_j, e'_j\}$ and $E_{2k} = \{e_k, e'_k\}$. Now, arguing by contradiction, suppose that $(z_1, c_1) \notin R \times [(ab) \cap C]$. Then $(z, c) \neq (z_1, c_1) = (z, c_1)$. Hence $c \neq c_1$. Since $(z_1, c_1) \notin R \times [(ab) \cap C]$ and $z_1 \in R$ we must have $c_1 \notin (ab) \cap C$. But as

$(z_1, c_1) \in \Pi_{\mathcal{F}}^{-1}(\Pi_{\mathcal{F}}(z,c)) \subseteq \{z\} \times C$, we have $c_1 \in C$. Hence $c_1 \notin (ab)$. Then

$$c_1 \notin (ab) \text{ and } c \in (ab).$$

By choice of (z_1, c_1), $\Pi_{\mathcal{F}}(z,c) = \Pi_{\mathcal{F}}(z_1,c_1) = \Pi_{\mathcal{F}}(z,c_1)$. Thus $\pi(c) = \pi(c_1)$. Therefore, $\{c, c_1\} \in \mathcal{E}_{-1} \cup \mathcal{E}_1$. We claim that either $\{c, c_1\} = \{e_j,\, e_j'\}$ or $\{c, c_1\} = \{e_k,\, e_k'\}$. For otherwise the open arcs A_{2j}, A_{2k}, and one of the two open arcs with endpoints c and c_1 are three distinct (and hence disjoint) components of $S^1 \backslash C$. We can denote these open arcs by $(e_j e_j')$, $(e_k e_k')$ and (cc_1). We have $e_j,\, e_k,\, c_1 \notin (ab)$ and $e_j',\, e_k',\, c \in (ab)$. However, as the three component arcs are disjoint, this is impossible. Thus, either $\{c, c_1\} = \{e_j,\, e_j'\}$ or $\{c, c_1\} = \{e_k,\, e_k'\}$. Now if $\{c, c_1\} = \{e_j,\, e_j'\}$ then, since $\pi(c) = \pi(c_1)$, we have $\pi(e_j') = \pi(e_j)$; thus

$$\Pi_{\mathcal{F}}(z, e_j') = \Pi_{\mathcal{F}}(z, e_j).$$

However, this contradicts the definition of $\Pi_{\mathcal{F}}$, since, by hypothesis, $z \in R \subseteq Z \backslash F(1) = Z \backslash F((-1)^{2j})$. A similar contradiction is reached if $\{c, c_1\} = \{e_k,\, e_k'\}$. This establishes (3). By a symmetric argument we also have:

$$\begin{aligned}&\text{If } c_0 \in C,\ (ab) \in \mathcal{I}(c_0, -1), \text{ and } R \subseteq Z \text{with } R \cap F(-1) = \emptyset,\\ &\text{then } R \times [(ab) \cap C] \text{ is an inverse set of } \Pi_{\mathcal{F}}.\end{aligned} \tag{5}$$

Define

$$\begin{aligned}\mathcal{I}'(1) &= \{(ab) \subseteq S^1 :\ a, b \in (C \backslash E) \cup \textstyle\bigcup_{j=1}^{\infty} A_{2j}\}\\ \mathcal{I}'(-1) &= \{(ab) \subseteq S^1 :\ a, b \in (C \backslash E) \cup \textstyle\bigcup_{j=1}^{\infty} A_{2j-1}\}.\end{aligned}$$

In a manner similar to the proof of (3), one can show that

$$\begin{aligned}&\text{If } i \in \{-1, 1\},\ (ab) \in \mathcal{I}'(i), \text{ and } R \subseteq Z \text{ with}\\ &R \cap F(i) = \emptyset, \text{ then } R \times [(ab) \cap C] \text{ is an inverse set of } \Pi_{\mathcal{F}}.\end{aligned} \tag{6}$$

There is also the following lemma.

Lemma 1 *If* $(z,c) \in Z \times C$ *and* $\Pi_{\mathcal{F}}(z,c) \subseteq U$, *where* U *is open in* $Z \times C$, *then there is an open arc* $(ab) \subseteq S^1$ *with* $c \in (ab) \not\supseteq C$ *and an open* $O \subseteq Z$ *with* $z \in O$ *such that for each subset* L *of* O *with* $z \in L$ *one has*

$$\begin{aligned} &(A) \quad \Pi_{\mathcal{F}}(z,c) \subseteq L \times [(ab) \cap C] \subseteq U, \text{ and} \\ &(B) \quad L \times [(ab) \cap C] \text{ is an inverse set of } \Pi_{\mathcal{F}}. \end{aligned}$$

Proof: We consider first the case when $\pi(c) = \{c\}$. Here, $\Pi_{\mathcal{F}}(z,c) = \{(z,c)\}$. As $C \backslash E$ is dense in C, there exists an open arc (ab) containing c with $a, b \in C \backslash E$ and an open neighborhood O of z in Z such that the basic open set $O \times [(ab) \cap C]$ is contained in U. Observe that $(ab) \not\supseteq C$, and if L is any subset of O containing z then (A) holds. Also, (B) holds for every subset L of O with $z \in L$ since, by (2), (B) holds for every $L \subseteq Z$.

Suppose next that $\pi(c) = \{c, c'\} = E_n$ for some n and $\Pi_{\mathcal{F}}(z,c) = \{(z,c)\}$. This means that $z \notin F((-1)^n)$. The set U contains a basic open neighborhood $O_0 \times [(ab) \cap C]$ of (z,c) where $(ab) \cap E_n = \{c\}$ and $b \in C \backslash E$. Note that, since $b \in C \backslash E$, $(ab) \not\supseteq C$. Let $O = O_0 \cap (Z \backslash F((-1)^n))$. Then, since the compactum $F((-1)^n)$ does not contain z, O is an open neighborhood of z, and $(z,c) \in O \times [(ab) \cap C] \subseteq U$. By (6), (B) holds for any $L \subseteq O$. Moreover, as $O \subseteq O_0$, (A) holds for any subset L of O with $z \in L$, since then $L \times [(ab) \cap C] \subseteq O_0 \times [(ab) \cap C] \subseteq U$.

Finally, assume that $\pi(c) = \{c, c'\} = E_n$ for some n, and that $\Pi_{\mathcal{F}}(z,c) = \{z\} \times \{c, c'\}$. The set U contains a basic open neighborhood of (z,c) of the form $O_0 \times [(ac') \cap C]$, where $a \in C \backslash E$. Likewise, U contains a basic open neighborhood of (z, c') of the form $O_1 \times [(bc) \cap C]$, where $b \in C \backslash E$. We can also select b

and a so that the open arc $(ab) = (ac') \cup (bc)$ does not contain C. Let $O = O_0 \cap O_1$. Then, if $z \in L \subseteq O$,

$$\Pi_{\mathcal{F}}(z,c) = \{z\} \times \{c,c'\} \subseteq L \times [(ab) \cap C] = (L \times [(ac') \cap C]) \cup (L \times [(bc) \cap C]) \subseteq (O_0 \times [(ac') \cap C]) \cup (O_1 \times [(bc) \cap C]) \subseteq U.$$

Thus (A) holds whenever $L \subseteq O$ and $z \in L$. Also, (B) holds for every subset L of O with $z \in L$ since, by (2), (B) holds for every $L \subseteq Z$. □

By Lemma 1 and by (3) of Proposition 3.7 in [6], it follows that $\mathcal{D}$ is an upper semicontinuous decomposition of $Z \times C$. Therefore, by Theorem 3.9 in [6], $\mathcal{Z}_{\mathcal{F}} = \Pi_{\mathcal{F}}(Z \times C)$ is a compact metric space. We will now prove:

Lemma 2 *If Y is a connected subset of Z with $Y \cap F(-1) \neq \emptyset \neq Y \cap F(1)$, and if K is a connected subset of S^1 with $K \cap C \neq \emptyset$, then $\Pi_{\mathcal{F}}(Y \times (K \cap C))$ is connected.*

Proof: For suppose $\Pi_{\mathcal{F}}(Y \times (K \cap C))$ is not connected. Then, because $\Pi_{\mathcal{F}}(Y \times \{c\})$ is connected for every $c \in C$ (by the continuity of $\Pi_{\mathcal{F}}$), we have

$$\Pi_{\mathcal{F}}(Y \times (K \cap C)) = \Pi_{\mathcal{F}}(Y \times C_1) \,|\, \Pi_{\mathcal{F}}(Y \times C_2), \qquad (7)$$

for some sets $C_1, C_2 \subseteq C$ with $K \cap C = C_1 \cup C_2$. Then, as $\Pi_{\mathcal{F}}$ is continuous,

$$Y \times (K \cap C) = (Y \times C_1) \,|\, (Y \times C_2).$$

Therefore, by the continuity of the projection from $Z \times C$ onto C,

$$K \cap C = C_1 \,|\, C_2. \qquad (8)$$

Now, K is connected implies there exists a closed arc $A = \{e^{it} : a \leq t \leq b\} \subseteq K$, where $0 \leq a < b < 2\pi$ and either $e^{ia} \in C_1$, $e^{ib} \in C_2$ or $e^{ia} \in C_2$, $e^{ib} \in C_1$. We assume the former case, without loss of generality. Note that

$$A \cap C = (A \cap C_1) \,|\, (A \cap C_2). \qquad (9)$$

Let $s(2) = \inf\{s \in [a,b] : e^{is} \in C_2\}$. Then $a \leq s(2) \leq b$ and $e^{is(2)} \in C_2$. Let $s(1) = \sup\{s \in [a, s(2)] : e^{is} \in C_1\}$. Then $a \leq s(1) < s(2)$ and $e^{is(1)} \in C_1$. Furthermore, the closed arc $A' = \{e^{is} : s(1) \leq s \leq s(2)\}$ has the property that $A \cap C_1 = \{e^{is(1)}\}$ and $A \cap C_2 = \{e^{is(2)}\}$. Hence, by (9), $e^{is} \notin C$ for $s(1) < s < s(2)$. Thus $\{e^{is(1)}, e^{is(2)}\} = E_n$ for some $n \in \{1, 2, 3, ...\}$. Suppose first that this n is even. Since $\emptyset \neq Y \cap F(1)$ there exists $y \in Y \cap F(1)$. Then $\{y\} \times E_n \subseteq F(1) \times E_n = F((-1)^n) \times E_n$, so that $\Pi_{\mathcal{F}}(y, e^{is(1)}) = \{y\} \times E_n = \Pi_{\mathcal{F}}(y, e^{is(2)})$. Thus, $\Pi_{\mathcal{F}}(Y \times C_1) \cap \Pi_{\mathcal{F}}(Y \times C_2) \neq \emptyset$. But this contradicts (7). Similarly if n is odd, by using the hypothesis that $Y \cap F(-1) \neq \emptyset$, we again arrive at a contradiction to (7). This establishes Lemma 2.□

It follows from Lemma 2 with $Y = Z$ and $K = S^1$ that $\Pi_{\mathcal{F}}(Z \times (S^1 \cap C)) = \Pi_{\mathcal{F}}(Z \times C) = \mathcal{Z}_{\mathcal{F}}$ is connected. Therefore, as we have already seen that $\mathcal{Z}_{\mathcal{F}}$ is a compact metric space (and is clearly nonempty), $\mathcal{Z}_{\mathcal{F}}$ is a continuum.

Now if Z is degenerate, i.e., $Z = \{p\} = F(-1) = F(1)$, then $\Pi_{\mathcal{F}}(Z \times C)$ is homeomorphic to $\pi(C)$. This continuum, $\pi(C)$, has the property that if $c, c' \in C$ and $\pi(c) \neq \pi(c')$, then $C \backslash (\pi(c) \cup \pi(c'))$ is the union of two disjoint open inverse sets of $\Pi_{\mathcal{F}}$, O_1 and O_2. Thus, by (1), $\pi(C) \backslash \{\pi(c), \pi(c')\} = \pi(O_1) \,|\, \pi(O_2)$.Hence, by Theorem 9.31 in [6], $\pi(C)$ is a simple closed curve.

Lemma 3 *Suppose each $F(i)$ is nondegenerate (i.e., $|F(i)| > 1$ for each $i \in \{-1, 1\}$) and Z is colocally connected. Then $\mathcal{Z}_{\mathcal{F}}$ is colocally connected.*

Proof: Suppose $D = \Pi_{\mathcal{F}}(z, c) \in \mathcal{U} \in T(\mathcal{D})$. We need to find $\mathcal{V} \in T(\mathcal{D})$ such that $\Pi_{\mathcal{F}}(z, c) \in \mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{Z}_{\mathcal{F}} \backslash \mathcal{V}$ is connected. To do so, let $U = \bigcup \mathcal{U}$. Then U is open in $Z \times C$, because $\mathcal{U} \in T(\mathcal{D})$. Since $\Pi_{\mathcal{F}}(z, c) \subseteq \bigcup \mathcal{U} = U$, we can find an open arc $(ab) \subseteq S^1$ and an open $O \subseteq Z$ as in Lemma 1. Since Z is colocally connected and $F(-1) \neq \{z\} \neq F(1)$ (because $F(-1)$

and $F(1)$ are nondegenerate), there is an open $O_Z \subseteq O$ such that $z \in O_Z$ and

$$Z \backslash O_Z \text{ is a subcontinuum of } Z \text{ meeting both } F(-1) \text{ and } F(1). \tag{10}$$

Let $\mathcal{V} = \Pi_{\mathcal{F}}(O_Z \times [(ab) \cap C)])$. Then $\Pi_{\mathcal{F}}(z, c) \in \mathcal{V}$. Also, by (B) of Lemma 1, we have

$$\Pi_{\mathcal{F}}^{-1}(\mathcal{V}) = \Pi_{\mathcal{F}}^{-1}(\Pi_{\mathcal{F}}(O_Z \times [(ab) \cap C])) = O_Z \times [(ab) \cap C]. \tag{11}$$

Hence $O_Z \times [(ab) \cap C]$ is an open inverse set of $\Pi_{\mathcal{F}}$. Thus $\mathcal{V} \in T(\mathcal{D})$ by (1). Furthermore, by (A) of Lemma 1 with $L = O_Z$, $O_Z \times [(ab) \cap C] \subseteq U$. Then $\mathcal{V} = \Pi_{\mathcal{F}}(O_Z \times [(ab) \cap C]) \subseteq \Pi_{\mathcal{F}}(U) = \mathcal{U}$, so that $\Pi_{\mathcal{F}}(z, c) \in \mathcal{V} \subseteq \mathcal{U}$. Now it remains only to show that $\mathcal{Z}_{\mathcal{F}} \backslash \mathcal{V}$ is connected. Notice first that the set $Z \backslash O_Z$ is connected, and that $(ab) \cap C$ and $C \backslash (ab)$ are nonempty, since (ab) was obtained via Lemma 1. Thus, by Lemma 2 and (10), the sets $M \equiv \Pi_{\mathcal{F}}((Z \backslash O_Z) \times C)$ and $N \equiv \Pi_{\mathcal{F}}(Z \times (C \backslash (ab)))$ are subcontinua of $\mathcal{Z}_{\mathcal{F}}$. Note too that

$$(Z \times C) \backslash (O_Z \times [(ab) \cap C]) = [Z \times (C \backslash (ab))] \cup [(Z \backslash O_Z) \times C]. \tag{12}$$

Also, $\mathcal{Z}_{\mathcal{F}} \backslash \mathcal{V} = \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(O_Z \times [(ab) \cap C]) = \Pi_{\mathcal{F}}((Z \times C) \backslash (O_Z \times [(ab) \cap C])$, where the second equality follows from (11). So, by (12),

$$\mathcal{Z}_{\mathcal{F}} \backslash \mathcal{V} = \Pi_{\mathcal{F}}(Z \times (C \backslash (ab))) \cup \Pi_{\mathcal{F}}((Z \backslash O_Z) \times C) = N \cup M.$$

Choose $z \in Z \backslash O_Z$. Then, as $a \in C \backslash E$, we have $(z, a) \in [Z \times (C \backslash (ab))] \cap [(Z \backslash O_Z) \times C]$. Hence $\Pi_{\mathcal{F}}(z, a) \in N \cap M$. Thus, because $\mathcal{Z}_{\mathcal{F}} \backslash \mathcal{V}$ is the union of the intersecting continua M and N, $\mathcal{Z}_{\mathcal{F}} \backslash \mathcal{V}$ is a subcontinuum of $\mathcal{Z}_{\mathcal{F}}$. This completes the proof of Lemma 3.$\square$

We let π_1 denote the continuous mapping $\pi \circ p$, where p is the projection of $Z \times C$ onto C, i.e., $p(z, c) = c$. Define a mapping $\Phi_{\mathcal{F}} : \mathcal{Z}_{\mathcal{F}} \to \pi(C)$ by $\Phi_{\mathcal{F}}(\Pi_{\mathcal{F}}(z,c)) = \pi_1(z,c)$, for $(z,c) \in Z \times C$. Then $\Phi_{\mathcal{F}}$ is well-defined (single-valued), for if $\Pi_{\mathcal{F}}(z,c) = \Pi_{\mathcal{F}}(z',c')$ then $\pi(c) = \pi(c')$, and hence $\pi_1(z,c) = \pi_1(z',c')$. The mapping $\Phi_{\mathcal{F}} = \pi_1 \circ \Pi_{\mathcal{F}}^{-1}$ is continuous by the Transgression Lemma (page 45 of [6]).

Lemma 4 *Suppose the ordering $A_1, A_2, A_3, \ldots$ of the components of $S^1 \backslash C$ is such that both $\bigcup \mathcal{E}_{-1}$ and $\bigcup \mathcal{E}_1$ are dense subsets of C. Assume that the compacta $F(-1)$ and $F(1)$ are disjoint. Suppose also that (*) if R and R' are subcontinua of Z each of which intersects both $F(1)$ and $F(-1)$, then $R \cap R' \cap F(1) \neq \emptyset \neq R \cap R' \cap F(-1)$. Then $\Phi_{\mathcal{F}}$ is an hereditarily monotone mapping from $\mathcal{Z}_{\mathcal{F}}$ onto $\pi(C)$.*

Proof: Since π_1 maps $Z \times C$ onto C, the continuous $\Phi_{\mathcal{F}}$ maps $\mathcal{Z}_{\mathcal{F}}$ onto $\pi(C)$. Let K be a subcontinuum of $\mathcal{Z}_{\mathcal{F}}$ and let φ denote the restriction of $\Phi_{\mathcal{F}}$ to K. We must show that φ is monotone. Clearly, φ is continuous and $\varphi(K)$ is a subcontinuum of $\pi(C)$. If $\varphi(K)$ is degenerate then φ is certainly monotone. Hence, it can be assumed that

$$\varphi(K) \text{ is a nondegenerate subcontinuum of } \pi(C). \tag{13}$$

To show that φ is monotone the following equality, a consequence of the definition of the surjective map $\Pi_{\mathcal{F}}$, will be freely used.

$$\mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(T \times C) = \Pi_{\mathcal{F}}([Z \backslash T] \times C \text{ for all } T \subseteq Z.$$

We aim to establish assertions (14), (21), (22), (24), and (29) below.

If $i \in \{-1, 1\}$, S is a component of $K \cap (\mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C))$ and $\Pi_{\mathcal{F}}(z,c) \in S$ for some $(z,c) \in Z \times C$, then S is a component of $K \cap (\Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c)))$. (14)

Here we first note that

$$z \in Z \backslash F(i). \tag{15}$$

Observe too that

$$\mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c)) =$$

$$\Pi_{\mathcal{F}}(F(i) \times C) \cup \Pi_{\mathcal{F}}([Z \backslash F(i)] \times [C \backslash \pi(c)]).$$

Hence, $\Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c)) \subseteq \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C)$. Then, as S is a component of $K \cap (\mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C))$, to prove S is a component of $K \cap (\Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c)))$ it suffices to show $S \subseteq \Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c))$. Thus it suffices to show that if $\Pi_{\mathcal{F}}(z_0, c_0) \in \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c))$ then $\Pi_{\mathcal{F}}(z_0, c_0) \notin S$. Also, as $\Pi_{\mathcal{F}}(F(i) \times C) \cap S = \emptyset$, we can assume $\Pi_{\mathcal{F}}(z_0, c_0) \in \Pi_{\mathcal{F}}([Z \backslash F(i)] \times [C \backslash \pi(c)])$. So $\Pi_{\mathcal{F}}(z_0, c_0) = \Pi_{\mathcal{F}}(z_1, c_1)$ for some $(z_1, c_1) \in [Z \backslash F(i)] \times [C \backslash \pi(c)]$. Hence,

$$z_0 = z_1 \in Z \backslash F(i). \tag{16}$$

Now $\pi(c_0) = \Pi_{\mathcal{F}}(z_0, c_0) = \Pi_{\mathcal{F}}(z_1, c_1) = \pi(c_1) \neq \pi(c)$. Thus $\pi(c_0) \cap \pi(c) = \emptyset$. Then, as $\bigcup \mathcal{E}_i$ is dense in C by hypothesis, there exists $(ab) \in \mathcal{I}(c, i)$ with

$$\pi(c_0) \subseteq C \backslash (ab). \tag{17}$$

We have

$$c \in \pi(c) \subseteq (ab) \text{ and } a, b \notin C. \tag{18}$$

Let $U_i = [Z \backslash F(i)] \times [(ab) \cap C]$ and $V_i = [Z \backslash F(i)] \times [C \backslash (ab)]$. Then, by (3) or (5), U_i and V_i are open inverse sets of $\Pi_{\mathcal{F}}$ whose union is $[Z \backslash F(i)] \times C$. Also, $U_i \cap V_i = \emptyset$, so from (1) and the initial hypothesis on S in (14) we have

$$S \subseteq \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C) = \Pi_{\mathcal{F}}([Z \backslash F(i)] \times C) = \Pi_{\mathcal{F}}(U_i) \,|\, \Pi_{\mathcal{F}}(V_i). \tag{19}$$

Then by (15) and (18) we have $(z,c) \in U_i$. So $\Pi_{\mathcal{F}}(z,c) \in S \cap \Pi_{\mathcal{F}}(U_i)$. Thus, as S is connected, it follows from (19) that

$$S \subseteq \Pi_{\mathcal{F}}(U_i) \subseteq \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(V_i). \tag{20}$$

Now, by (16) and (17), $(z_0, c_0) \in V_i$. Hence $\Pi_{\mathcal{F}}(z_0, c_0) \in \Pi_{\mathcal{F}}(V_i)$. Therefore, by (20), $\Pi_{\mathcal{F}}(z_0, c_0) \notin S$. This completes the proof of (14). We now prove the following.

$$\text{If } i \in \{-1,1\}, \text{ then } K \cap \Pi_{\mathcal{F}}(F(i) \times C) \neq \emptyset. \tag{21}$$

Assume $K \cap \Pi_{\mathcal{F}}(F(i) \times C) = \emptyset$. Then $K = K \cap \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C)$, so K is a component of $K \cap \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C)$. Moreover, since $\emptyset \neq K \subseteq \mathcal{Z}_{\mathcal{F}} = \Pi_{\mathcal{F}}(Z \times C)$, $\Pi_{\mathcal{F}}(z,c) \in K$ for some $(z,c) \in Z \times C$. Therefore, by (14) (with $S = K$) we have that K is a component of $K \cap (\Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c)))$. Hence $K \subseteq \Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c))$. Thus, $\varphi(K) = \Phi_{\mathcal{F}}(K) = \{\pi(c)\}$. But this contradicts (13). So (21) holds. We also make this claim:

If $i \in \{-1,1\}$ and S is a component of $K \cap (\mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C))$, then $Cl_K(S) \cap \Pi_{\mathcal{F}}(F(i) \times C) \neq \emptyset$ and S is not compact. (22)

For, since $F(i) \times C$ is compact and $\Pi_{\mathcal{F}}$ is continuous, $\Pi_{\mathcal{F}}(F(i) \times C)$ is compact. Hence, by (21),

$$\Pi_{\mathcal{F}}(F(i) \times C) \text{ is closed in } \mathcal{Z}_{\mathcal{F}} \text{ and intersects } K. \tag{23}$$

Let $E' = K \cap \Pi_{\mathcal{F}}(F(i) \times C)$. Then, since $S \subseteq \mathcal{Z}_{\mathcal{F}} \backslash \Pi_{\mathcal{F}}(F(i) \times C)$,

$$S \cap E' = \emptyset.$$

Now, by (21), $K \cap \Pi_{\mathcal{F}}(F(-i) \times C) \neq \emptyset$. Moreover, $F(-1) \cap F(1) = \emptyset$ by hypothesis, and hence $\emptyset \neq K \cap \Pi_{\mathcal{F}}(F(-i) \times C) \subseteq K \cap \Pi_{\mathcal{F}}([Z \backslash F(i)] \times C) =$

$K\cap(\mathcal{Z}_{\mathcal{F}}\backslash\Pi_{\mathcal{F}}(F(i)\times C)) = K\backslash E'$. Also, by (23), E' is a nonempty closed subset of K. Thus, $K\backslash E'$ is a nonempty, proper open subset of the continuum K. Then, as S is a component of $K\backslash E'$, it follows (from Theorem 2-16 in [2]) that $Cl_K(S)\cap E'\neq\emptyset$. Therefore, since $E' = K\cap\Pi_{\mathcal{F}}(F(i)\times C)$, we have $Cl_K(S)\cap\Pi_{\mathcal{F}}(F(i)\times C)\neq\emptyset$. Furthermore, because $S\cap E' = \emptyset\neq Cl_K(S)\cap E'$, we have $S\neq Cl_K(S)$. Hence S is not compact. This completes the proof of (22). There are two more facts that we will need to show φ is monotone.

> Assume $i\in\{-1,1\}$, $c\in C$, $\pi(c)\in\varphi(K)$, and N is a component of $\varphi^{-1}(\pi(c))$. Then $N\cap\Pi_{\mathcal{F}}(F(i)\times\pi(c))\neq\emptyset$. (24)

To see this, first note that $\varphi^{-1}(\pi(c))$ is a closed subset of K and N is a closed subset (being a component) of $\varphi^{-1}(\pi(c))$. Thus

$$N \text{ is a closed subset of } K. \tag{25}$$

Now suppose $N\cap\Pi_{\mathcal{F}}(F(i)\times\pi(c)) = \emptyset$. By the definition of N (and the definition of φ as a restriction of $\Phi_{\mathcal{F}}$), we have $N\subseteq\Pi_{\mathcal{F}}([Z\backslash F(i)]\times\pi(c))$. Hence, as $\pi(c)\in\varphi(K)$,

$$\emptyset\neq N\subseteq\Pi_{\mathcal{F}}([Z\backslash F(i)]\times\pi(c))\subseteq\mathcal{Z}_{\mathcal{F}}\backslash\Pi_{\mathcal{F}}(F(i)\times C). \tag{26}$$

Thus there exist $z'\in Z\backslash F(i)$ and $c'\in\pi(c)$ with $\Pi_{\mathcal{F}}(z',c')\in N$. Then, by (25) and (26), we have

$$\Pi_{\mathcal{F}}(z',c')\in K\cap N\cap[\mathcal{Z}_{\mathcal{F}}\backslash\Pi_{\mathcal{F}}(F(i)\times C)]. \tag{27}$$

Now let S be the component of $K\cap[\mathcal{Z}_{\mathcal{F}}\backslash\Pi_{\mathcal{F}}(F(i)\times C)]$ containing $\Pi_{\mathcal{F}}(z',c')$. Then, by (14), S is a component of $K\cap\Pi_{\mathcal{F}}([Z\backslash F(i)]\times\pi(c'))$. But $\pi(c') = \pi(c)$ and $K\cap\Pi_{\mathcal{F}}([Z\backslash F(i)]\times\pi(c))\subseteq K\cap\Pi_{\mathcal{F}}(Z\times\pi(c)) = \varphi^{-1}(\pi(c))$, so S is a connected subset of $\varphi^{-1}(\pi(c))$ containing $\Pi_{\mathcal{F}}(z',c')$. Therefore, as N is the component of $\varphi^{-1}(\pi(c))$ containing $\Pi_{\mathcal{F}}(z',c')$,

$$\emptyset\neq S\subseteq N. \tag{28}$$

By (22), $Cl_K(S) \cap \Pi_{\mathcal{F}}(F(i) \times C) \neq \emptyset$. Then $Cl_K(N) \cap \Pi_{\mathcal{F}}(F(i) \times C) \neq \emptyset$, by (28). Therefore, by (25), $N \cap \Pi_{\mathcal{F}}(F(i) \times C) \neq \emptyset$. Thus, there exists $(z_0, c_0) \in F(i) \times C$ with $\Pi_{\mathcal{F}}(z_0, c_0) \in N$. Then, as $\Pi_{\mathcal{F}}(z_0, c_0) \in N \subseteq \varphi^{-1}(\pi(c))$, we have $\varphi(\Pi_{\mathcal{F}}(z_0, c_0)) = \pi(c)$. But $\varphi(\Pi_{\mathcal{F}}(z_0, c_0)) = \Phi_{\mathcal{F}}(\Pi_{\mathcal{F}}(z_0, c_0)) = \pi(c_0)$. Hence $\pi(c_0) = \pi(c)$, and $c_0 \in \pi(c)$. Thus $\Pi_{\mathcal{F}}(z_0, c_0) \in N \cap \Pi_{\mathcal{F}}(F(i) \times \pi(c))$. Therefore, $N \cap \Pi_{\mathcal{F}}(F(i) \times \pi(c)) \neq \emptyset$, as desired. The last fact to be proved is the following.

Suppose $i \in \{-1, 1\}$ and $c \in (C \backslash E) \cup \bigcup \mathcal{E}_i$. Let B be the set $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z \backslash F(i)] \times \pi(c))$. Then the restriction $\Upsilon = \Pi_{\mathcal{F}}|_B$ maps B homeomorphically onto $K \cap \Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c))$. (29)

To establish (29), note first that $\Pi_{\mathcal{F}}$ maps compact subsets of $Z \times C$ onto compact subsets of $\mathcal{Z}_{\mathcal{F}}$. Hence, $\Pi_{\mathcal{F}}$ is a closed mapping. Moreover, B is an inverse set of $\Pi_{\mathcal{F}}$. Thus, by Theorem 1 in Section 13, Chapter I of [4], Υ is a closed mapping. Notice too that Υ is continuous and, because $c \in (C \backslash E) \cup \bigcup \mathcal{E}_i$, Υ is one-to-one. So Υ is indeed a homeomorphism from B onto $K \cap \Pi_{\mathcal{F}}([Z \backslash F(i)] \times \pi(c))$.

We now show φ is monotone. Suppose $c \in C$. If $\pi(c) \notin \varphi(K)$ then $\varphi^{-1}(\pi(c))$ is empty, and hence is connected. So assume $\pi(c) \in \varphi(K)$. Let N and N' be components of $\varphi^{-1}(\pi(c))$. We will show $N = N'$ by proving that $N \cap N' \neq \emptyset$. Observe that

$$N \text{ and } N' \text{ are subcontinua of } K.$$

Consider first the case that

$$c \in (C \backslash E) \cup \bigcup \mathcal{E}_1.$$

By (24) there exist, for each $i \in \{-1, 1\}$, $z(i) \in F(i)$ and $c(i) \in \pi(c)$ with $\Pi_{\mathcal{F}}(z(i), c(i)) \in N$. Similarly by (24), there exist, for each $i \in \{-1, 1\}$, $z'(i) \in F(i)$ and $c'(i) \in \pi(c)$ with

$\Pi_{\mathcal{F}}(z'(i), c'(i)) \in N'$. Then, since $F(-1)$ and $F(1)$ are assumed to be disjoint,

$$\begin{aligned}&\Pi_{\mathcal{F}}(z(-1), c(-1)) \in N \cap K \cap \Pi_{\mathcal{F}}([Z\backslash F(1)] \times \pi(c)), \text{ and}\\ &\Pi_{\mathcal{F}}(z'(-1), c'(-1)) \in N' \cap K \cap \Pi_{\mathcal{F}}([Z\backslash F(1)] \times \pi(c)).\end{aligned}$$

Let S be the component of $K \cap \Pi_{\mathcal{F}}([Z\backslash F(1)] \times C)$ that contains the point $\Pi_{\mathcal{F}}(z(-1), c(-1))$, and S' be the component of $K \cap \mathcal{Z}_{\mathcal{F}}\backslash\Pi_{\mathcal{F}}(F(1) \times C)$ containing $\Pi_{\mathcal{F}}(z'(-1), c'(-1))$. Let H be the component of $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \pi(c))$ containing $(z(-1), c(-1))$, and let H' be the component of $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \pi(c))$ containing $(z'(-1), c'(-1))$. Then

$$\begin{aligned}&H \subseteq \Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \{c(-1)\}) \text{ , and}\\ &H' \subseteq \Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \{c'(-1)\}).\end{aligned}$$

Let $Cl(H)$ denote the closure of H in $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \{c(-1)\}$, and let $Cl(H')$ denote the closure of H' in $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \{c'(-1)\}$. We claim that

$$\begin{aligned}&Cl(H) \text{ is a subcontinuum of } Z \times \{c(-1)\} \text{ that intersects}\\ &\text{both } F(1) \times \{c(-1)\} \text{ and } F(-1) \times \{c(-1)\}.\end{aligned} \tag{30}$$

For consider the component S of $K \cap \mathcal{Z}_{\mathcal{F}}\backslash\Pi_{\mathcal{F}}(F(1) \times C)$ that contains $\Pi_{\mathcal{F}}(z(-1), c(-1))$. By (14) and (22), S is a non-compact component of the set $K \cap \Pi_{\mathcal{F}}([Z\backslash F(1)] \times \pi(c(-1))$. Thus, as $c(-1) \in \pi(c)$,

$$S \text{ is non-compact and a component of } K \cap \Pi_{\mathcal{F}}([Z\backslash F(1)] \times \pi(c)). \tag{31}$$

Let Υ be the restriction of $\Pi_{\mathcal{F}}$ to $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \pi(c))$. By (29), Υ maps $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \pi(c))$ homeomorphically onto $K \cap \Pi_{\mathcal{F}}([Z\backslash F(1)] \times \pi(c))$. Therefore, by (31),

$\Upsilon^{-1in}(S)$ is non-compact and a component of

$$\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \pi(c)).$$

Also, $(z(-1), c(-1)) \in [Z\backslash F(1)] \times \pi(c)$, and (from above) $\Pi_{\mathcal{F}}(z(-1), c(-1)) \in K \cap \Pi_{\mathcal{F}}([Z\backslash F(1)] \times \pi(c))$. Consequently $(z(-1), c(-1)) \in \Upsilon^{-1}(S)$. Hence, as $\Upsilon^{-1}(S)$ is the component of $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \pi(c))$ containing $(z(-1), c(-1))$, we have $\Upsilon^{-1}(S) = H$. Then, as Υ^{-1} is a homeomorphism, it follows from (31) that H is not compact. Moreover,

$$H \subseteq [Z\backslash F(1)] \times \{c(-1)\},$$

since $(z(-1), c(-1)) \in H \subseteq [Z\backslash F(1)] \times \pi(c)$ and H is connected. Then, as H is not compact, there exist points $(z_1, c(-1)), (z_2, c(-1)), \ldots,$ in H which converge to some $(z_0, c(-1)) \in [Cl(H)]\backslash H$. Note that $H \cup \{(z_0, c(-1))\}$ is connected. Also, $(z_0, c(-1)) \in \Pi_{\mathcal{F}}^{-1}(K)$, since $(z_k, c(-1)) \in H \subseteq \Pi_{\mathcal{F}}^{-1}(K)$ for $k \geq 1$, and $\Pi_{\mathcal{F}}^{-1}(K)$ is closed in $Z \times C$. Hence, as $(z_0, c(-1)) \in \Pi_{\mathcal{F}}^{-1}(K)$ and as $H \cup \{(z_0, c(-1))\}$ is a connected subset of $\Pi_{\mathcal{F}}^{-1}(K)$ that properly contains the component H of $\Pi_{\mathcal{F}}^{-1}(K) \cap ([Z\backslash F(1)] \times \{c(-1)\})$, we have $(z_0, c(-1)) \notin [Z\backslash F(1)] \times \{c(-1)\}$. That is, $z_0 \in F(1)$. Then, because $(z_0, c(-1)) \in Cl(H) \cap (F(1) \times \{c(-1)\})$ and $(z(-1), c(-1)) \in Cl(H) \cap F(-1) \times \{c(-1)\}$, (30) holds. Symmetric to (30), one also has

$Cl(H')$ is a subcontinuum of $Z \times \{c'(-1)\}$ that intersects both $F(1) \times \{c'(-1)\}$ and $F(-1) \times \{c'(-1)\}$. (32)

By (30) and (32), and since $c(-1), c'(-1) \in \pi(c)$,

$$\Pi_{\mathcal{F}}(Cl(H)) \subseteq \varphi^{-1}(\pi(c(-1))) = \varphi^{-1}(\pi(c))$$

$$= \varphi^{-1}(\pi(c'(-1))) \supseteq \Pi_{\mathcal{F}}(Cl(H')).$$

Moreover, $\Pi_{\mathcal{F}}(Cl(H))$ is, again by (30), a continuum in $\varphi^{-1}(\pi(c))$ containing $\Pi_{\mathcal{F}}(z(-1), c(-1))$. Thus, as N is the component of $\varphi^{-1}(\pi(c))$ containing $\Pi_{\mathcal{F}}(z(-1), c(-1))$, $\Pi_{\mathcal{F}}(Cl(H))$ is a subcontinuum of N. Similarly, $\Pi_{\mathcal{F}}(Cl(H'))$ is a subcontinuum of N'. Now let R and R' denote the projections of $Cl(H)$ and $Cl(H')$, respectively, onto Z. Then R and R' are subcontinua of Z each of which intersects both $F(1)$ and $F(-1)$ (by (30) and (32)). Thus, by hypothesis (*) of Lemma 4, there exists $\hat{z} \in R \cap R' \cap F(1)$. Consequently, $(\hat{z}, c(-1)) \in Cl(H)$ and $(\hat{z}, c'(-1)) \in Cl(H')$. Hence, as $c(-1), c'(-1) \in \pi(c)$ and $c \in (C \backslash E) \cup \bigcup \mathcal{E}_1$, we have

$$\Pi_{\mathcal{F}}(\hat{z}, c(-1)) = \Pi_{\mathcal{F}}(\hat{z}, c'(-1)) \in$$
$$\Pi_{\mathcal{F}}(Cl(H)) \cap \Pi_{\mathcal{F}}(Cl(H')) \subseteq N \cap N'.$$

Thus, $N \cap N' \neq \emptyset$. Hence $N = N'$.

The proof that $N = N'$ when $c \in (C \backslash E) \cup \bigcup \mathcal{E}_{-1}$ is symmetric to the argument just given. This completes the proof of Lemma 4.□

3 An Example

Let Δ denote a fixed plane triangle with vertices (x_i, y_i) for $i = 0, 1, 2$. For any real number z let $\Delta(z)$ denote the triangle $\Delta \times \{z\}$ in R^3, and let $V_i(z)$ be the vertex (x_i, y_i, z) of $\Delta(z)$. For $i = 0, 1, 2$, let I_i denote the closed vertical line segment joining $V_i(0)$ to $V_i(1)$. Let Z' be the continuum defined by

$$Z' = I_0 \cup I_1 \cup I_2 \cup \bigcup_{n=0}^{\infty} \Delta(n/(n+1)) \cup \Delta(1).$$

For $0 \leq n < \infty$ let $i(n) = n \; mod \; 3$, and let O_n be the open subarc of $I_{i(n)}$ with missing endpoints $V_{i(n)}(\frac{n}{n+1})$ and $V_{i(n)}(\frac{n+1}{n+2})$. Let $O = \bigcup_{n=0}^{\infty} O_n$ and

$$Z'' = Z' \backslash O.$$

Define Z to be the union of Z'' with its reflection through the plane $z = 0$. It is not difficult to verify that Z', Z'' and Z are colocally connected continua. Define

$$F(1) = \{V_0(1),\ V_1(1),\ V_2(1)\} \text{ and}$$
$$F(-1) = \{V_0(-1),\ V_1(-1),\ V_2(-1)\}.$$

Observe that

Each subcontinuum of Z intersecting both $F(1)$ and $F(-1)$ contains at least two of the three points in $F(1)$ and at least two of the three points in $F(-1)$.

(33)

Let $\mathcal{F} = \{F(-1), F(1)\}$, and let the ordering of the components of $S^1 \backslash C$ be as stated in Lemma 3. Then $\mathcal{Z}_{\mathcal{F}}$ is colocally connected by Lemma 3. Also, by (33), if R and R' are subcontinua of Z each of which intersects both $F(1)$ and $F(-1)$, then $R \cap R' \cap F(1) \neq \emptyset \neq R \cap R' \cap F(-1)$. Hence, by Lemma 4, $\Phi_{\mathcal{F}}$ is an hereditarily monotone mapping of $\mathcal{Z}_{\mathcal{F}}$ onto the simple closed curve $\pi(C)$. Choose a homeomorphism Γ from $\pi(C)$ onto S^1. Then $h = \Gamma \circ \Phi_{\mathcal{F}}$ is an hereditarily monotone mapping of $\mathcal{Z}_{\mathcal{F}}$ onto S^1.

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