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FAREY TREE AND DISTRIBUTION OF SMALL DENOMINATORS

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Abstract

We describe how a Farey tree partitioning of the rationals in $[0,1]$ can be used to find the fraction with smallest denominator that lies strictly between two given real numbers α, β. We then derive the probability distribution of the smallest denominator when α and β are randomly chosen, uniformly from the unit interval. A discussion of a naturally associated map is also included.

1 The Farey Tree

Let $0 < \alpha \leq 1$, $0 \leq \beta < \alpha$. We denote by $Q(\alpha, \beta)$ the rational number with smallest possible denominator lying strictly between α and β, and by $N(\alpha, \beta)$ the denominator of $Q(\alpha, \beta)$.

An explicit algorithm for computing $Q(\alpha, \beta)$ (based on the continued fraction expansion of α and β) is given in [2]. In this article, we use a Farey tree partitioning of the rationals in $[0,1]$ to describe both $Q(\alpha, \beta)$ and the algorithm, and we use the same tree to derive the probability distribution of $N(\alpha, \beta)$ when α and β are randomly chosen, uniformly from the triangle $\{(\alpha, \beta) : 0 < \alpha \leq 1, 0 \leq \beta < \alpha\}$.

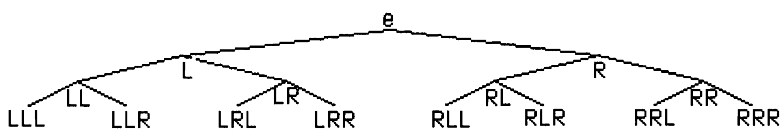
We begin by describing the Farey tree, an infinite binary tree with vertices labeled by the rationals. This construction has been used before by several authors, see for example [1], or [4].

The *Farey sequence of order n* is the ascending sequence of all rational numbers in $[0,1]$ whose denominator is at most n. So for example the Farey sequence of order 4 is

$$\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}.$$

Two fractions a/b, c/d are said to be *adjacent Farey fractions* if they occur in consecutive order in some Farey sequence. It is easy to check that a/b, c/d are adjacent Farey fractions if and only if $|ad - bc| = 1$. For any pair of fractions (in lowest terms), Farey addition is defined by $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$.

The Farey tree (like any other binary tree) starts at a vertex e (the root of the tree), and has branches consisting of sequences of left and right turns, as in the following picture.



Let B_n be the set of words of length n over the alphabet $\{L, R\}$. If $u \in B_n$ and $v \in B_m$, we can form $uv \in B_{n+m}$ by concatenation. Denote by e the empty word (we are at the root of the tree when we take no turns). So $we = w = ew$ for every word w, and $B_0 = \{e\}$. We identify $w \in B_n$ with the corresponding vertex of the tree reached by traveling along w. The length of a word is denoted by $|w|$, and we will write L^∞ for the infinite word consisting of all L's. We also set for convenience $B_{-1} = \{L^\infty, R^\infty\}$.

Given $u \in B_n$, the *left [right] child* of u is uL $[uR] \in B_{n+1}$. We now define *left* and *right ancestors* u^-, u^+ of u inductively as follows. Let $e^- = L^\infty$, $e^+ = R^\infty$, and given $u \in B_n$, $n \geq 0$, define

$$(uL)^- = u^-, \quad (uL)^+ = u = (uR)^-, \quad (uR)^+ = u^+.$$

So we have $L^- = (eL)^- = e^- = L^\infty$, $L^+ = (eL)^+ = e$, $R^- = (eR)^- = e$, $R^+ = (eR)^+ = e^+ = R^\infty$, and so on. Note that u^- is obtained by deleting all terminal L's from u, and then deleting one R. Similarly for u^+.

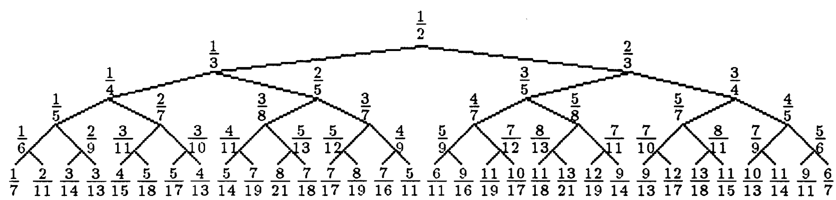
Next, we associate a rational number to every vertex in such a way that each of (uL, u), (u, uR), (u^-, u), (u, u^+), (u^-, u^+) correspond to adjacent Farey fractions.

Define $f : \bigcup_{n=-1}^{\infty} B_n \longrightarrow \mathbf{Z}^+ \times \mathbf{N}$ inductively by $f(L^\infty) = (0,1)$, $f(R^\infty) = (1,1)$, and for $u \in B_n$, $n \geq 0$, $f(u) = f(u^-) + f(u^+)$, where addition is coordinate-wise. So $f(e) = f(e^-) + f(e^+) = f(L^\infty) + f(R^\infty) = (1,2)$, and similarly we find $f(L) = (1,3)$, $f(R) = (2,3)$ and so on.

Lemma 1 *If* $w \in \bigcup_{n=0}^{\infty} B_n$ *and* $f(w^-) = (a,b)$, $f(w^+) = (c,d)$, *then* $bc - ad = 1$.

Proof: The lemma is true if $w = e$. If $w = uR$, with $u \in B_{n-1}$, then $u^- = (w^-)^-$, and $w^+ = u^+ = (w^-)^+$. Let $f(u^-) = (k,l)$. Since $(a,b) = f((w^-)^-) + f((w^-)^+) = (k+c, l+d)$, we find $bc - ad = lc - kd = 1$, by induction. Similarly if $w = uL$. □

In particular, (a,b) and (c,d) above are relatively prime. Applying Lemma 1 to wL, we find that every pair (a,b) in the image of f is relatively prime. So if we define $F : \bigcup_{n=-1}^{\infty} B_n \longrightarrow \mathbf{Q}$ by $F(w) = a/b$, where $f(w) = (a,b)$, then we have $F(w) = F(w^-) \oplus F(w^+)$ for all w, where $\oplus$ is Farey addition, and the pairs $(F(uL), F(u))$, $(F(u), F(uR))$, $(F(u^-), F(u))$, $(F(u), F(u^+))$, $(F(u^-), F(u^+))$ are all adjacent Farey fractions. It is clear from the definition of f that the denominator of $F(w)$ is strictly greater than that of $F(w^-)$ or $F(w^+)$, and it is easy to check that $0 < F(w) < 1$ for all $w \in \bigcup_{n=0}^{\infty} B_n$. The figure below shows the values of F on the tree for the first few levels.



Lemma 2 *Let $a \in \mathbf{Z}^+$, $b, c, d \in \mathbf{N}$, with $0 \le a/b < c/d \le 1$, and $bc - ad = 1$. Then there is some $w \in \bigcup_{n=0}^{\infty} B_n$ such that $F(w^-) = a/b$ and $F(w^+) = c/d$.*

Proof: If $a/b = 0$, then $c = d = 1$ and we let $w = e$. If $c/d = 1$, then $b = a + 1$, and we let $w = R^{a-1}$, (because $F(R^k) = (k+1)/(k+2)$ for all k).

Suppose now that $0 < a/b < c/d < 1$. Note that $(a-c)/(b-d) > 0$. If $a - c$ is positive, then $b + c - a - d = (1 + (a-c)(d-c))/c > 0$, so that $b - d > a - c$, and $0 < (a-c)/(b-d) < 1$, that is, $(a-c)/(b-d)$ and c/d are adjacent Farey fractions in $[0,1]$. By induction (on the sum of the denominators), there is a u such that $F(u^-) = (a-c)/(b-d)$, $F(u^+) = c/d$. Then $F(u) = F(u^-) \oplus F(u^+) = a/b$, and if we let $w = uR$, then $F(w^-) = F(u) = a/b$, $F(w^+) = F(u^+) = c/d$. The case that $a-c$ is negative is similar. □

We have thus established a bijective correspondence between the vertices of the Farey tree and the rational numbers in $(0,1)$. This correspondence can be extended to infinite paths on the tree, as follows.

Denote by $N(w)$ the denominator of $F(w)$. It is easy to check that $N(w) \geq |w| + 2$. Also, $N(wR) = N(w) + N(w^+)$, by definition, and since $F(w)$ and $F(w^+)$ are adjacent Farey fractions, we have

$$F(w^+) - F(w) = \frac{1}{N(w)N(w^+)}.$$

It then follows that

$$F(w^+) - F(wR^m) = \frac{1}{N(w^+)(N(w) + mN(w^+))}$$

for all m. Also note that

$$F(w^-) < F(wL) < F(w) < F(wR) < F(u^+). \qquad (1)$$

So the sequence $F(wR^m)$ increases to $F(w^+)$, and similarly the sequence $F(wL^m)$ decreases to $F(w^-)$.

Suppose now that $x = x_1 x_2 \cdots$ is an infinite path on the Farey tree, starting at the root. Let $w_n = x_1 x_2 \cdots x_n$. Using the above, one proves that $\{F(w_n)\}$ is a Cauchy sequence, and so there is some α in $[0,1]$ such that $\lim_{n\to\infty} F(w_n) = \alpha$. We extend F to infinite paths by letting $F(x) = \alpha$.

2 Continued Fractions

We now summarize the properties of the correspondence between $[0,1]$ and finite or infinite paths on the Farey tree, together with its connections with continued fractions. Proofs of all statements follow easily from the previous results, and from elementary properties of continued fractions.

Any rational number in $[0,1]$ corresponds to a unique, finite path on the tree, and it also corresponds to precisely two infinite paths, obtained by attaching either LR^∞ or RL^∞ to the finite path. For example,

$$\frac{7}{19} = F(LRLLR) = F(LRLLRLR^\infty) = F(LRLLRRL^\infty).$$

To obtain the continued fraction expansion corresponding to a finite path, first rewrite it as an infinite path. Then the first digit of the continued fraction expansion is 1 if the path starts with R, and is $k+1$ if it starts with k consecutive L's. Then the other digits are precisely the number of the following consecutive L's and R's. So for example $LRLLRLR^\infty$ corresponds to $[2,1,2,1,1] = \cfrac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1}}}}}$, and $RRLLLRL^\infty$ corresponds to $[1,2,3,1] = \cfrac{1}{1+\frac{1}{2+\frac{1}{3+\frac{1}{1}}}}$, which are the continued fraction expansions of $7/19$ and $9/13$.

An irrational number in $[0,1]$ corresponds to a unique infinite path that does not terminate with an infinite string or L's or R's.

Given two numbers α, β in $[0,1]$, to find $Q(\alpha,\beta)$ we proceed as follows. Represent both α and β as infinite paths on the Farey tree. If there is more than one choice (that is, if at least one of α, β is rational), choose a pair of representations that have the longest possible initial overlap. Then $Q(\alpha,\beta)$

corresponds to the last common vertex on the paths. We illustrate with an example. Let $\alpha = 3/8$, $\beta = 5/13$. Then α corresponds to $LRLRL^\infty$ and $LRLLR^\infty$, and β corresponds to $LRLRRL^\infty$ and $LRLRLR^\infty$. The maximum overlap occurs for the pairs $LRLRL^\infty$, $LRLRLR^\infty$, with initial overlap $LRLRL$, and so $Q(3/8, 5/13) = F(LRLRL) = 8/21$. This procedure is essentially equivalent to the algorithm of Section II of [2].

To derive the same result using only continued fraction expansions, suppose that $[a_1, a_2, \ldots]$ and $[b_1, b_2, \ldots]$ are expansions for α and β. If an expansion is finite, attach ∞ at the end. So the expansions for 7/19 are $[2, 1, 2, 2, \infty]$ and $[2, 1, 2, 1, 1, \infty]$. This is to ensure that d and m below are well-defined even if one expansion is an initial segment of the other. Let $d = d(\alpha, \beta) = \min\{k : a_k \neq b_k\}$, $m = m(\alpha, \beta) = \min\{a_d, b_d\}$, and $M(\alpha, \beta) = m + \sum_{i=1}^{d-1} a_i$ (so that $M(\alpha, \beta) - 1$ is the length of the overlap of the corresponding paths on the Farey tree). Choose a pair of expansions that maximizes $M(\alpha, \beta)$. Then $Q(\alpha, \beta)$ is given by the continued fraction $[a_1, a_2, \ldots, a_{d-1}, m + 1]$. This procedure is essentially equivalent to the algorithm of Section III of [2].

As example, 3/8 has expansions $[2, 1, 1, 1, \infty]$ and $[2, 1, 2, \infty]$, and 5/13 has expansions $[2, 1, 1, 2, \infty]$ and $[2, 1, 1, 1, 1, \infty]$. The maximum value for M is 6 and occurs for the pairs $[2, 1, 1, 1, \infty]$ and $[2, 1, 1, 1, 1, \infty]$. So we find again

$$Q(3/8, 5/13) = [2, 1, 1, 1, 2] = 8/21.$$

3 Distribution of Smallest Denominator

We now derive a formula for the probability that the smallest possible denominator of any fraction between α and β is n, when (α, β) is randomly chosen uniformly from the triangle $\{(\alpha, \beta) : 0 < \alpha \leq 1; \;\; 0 \leq \beta < \alpha\}$.

Lemma 3 *Let $n \geq 2$, and $0 < k < n$, with $\gcd(n, k) = 1$. Then there are unique integers a, b such that $0 < a < n$, $0 \leq b < k$ and $ak - bn = 1$*

Proof: Let a be the unique integer in $\{1, \ldots, n-1\}$ such that $ak \equiv 1 \pmod{n}$. Then we have $ak = 1 + bn$ for some b, and since $0 \leq b = k(a/n) - 1/n \leq k - 1$, the lemma is proved. □

As in Section 1, we denote by B_n the set of all words of length n over the alphabet $\{L, R\}$, and we identify the vertices of the tree with the elements of $\bigcup_{n=0}^{\infty} B_n$.

Let $w \in \bigcup_{n=0}^{\infty} B_n$. Define $\mathcal{R}(w) = (F(w), F(w^+)] \times [F(w^-), F(w))$. So $\mathcal{R}(w)$ is a subset of the triangle $\{(\alpha, \beta) : 0 < \alpha \leq 1, 0 \leq \beta < \alpha\}$.

Lemma 4 *If $w \neq u$, then $\mathcal{R}(w) \cap \mathcal{R}(u) = \emptyset$.*

Proof: From Lemma 1, we have for any word w

$$F(w^-) = \lim_{n \to \infty} F(wL^n) = F(wL^\infty) <$$

$$F(wL) < F(w) < F(wR) < F(wR^\infty) =$$

$$\lim_{n \to \infty} F(wR^n) = F(w^+).$$

Assume without losing generality that $F(w) < F(u)$. Suppose first that w is not a subword of u and u is not a subword of w. Then there are (possibly empty) words v, s, t such that $w = vLs$, $u = vRt$, and we find

$$F(w^+) = F(vLsR^\infty) \leq F(vLR^\infty) = F(v) < F(u),$$

so that $(F(w), F(w^+)] \cap (F(u), F(u^+)] = \emptyset$.

Suppose now that u is a subword of w. Then we have $w = uLv$ for some (possibly empty) word v, and so

$$F(w^+) = F(uLvR^\infty) \le F(uLR^\infty) = F(u),$$

and again we find $(F(w), F(w^+)] \cap (F(u), F(u^+)] = \emptyset$. The case that w is a subword of u is similar, and gives $[F(w^-), F(w)) \cap [F(u^-), F(u)) = \emptyset$. □

Lemma 5 *Let $(\alpha, \beta) \in \mathbf{R}^2$, with $0 < \alpha \le 1$, $0 \le \beta < \alpha$. Then there is some $w \in \bigcup_{n=0}^\infty B_n$ such that $(\alpha, \beta) \in \mathcal{R}(w)$, and $F(w)$ is the fraction with lowest denominator between α and β.*

Proof: Let u, v be words such that $F(u) = \alpha$, $F(v) = \beta$. If u is not a subword of v and v is not a subword of u, let w be the first common ancestor of u and v. Then we can write $u = wRs$, $v = wLt$ for some words s, t that do not end in L^∞ or R^∞, and we have $F(w^-) < F(w) < F(w)$, $F(w) < F(u) < F(w^+)$, so that $(\alpha, \beta) = (F(u), F(v)) \in \mathcal{R}(w)$. If v is a subword of u, then v must be a finite word. Choose an infinite word v' such that $F(v') = F(v)$ and the words v', u have maximum possible overlap w, say. Since $F(u) > F(v)$, we must have $u = wRt$ and $v = wLs$ for some words t and s. Then $F(w) < F(wRt) \le F(wR^\infty) = F(w^+)$, and $F(w^-) = F(wL^\infty) \le F(wLs) < F(w)$, i.e. $(\alpha, \beta) = (F(u), F(v)) \in \mathcal{R}(w)$. The case that u is a subword of v is similar, and the last assertion follows from the algorithm described in Section 2. □

Lemmas 4 and 5 show that $\{\mathcal{R}(w) : w \in B_n, n \ge 0\}$ is a partition of the triangle $\{(\alpha, \beta) : 0 < \alpha \le 1, 0 \le \beta < \alpha\}$.

There is a bijection $\theta : \{(n, k) : n \ge 2,\ 0 < k < n, \gcd(n, k) = 1\} \longrightarrow \bigcup_{n=0}^\infty B_n$ given by $\theta(n, k) = w$, where $F(w) = a/n$, $F(w^-) = b/k$ and a, b are such that $ak - bn = 1$, $0 < a < n$, $0 \le b < k < n$, as provided by Lemma 3 (see also the comments following Lemma 1).

Proposition 6 *Let $n \geq 2$, $0 < k < n$ with $\gcd(n,k) = 1$ be given, let $w = \theta(n,k)$, and let a, b be such that $ak - bn = 1$, as in Lemma 3. Then for any α, β with $0 < \alpha \leq 1, 0 \leq \beta < \alpha$, the fraction with lowest denominator between α and β is a/n if and only if $(\alpha, \beta) \in \mathcal{R}(w)$.*

Proof: Suppose that the fraction with lowest denominator between α and β is a/n. If $(b-a)/(n-k) < \alpha$, then we would have $\beta < a/n < (b-a)/(n-k) < \alpha$, a contradiction. So we must have $a/n = F(w) < \alpha \leq (b-a)/(n-k) = F(w^+)$, and in a similar fashion we find $F(w^-) \leq \beta < F(w)$, i.e. $(\alpha, \beta) \in \mathcal{R}(w)$. The converse is Lemma 5. □

Theorem 7 *If the point (α, β) is randomly chosen uniformly from the triangle $\{(\alpha, \beta) : 0 < \alpha \leq 1, 0 \leq \beta < \alpha\}$, then the probability that the lowest possible denominator of any fraction between α and β is n is given by*

$$\frac{4}{n^3} \sum_{\substack{k<n; \\ \gcd(k,n)=1}} \frac{1}{k}.$$

Proof: Since the area of the triangle $\{(\alpha, \beta) : 0 < \alpha \leq 1, 0 \leq \beta < \alpha\}$ is $1/2$, by Proposition 6 the probability in question is

$$2 \sum \text{area of } \mathcal{R}(w) = 2 \sum (F(w^+) - F(w))(F(w) - F(w^-)),$$

where the sum is over all words w such that $F(w) = a/n$ for some a relatively prime to n. Using the bijective correspondence described after Lemma 5, we have $F(w^-) = b/k$, where $0 \leq b < k < n$, $ak - bn = 1$, and then by definition $F(w^+) = (a-b)/(n-k)$. We then find

$$\sum (F(w^+) - F(w))(F(w) - F(w^-)) =$$

$$\sum_{\substack{k<n;\\ \gcd(k,n)=1}} \left(\frac{a-b}{n-k} - \frac{a}{n}\right)\left(\frac{a}{n} - \frac{b}{k}\right) = \sum_{\substack{k<n;\\ \gcd(k,n)=1}} \frac{1}{n(n-k)}\frac{1}{nk} =$$

$$\frac{1}{n^3}\sum_{\substack{k<n;\\ \gcd(k,n)=1}} \left(\frac{1}{k} + \frac{1}{n-k}\right) = \frac{2}{n^3}\sum_{\substack{k<n;\\ \gcd(k,n)=1}} \frac{1}{k}.$$

□

Corollary 8

$$\sum_{n=2}^{\infty} \frac{4}{n^3} \sum_{\substack{k<n;\\ \gcd(k,n)=1}} \frac{1}{k} = 1. \tag{2}$$

The question arises: what is the expected value of $N(\alpha, \beta)$? That is, what is

$$\sum_{n=2}^{\infty} \frac{4}{n^2} \sum_{\substack{k<n;\\ \gcd(k,n)=1}} \frac{1}{k}? \tag{3}$$

Computer estimates suggest it is close to 4. It is also natural to ask whether there is a simpler, more direct proof of (2). Both questions have been answered by Sam Northshield [3], who made use of the formula

$$\sum_{n=1}^{\infty} \sum_{\substack{1\leq k<n;\\ \gcd(k,n)=1}} \frac{1}{n^m k} = \frac{1}{\zeta(m+1)} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^m k}$$

(where ζ is the Riemann Zeta function) to show that (2) is equivalent to the identity $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^2k}=\frac{2}{\zeta(3)}$, while the identity $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^3k}=\frac{5}{4\zeta(4)}$ is used to show that the sum in (3) is indeed 4.

4 The Shift Map on the Farey Tree

The natural shift map σ on $\{L,R\}^N$ (defined by $(\sigma(x))_i = x_{i+1}$) induces a map T on $[0,1]\setminus\{1/2\}$ via the correspondence provided by the map F defined in section 1. So T is a map such that $T(F(x)) = F(\sigma(x))$ holds for infinite sequences x in $\{L,R\}^N$ (except for the two sequences LR^∞ and RL^∞, corresponding to 1/2).

It is easy to check that if the continued fraction expansion of x is $[a_1, a_2, a_3, \ldots]$, then

$$T(x) = \begin{cases} [a_1-1, a_2, a_3, \ldots] & \text{if } \ a_1 > 1 \\ [1, a_2-1, a_3, a_4, \ldots] & \text{if } \ a_1 = 1, a_2 > 1 \\ [a_3+1, a_4, \ldots] & \text{if } \ a_1 = a_2 = 1 \end{cases}$$

It then follows that

$$T(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } \ 0 \le x < 1/2 \\ 2-\dfrac{1}{x} & \text{if } \ 1/2 < x \le 1 \end{cases}$$

From the above formula for T we see that if a/b is a rational number in lowest terms, then $T(a/b)$ is a rational number with denominator strictly less than b. So it is evident that iteration of T on a rational number eventually produces one of the two boundary points 0, 1. Iteration on the irrationals is more interesting. There is a T- invariant absolutely continuous measure, with density given by $\rho(t) = \frac{1}{t} + \frac{1}{1-t}$ (see [4, p.93]). We are currently investigating how the map T transforms sets with constant $N(\alpha, \beta)$ (the rectangles $\mathcal{R}(w)$ of Section 3).

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