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ORDERED UNIFORM COMPLETIONS OF GO-SPACES

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Abstract

In this paper we define GO-uniform spaces and prove that the uniform completion of a GO-uniform space is a GO-d-extension of the initial GO-space. A characterization of non-convergent minimal Cauchy filters in a GO-uniform space is given. We also characterize GO-spaces which have only one compatible GO-uniformity and show that there is a 1–1 correspondence between GO-paracompactifications and GO-uniformity classes. Finally we give several examples corresponding to the above results.

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1 Introduction

Throughout the paper by a uniformity on a set X we understand a uniformity defined by covers of X. For a uniformity $\mathcal{U}$, by $\tau_{\mathcal{U}}$ we understand the topology on X generated by this uniformity ([4], [12]). For a collection α of subsets of a set X and $S \subset X$ we have:

$$St(S, \alpha) = \cup\{A \in \alpha : S \cap A \neq \emptyset\}.$$

For covers α and β of a set X, the symbols $\beta < \alpha$ and $\beta \overset{\star}{<} \alpha$ mean respectively, that the cover β is a refinement of the cover α and that $\{St(B, \beta) : B \in \beta\} < \alpha$.

A *linearly ordered topological space* (abbreviated LOTS) is a triple $(X, \lambda(\leqslant), \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set and $\lambda(\leqslant)$ is the usual interval topology defined by $\leqslant$ (i.e., $\lambda(\leqslant)$ is the topology generated by $\{\ ]a, \rightarrow [\ : a \in X\} \cup \{\ ] \leftarrow, a[\ : a \in X\}$ as a subbase, where $]a, \rightarrow [\ = \{x \in X : a < x\}$ and $] \leftarrow, a[\ = \{x \in X : x < a\}$). A *generalized ordered space* (abbreviated GO-space) is a triple $(X, \tau, \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set and τ is a topology on X such that $\lambda(\leqslant) \subset \tau$ and τ has a base consisting of order convex sets, where a subset A of X is called *order convex* or simply *convex* if $x \in A$ for every x lying between two points of A.

It is well known that a topological space (X, τ) is a GO-space together with some ordering $\leqslant_X$ on X if and only if (X, τ) is a topological subspace of some LOTS $(Y, \lambda(\leqslant_Y), \leqslant_Y)$ with $\leqslant_X = \leqslant_Y|_X$, where the symbol $\leqslant_Y|_X$ is the restriction of the order $\leqslant_Y$ to X, so any GO-space has a linearly ordered extension. Note that a LOTS $(Y, \lambda(\leqslant_Y), \leqslant_Y)$ is called a *linearly ordered extension* of a GO-space $(X, \tau, \leqslant_X)$ if $X \subset Y, \tau = \lambda(\leqslant_Y)|_X$ and $\leqslant_X = \leqslant_Y|_X$, ([11]). Any GO-space X has a linearly ordered extension Y such that X is dense in Y (such an extension is called a linearly ordered d-extension in [11]). In this paper the extensions that we will consider will not always be linearly ordered extensions and so we will use the term *GO-extension* of the GO-space $(X, \tau_X, \leqslant_X)$ to mean a GO-space $(Y, \tau_Y, \leqslant_Y)$ such that $X \subset Y$, $\tau_X = \tau_Y|_X$ and $\leqslant_X = \leqslant_Y|_X$. Similarly we say *GO-d-extension* for the case when X is dense in Y. The extensions that we will consider are all GO-d-extensions, so by an extension we always mean a GO-d-extension. We will be interested in such extensions which are completions of $(X, \tau, \leqslant)$ with respect to some GO-uniformity, the definition of which will be given in 2.

For the sake of completeness we give the following definition:

Let $(X, \tau, \leqslant)$ be a GO-space and (A, B) an ordered pair of disjoint open sets of X such that:

(i) $X = A \cup B$,

(ii) $a < b$ whenever $a \in A$ and $b \in B$.

Then (A, B) is called a *gap* if it satisfies (i), (ii) and

(iii) A has no maximal point, and B has no minimal point.

If furthermore $A = \emptyset$ or $B = \emptyset$, then (A, B) is called an *endgap*. (A, B) is called a *pseudo-gap* if it satisfies (i), (ii),

(iv) $A \neq \emptyset, B \neq \emptyset$,

and, $(\text{iv})_l$ or $(\text{iv})_r$ stated by

$(\text{iv})_l$ A has no maximal point, and B has a minimal point,

$(\text{iv})_r$ A has a maximal point, and B has no minimal point.

Suppose (A, B) is a (pseudo-)gap of a GO-space $(X, \tau, \leqslant)$. If there are discrete subsets A' of A which is cofinal in A and a discrete subset B' of B which is coinitial in B, then (A, B) is called a *Q-(pseudo-)gap*. It is well known that a GO-space X is paracompact if and only if every gap of X is a Q-gap, and every pseudo-gap is a Q-pseudo-gap ([5],[8]).

We will also need the following linearly ordered extension of an arbitrary GO-space $(X, \tau, \leqslant)$. Define $L(X)$ to be a subset of $X \times \{-1, 0, 1\}$ by

$$L(X) = (X \times \{0\}) \cup \{(x, -1) : x \in X \text{ and } [x, \rightarrow [\, \in \tau - \lambda(\leqslant)\} \\ \cup \{(x, 1) : x \in X \text{ and }] \leftarrow, x] \in \tau - \lambda(\leqslant)\}.$$

Let $L(X)$ be a LOTS by the lexicographic order on $L(X)$. Then it is easily seen that $L(X)$ is a linearly ordered d-extension of X ([1],[11]). In addition, $L(X)$ is a minimal linearly ordered d-extension of X in the sense that $L(X)$ embeds by a monotonic homeomorphism into any linearly ordered d-extension of X ([11]).

For further reading on the topic of uniformities and ordered spaces, see [1], [9] and [13].

2 Generalized Ordered Uniformities

Let X be a set, $\mathcal{U}$ a uniformity on X, τ a topology on X and $\leqslant$ a linear order on X.

Definition 2.1 The topology τ is said to be $\leqslant$-*convex* (or just *convex*) if τ has a base consisting of convex (w.r.t. $\leqslant$) sets.

The topology τ can be either coarser or finer than $\lambda(\leqslant)$. For example, both the anti-discrete (trivial) topology and the discrete topology on X are $\leqslant$-convex topologies on $(X, \leqslant)$.

Proposition 2.2 *If τ is a T_1 convex topology on $(X, \leqslant)$ then $\lambda(\leqslant) \subset \tau$.*

Proof: We prove that for every $a, b \in X$, $]a, b[$ is open. Say $x \in]a, b[$. Since X is a T_1-space, there exists open convex sets A_x, B_x such that $x \in A_x \cap B_x, a \notin A_x, b \notin B_x$. Thus we have that the set $A_x \cap B_x$ is open, convex and $x \in A_x \cap B_x \subset]a, b[$. □

Corollary 2.3 *A T_1 convex topology on $(X, \leqslant)$ is a GO-topology.*

As Example 5.8 shows, one cannot replace T_1 by T_0 in Proposition 2.2 and Corollary 2.3.

Definition 2.4 The triple $(X, \mathcal{U}, \leqslant)$ is called a *GO-uniform space* if the uniformity $\mathcal{U}$ has a base $\mathcal{B}$, each of the covers of which consists of convex sets. In this case $\mathcal{U}$ is called a *GO-uniformity* on $(X, \leqslant)$.

It is evident that if $\mathcal{U}$ is a GO-uniformity then $\tau_{\mathcal{U}}$ is a T_1 convex topology and hence every GO-uniformity induces a GO-topology on $(X, \leqslant)$. We say that the GO-uniformity $\mathcal{U}$ is a GO-uniformity of the GO-space $(X, \tau, \leqslant)$ if $\tau_{\mathcal{U}} = \tau$.

Proposition 2.5 *Let $\{\mathcal{U}_a : a \in \mathcal{A}\}$ be an arbitrary family of GO-uniformities of a GO-space $(X, \tau, \leqslant)$. Then $\mathcal{U} = \sup\{\mathcal{U}_a : a \in \mathcal{A}\}$ is a GO-uniformity of the GO-space $(X, \tau, \leqslant)$. If $\mathcal{U}_a$ is precompact for all $a \in \mathcal{A}$, then $\mathcal{U}$ is also precompact.*

Proof: It is known that the base of the uniformity $\mathcal{U}$ consists of covers of the form $\wedge_{i=1}^{n} \alpha_{a_i}$, where $\alpha_{a_i} \in \mathcal{U}_{a_i}$ and $\{a_1, a_2, \ldots, a_n\}$ is an arbitrary finite subset of $\mathcal{A}$. If the covers α_{a_i}, $i = 1, 2, \ldots, n$ consists of open convex sets, then so does the cover $\wedge_{i=1}^{n} \alpha_{a_i}$. Also, if each cover α_{a_i} is finite, then $\wedge_{i=1}^{n} \alpha_{a_i}$ is also finite. □

Corollary 2.6 *In all the GO-(*precompact)uniformities of the GO-space $(X, \tau, \leqslant)$, there exists a largest GO-(*precompact) uniformity.*

Moreover, we have the following result, which in particular shows that the universal uniformity of $(X, \tau, \leqslant)$ is always a GO-uniformity.

For a cover α of the space $(X, \tau, \leqslant)$, by $\widehat{\alpha}$ we denote the cover consisting of the convex components of the elements of the cover α. If α is an open cover, then so is the cover $\widehat{\alpha}$ and we always have that $\widehat{\alpha} < \alpha$.

Proposition 2.7 *Let $(X, \tau, \leqslant)$ be a GO-space. If $\mathcal{U}$ is any uniformity compatible with τ, then $\widehat{\mathcal{B}} = \{\widehat{\alpha} : \alpha \in \mathcal{U}\}$ is a base for a GO-uniformity, finer than $\mathcal{U}$ and compatible with τ.*

Proof: First of all, if $\beta \overset{\star}{<} \alpha$ then $\widehat{\beta} \overset{\star}{<} \widehat{\alpha}$. This follows from the fact that for all $B' \in \widehat{\beta}$ there exists a $B \in \beta$ such that B' is a convex component of B and $St(B', \widehat{\beta}) \subset St(B, \beta)$. But $\beta \overset{\star}{<} \alpha$ implies that $St(B, \beta) \subset A$ for some $A \in \alpha$, and since $St(B', \widehat{\beta})$ is convex, we get that there exists a convex component A' of A with $St(B', \widehat{\beta}) \subset A'$.

Also, for $\alpha, \beta \in \mathcal{U}$ we have that $\widehat{\alpha \wedge \beta} < \widehat{\alpha} \wedge \widehat{\beta}$. Thus $\widehat{\mathcal{B}}$ defines a uniformity $\widehat{\mathcal{U}}$ which is a GO-uniformity.

Since $\widehat{\mathcal{U}}$ has a base consisting of open covers and $\widehat{\mathcal{U}} \supset \mathcal{U}$, which follows from the fact that $\widehat{\alpha} < \alpha$, we get that $\widehat{\mathcal{U}}$ is compatible with τ and finer than $\mathcal{U}$. □

Corollary 2.8 *Let $(X, \tau, \leqslant)$ be a GO-space. Then the universal uniformity is a GO-uniformity.*

Let $U(X, \tau, \leqslant)$ be the set of all GO-uniformities of a GO-space $(X, \tau, \leqslant)$. It is partially ordered by inclusion. If $\mathcal{U} \in$

$U(X, \tau, \leqslant)$, then by $\Phi(\mathcal{U})$ we denote the set of all minimal Cauchy filters of the uniform space $(X,\mathcal{U})$. In $U(X, \tau, \leqslant)$ an equivalence relation is defined in the following manner: $\mathcal{U}_1 \sim \mathcal{U}_2$ if and only if $\Phi(\mathcal{U}_1) = \Phi(\mathcal{U}_2)$. By $E(\mathcal{U})$ we denote the equivalence class containing the uniformity $\mathcal{U}$ and let $\mathcal{U}_E = \sup\{\mathcal{U}' : \mathcal{U}' \in E(\mathcal{U})\}$.

Let $(X,\mathcal{U}, \leqslant)$ be a GO-uniform space. The GO-uniformity $\mathcal{U}_E$ is called *E-leader* of the GO-uniformity $\mathcal{U}$. The GO-uniformity $\mathcal{U}$ is called *preuniversal GO-uniformity* if $\mathcal{U} = \mathcal{U}_E$.

If a cover α of a GO-space $(X, \tau, \leqslant)$ consists of open convex sets, then the cover α is called an *open convex cover.*

Remember that a topological space X is *Dieudonné complete* if there exists a complete uniformity on the space X. This is equivalent to X being Tychonoff and the universal uniformity on the space X being complete. Since no stationary subset S of the space $W(\kappa)$, where κ is some regular uncountable cardinal, is Dieudonné complete, we have that a GO-space $(X, \tau, \leqslant)$ is Dieudonné complete if and only if it is paracompact ([4], [5]).

Proposition 2.5 and Corollary 2.6 where obtained by Borubaev ([3]) for the case when $\tau = \lambda(\leqslant)$, that is when X is a LOTS. The following theorem was also obtained by Borubaev ([3]) for the case when X is a LOTS. Since the proof is similar to the one for LOTS, we only give a short proof to show the structure of the GO-d-extension.

Theorem 2.9 *Let $(X,\mathcal{U}, \leqslant)$ be a GO-uniform space and $(\widetilde{X},\widetilde{\mathcal{U}})$ the completion of the uniform space $(X,\mathcal{U})$. Then there exists a linear order $\widetilde{\leqslant}$ on $\widetilde{X}$ such that the following holds:*

(1) *The order $\widetilde{\leqslant}$ induces on X the initial order $\leqslant$;*

(2) *$(\widetilde{X},\widetilde{\mathcal{U}}, \widetilde{\leqslant})$ is a GO-uniform space;*

(3) *$(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$ is a paracompact extension of the GO-space $(X, \tau, \leqslant)$, where $\widetilde{\tau} = \tau_{\widetilde{\mathcal{U}}}$ and $\tau = \tau_{\mathcal{U}}$.*

Furthermore, if $\mathcal{U}$ is a preuniversal GO-uniformity, then $\widetilde{\mathcal{U}}$ is the universal uniformity of the GO-space $(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$.

Proof: Let $\widetilde{X}$ be the set of all minimal Cauchy filters in $(X, \mathcal{U})$. Define on $\widetilde{X}$ a linear order in the following manner: for $\mathcal{F}_1, \mathcal{F}_2 \in \widetilde{X}$, we write $\mathcal{F}_1 \widetilde{<} \mathcal{F}_2$ if and only if there exist open convex (in X) sets $I_1 \in \mathcal{F}_1, I_2 \in \mathcal{F}_2$ such that $x_1 < x_2$ for $x_1 \in I_1, x_2 \in I_2$. It can be easily verified that $\widetilde{<}$ is a linear order on $\widetilde{X}$. For every $x \in X$ by $\mathcal{F}_x$ we denote the neighbourhood filter of x, which is a minimal Cauchy filter in $(X, \mathcal{U})$. Note that $x < y \Leftrightarrow \mathcal{F}_x \widetilde{<} \mathcal{F}_y$. By identifying the point $x \in X$ with its neighbourhood filter $\mathcal{F}_x$ one can look at X as a subset of $\widetilde{X}$ and by the above, the linear order $\widetilde{<}$ of $\widetilde{X}$ induces on X the initial order $<$.

For every open convex set I of the GO-space $(X, \tau, \leqslant)$ we put $\widetilde{I} = \{\mathcal{F} \in \widetilde{X} : I \in \mathcal{F}\}$. If $\mathcal{B}$ is a base of the uniformity $\mathcal{U}$ consisting of open convex (in X) covers, $\widetilde{\mathcal{B}} = \{\widetilde{\alpha} : \alpha \in \mathcal{B}\}$ is a base for the uniformity $\widetilde{\mathcal{U}}$, where $\widetilde{\alpha} = \{\widetilde{I} : I \in \alpha\}$. It can be easily proved that $\widetilde{I}$ is an open convex set of $(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$, and the fact that $\widetilde{\mathcal{U}}$ induces a GO-topology on $(\widetilde{X}, \widetilde{\leqslant})$ follows from (i) the uniformity $\widetilde{\mathcal{U}}$ has a base $\widetilde{\mathcal{B}}$ consisting of open convex (in $\widetilde{X}$) covers, and (ii) for every $\mathcal{F} \in \widetilde{X}$ the system $\{\widetilde{I} : I \in \mathcal{F}, I$ is an open convex (in X) set $\}$ is a base of the point $\mathcal{F}$ in the space $(\widetilde{X}, \widetilde{\tau})$. Hence $(\widetilde{X}, \widetilde{\mathcal{U}}, \widetilde{\leqslant})$ is a GO-uniform space, and so (1) and (2) are proved.

(3) follows from the remark on Dieudonné complete GO-spaces above. From (3) and Corollary 2.8 follows that if $\mathcal{U}$ is a preuniversal GO-uniformity then $\widetilde{\mathcal{U}}$ is the universal uniformity of the GO-space $(\widetilde{X}, \widetilde{\tau}, \widetilde{\leqslant})$.□

As is proved in [3] if $(X, \tau, \leqslant)$ is a LOTS, the completion is also a LOTS with respect to the order $\widetilde{\leqslant}$, that is $\widetilde{\mathcal{U}}$ induces the usual open interval topology with respect to $\widetilde{\leqslant}$.

3 Minimal Cauchy Filters

We first give some definitions concerning (pseudo-)gaps of a GO-space $(X, \tau, \leqslant)$. Since we are concerned with GO-uniformities, we give the following definition which only slightly differs from that given in Nagata ([12]).

Definition 3.1 A (pseudo-)gap (A, B) of a GO-space $(X, \tau, \leqslant)$ is said to be *covered* by the convex set V if $V \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. A cover α of X is said to *cover* the (pseudo-) gap (A, B) if α has an element, a convex component of which covers (A, B). If (A, B) is an endgap, then by 'covered' we mean 'almost covered' (cf. Definition 3.2).

Definition 3.2 A (pseudo-)gap (A, B) is said to be *almost covered* by the convex set V if either

(i) $V \subset A$ and V has no upper bound in A, or

(ii) $V \subset B$ and V has no lower bound in B.

A cover α of X is said to *almost cover* the (pseudo-)gap (A, B) if α has an element, a convex component of which almost covers (A, B).

From the definitions above one can see that a cover α can both cover and almost cover a (pseudo-)gap (A, B).

Definition 3.3 Let $(X, \mathcal{U}, \leqslant)$ be a GO-uniform space. A (pseudo-)gap (A, B) is said to be a *$\mathcal{U}$-(pseudo-)gap* if there exists $\alpha \in \mathcal{U}$ such that α does not cover (A, B) nor almost cover (A, B).

Thus a gap (A, B) is a $\mathcal{U}$-gap provided there exists $\alpha \in \mathcal{U}$ such that:

$$U \in \widehat{\alpha} \rightarrow \begin{array}{ll} U & \subset A \text{ and } U \text{ is not cofinal in } A; \text{ or} \\ U & \subset B \text{ and } U \text{ is not co-initial in } B. \end{array}$$

Similarly, a pseudo-gap (A, B), say $(A, B) = (] \leftarrow, a_0],]a_0, \rightarrow [)$ where $] \leftarrow, a_0] \in \tau - \lambda(\leqslant)$, is a $\mathcal{U}$-pseudo-gap provided there exists $\alpha \in \mathcal{U}$ such that:

$$U \in \widehat{\alpha} \rightarrow \begin{array}{l} U \subset A; \text{ or} \\ U \subset B \text{ and } U \text{ is not co-initial in } B. \end{array}$$

The next theorem characterizes non-convergent Cauchy filters, in particular minimal Cauchy filters.

Theorem 3.4 *Let $(X, \mathcal{U}, \leqslant)$ be a GO-uniform space and $\mathcal{F}$ a Cauchy filter. The following are equivalent:*

(i) *$\mathcal{F}$ does not converge to any point in X;*

(ii) *there exists a unique (pseudo-)gap (A, B) such that for every $a \in A, b \in B$, we have $]a, b[\in \mathcal{F}$;*

(iii) *there exists a (pseudo-)gap (A, B) such that for every $a \in A, b \in B$, we have $]a, b[\in \mathcal{F}$;*

(iv) *there exists a (pseudo-)gap (A, B) such that if $\mathcal{B}$ is any base of open convex covers for $(X, \mathcal{U}, \leqslant)$ and if $\alpha \in \mathcal{B}$, then some $U \in \alpha \cap \mathcal{F}$ covers, or almost covers, (A, B).*

Proof: (i) $\Rightarrow$ (ii). Let $\mathcal{F}$ be a Cauchy filter which does not converge to any point in X. Since $\mathcal{F}$ is Cauchy, we have that for every $\alpha \in \mathcal{U}$, $\alpha \cap \mathcal{F} \neq \emptyset$. Also, for every $x \in X$ there exists $\alpha_x \in \mathcal{U}$ such that $St(x, \alpha_x) \notin \mathcal{F}$, since $\mathcal{F}$ does not converge. If $\beta_x \overset{\star}{<} \alpha_x$ then $St(x, \beta_x) \notin \mathcal{F}$ and if $B \in \beta_x \cap \mathcal{F}$ we have that $B \cap St(x, \beta_x) = \emptyset$, as otherwise $B \subset St(x, \alpha_x) \notin \mathcal{F}$.

Let $A = \{x \in X$: there exist $\alpha \in \mathcal{U}$ with $St(x, \alpha) \notin \mathcal{F}$, and a convex set $A_x \in \mathcal{F}$ such that $x < a$ for every $a \in A_x$ and $St(x, \alpha) \cap A_x = \emptyset\}$. A is open in X, because if $x \in A$ and $\alpha \in \mathcal{U}$ satisfies the condition given in the definition of the set A, then there exists $\beta \in \mathcal{U}$ with $\beta \overset{\star}{<} \alpha$. So for every $y \in St(x,\beta)$ we have $St(y,\beta) \subset St(x,\alpha)$ which implies that $St(y,\beta) \notin \mathcal{F}$. Since one can consider only covers from some base $\mathcal{B}$ of $\mathcal{U}$ consisting of open convex covers, we get that $y \in A$, as $St(y,\beta) \cap A_x = \emptyset$, and $y < a$ for every $a \in A_x$. Similarly one defines the set B to be $B = \{y \in X$: there exist $\alpha \in \mathcal{U}$ with $St(y,\alpha) \notin \mathcal{F}$, and a convex set $B_y \in \mathcal{F}$ such that $y > b$ for every $b \in B_y$ and $St(y,\alpha) \cap B_y = \emptyset\}$.

From the first paragraph of the proof and considering only open convex covers we get that (A, B) is a gap or pseudo-gap of X. Note that this cannot be a jump. If (A, B) is not an endgap then for every $a \in A$ and $b \in B$ take A_a and B_b which belong to $\mathcal{F}$, then we have that $A_a \cap B_b \in \mathcal{F}$ and $]a,b[\, \supset A_a \cap B_b \in \mathcal{F}$. This implies that $]a,b[\, \in \mathcal{F}$. If $A = \emptyset$ (or $B = \emptyset$) then for every $b \in B$ ($a \in A$) we have that $]\leftarrow, b[$ ($]a, \rightarrow[$) is in $\mathcal{F}$. Similarly for pseudo-gaps, say A has a maximal element a_0, then $]a_0, b[\, \in \mathcal{F}$ for every $b \in B$. From the properties of a filter there can be only one such (pseudo-)gap with the above property.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (iv). Let (A, B) be a (pseudo-)gap such that for every $a \in A, b \in B$ we have $]a,b[\in \mathcal{F}$. Let $\mathcal{B}$ be any base of $\mathcal{U}$ consisting of interval covers. Consider the case when (A, B) is an internal gap. Similar arguments hold for the case when (A, B) is an endgap or pseudo-gap. Since $\mathcal{F}$ is Cauchy we have that for every $\alpha \in \mathcal{B}$, $\alpha \cap \mathcal{F} \neq \emptyset$, say $U \in \alpha \cap \mathcal{F}$. If U does not cover (A, B) then either $U \subset A$ or $U \subset B$. Without loss of generality, assume that $U \subset A$. If U has an upper bound in A, say a_0, then $]a_0, b[\, \in \mathcal{F}$ for every $b \in B$ and $U \cap\,]a_0, b[\, = \emptyset$, which is a contradiction.

(iv) $\Rightarrow$ (i). Suppose (iv) holds and $\mathcal{F}$ converges to the point $x \in X$. Let us first consider the case when (A, B) is a gap. If $x \in A$, then there are $a_1, a_2 \in A$ such that $a_1 < x < a_2$ (unless x is the first element, but then the argument still holds with only slight modifications). There exists $\alpha_0 \in \mathcal{B}$ with $St(x, \alpha_0) \subset]a_1, a_2[$ and $U_0 \in \beta_0 \cap \mathcal{F}$ satisfying the hypothesis of (iv), where $\beta_0 \overset{\star}{<} \alpha_0$. Thus $St(x, \beta_0) \cap U_0 = \emptyset$, otherwise $U_0 \subset St(x, \alpha_0)$. Hence $\mathcal{F}$ does not converge to x. Next suppose (A, B) is a pseudo-gap, say $(A, B) = (] \leftarrow, a_0],]a_0, \rightarrow [)$ where $] \leftarrow, a_0] \in \tau - \lambda(\leqslant)$. An argument parallel to the one above for gaps will show that $\mathcal{F}$ cannot converge to any point of X other than, perhaps, the point a_0. Suppose $\mathcal{F}$ converges to a_0. Consider the case where a_0 is not the left end point of X. Let a be any point with $a < a_0$. Then there is a cover $\alpha_0 \in \mathcal{B}$ with $St(a_0, \alpha_0) \subset]a, a_0]$ and $\beta_0 \overset{\star}{<} \alpha_0$, with $\beta_0 \in \mathcal{B}$. Let $U_0 \in \beta_0 \cap \mathcal{F}$ satisfying the hypothesis of (iv), then again $St(a_0, \beta_0) \cap U_0 = \emptyset$, otherwise $U_0 \subset St(a_0, \alpha_0) \subset]a, a_0]$, which is a contradiction. Finally, if a_0 is the left end point of X then $\{a_0\}$ is an open set. By repeating the above argument but changing $]a, a_0]$ to the open set $\{a_0\}$, one again obtains a contradiction. □

From Theorem 3.4 one can make the following remarks:

Remark 3.5. *(On gaps)* For every internal gap (A, B) there can be at most two *minimal* Cauchy filters converging to it (at most one for endgaps): one having a base in A and one having a base in B. In this case the gap is turned into a jump in the completion $\widetilde{X}$, that is there exist $a_0, b_0 \in \widetilde{X} - X$ such that $a < a_0 < b_0 < b$ for every $a \in A, b \in B$ and $]a_0, b_0[= \emptyset$. It can be the case that there is only one *minimal* Cauchy filter converging to an internal gap (A, B), for example this will be the case when there is a base $\mathcal{B}$ of $\mathcal{U}$ consisting of open convex covers such that for every $\beta \in \mathcal{B}, \beta$ covers (A, B). In this case the gap is turned into one point in the completion, that is there is a point $c = (A, B) \in \widetilde{X} - X$ such that $a < c < b$ for every $a \in A, b \in B$. As we shall see later, it can also be the case that no minimal Cauchy filter converges to the gap (A, B), this can only happen if (A, B) is a Q-gap (see below).

The term *minimal* is essential here as there can be more than two Cauchy filters converging to an internal gap (A, B).

This also applies to *Remark* 3.6.

Now suppose (A, B) is a gap of $(X, \mathcal{U}, \leqslant)$ such that there exist open convex covers $\alpha, \beta \in \mathcal{U}$ which do not cover (A, B) and moreover

(a) there is a $V \in \alpha$ satisfying (i) of Definition 3.2 but there is no $U \in \alpha$ satisfying (ii) of Definition 3.2;

(b) there is no $V \in \beta$ satisfying (i) of Definition 3.2 but there is a $U \in \beta$ satisfying (ii) of Definition 3.2.

Then by considering the cover $\alpha \wedge \beta \in \mathcal{U}$ one can see that (A, B) is a $\mathcal{U}$-gap.

Remark 3.6 *(On pseudo-gaps)* Suppose $(X, \mathcal{U}, \leqslant)$ is a GO-uniform space, $\tau_{\mathcal{U}}$ is the topology induced by $\mathcal{U}$ and $\lambda(\leqslant)$ is the open interval topology on X. Then, as is well known, if $a_0 \in X$ such that $]\leftarrow, a_0] \in \tau - \lambda(\leqslant)$ then it defines a pseudo-gap. In this case there can be at most two *minimal* Cauchy filters connected with this pseudo-gap $(A, B) = (\,]\leftarrow, a_0],]a_0, \rightarrow[\,)$, one which converges to the point a_0 and one which does not converge to any point in X, and has a base in $]a_0, \rightarrow[$. This will be the case when $\mathcal{U}$ has a base $\mathcal{B}$ of open convex covers with the property that for every $\beta \in \mathcal{B}$, β either covers or almost covers (A, B). Then there is a point $a_0^+ \in \widetilde{X} - X$ such that $a_0 < a_0^+ < b$ for every $b \in B =]a_0, \rightarrow[$ and $]a_0, a_0^+[= \emptyset$. Similarly for the case of $a_0 \in X$ with $[a_0, \rightarrow[\in \tau - \lambda(\leqslant)$. For the case that both $]\leftarrow, a_0]$ and $[a_0, \rightarrow[\in \tau - \lambda(\leqslant)$, then there can be three *minimal* Cauchy filters connected with a_0, one converging to a_0, the other two do not converge to any point in X, one has a base in $]\leftarrow, a_0[$ and the other in $]a_0, \rightarrow[$. In this case there are $a_0^-, a_0^+ \in \widetilde{X} - X$ such that $a < a_0^- < a_0^+ < b$ for every $a \in]\leftarrow, a_0[$, $b \in]a_0, \rightarrow[$ and $]a_0^-, a_0[= \emptyset =]a_0, a_0^+[$. Again, it can also be the case that there is only the *minimal* Cauchy filter which converges to a_0, and again this can only happen if the respective pseudo-gap is a Q-pseudo-gap (see below).

Remark 3.7 Let $(X, \mathcal{U}, \leqslant)$ be such that $\mathcal{U}$ is a GO-uniformity with $\tau_{\mathcal{U}} = \lambda(\leqslant)$. Also, let $\mathcal{U}$ have a base $\mathcal{B}$ consisting of open convex covers with the property that for every $\beta \in \mathcal{B}$, β covers every gap of X. Then $\mathcal{U}$ is precompact, since an open cover α of a LOTS X has a finite subcover if every gap of X is covered by α ([12]). Thus the completion $(\widetilde{X}, \widetilde{\mathcal{U}})$ of $(X, \mathcal{U})$ is a compact uniform space, and $(\widetilde{X}, \tau_{\widetilde{\mathcal{U}}})$ is a compact LOTS and a linearly ordered compactification of (X, τ). One can easily see that $(\widetilde{X}, \tau_{\widetilde{\mathcal{U}}})$ is homeomorphic to the Dedekind compactification X^+ ([10], [12]).

Suppose $(X, \mathcal{U}, \leqslant)$ is a GO-uniform space and $\mathcal{U}$ has a base $\mathcal{B}$ of open convex covers that covers every gap of $(X, \leqslant)$ as above. Moreover, suppose that for every pseudo-gap (A, B) and every $\beta \in \mathcal{B}$, either β covers (A, B) or almost covers (A, B). Then $(\widetilde{X}, \tau_{\widetilde{\mathcal{U}}})$ is homeomorphic to the Dedekind compactification X^+ of X.

Remark 3.8 Suppose $(X, \tau, \leqslant)$ is not paracompact. Then (X, τ) is not Dieudonné complete, which implies that every GO-uniformity $\mathcal{U}$ on $(X, \tau, \leqslant)$ is not complete.

Since $(X, \tau, \leqslant)$ is not paracompact, there is at least one (pseudo-)gap which is not a Q-(pseudo-)gap. Say (A, B) is not a Q-(pseudo-)gap, then (A, B) is not a $\mathcal{U}$-(pseudo-)gap, that is we have:

For any GO-space $(X, \tau, \leqslant)$ and any GO-uniformity $\mathcal{U}$ compatible with τ, every $\mathcal{U}$-(pseudo-) gap is a Q-(pseudo-)gap.

This follows from the fact that any non-Q-(pseudo-)gap which is a $\mathcal{U}$-(pseudo-)gap will remain a non-Q-(pseudo-)gap in the completion, which is paracompact, and this cannot be in the light of the result mentioned after the definition of Q-(pseudo-)gaps concerning paracompact GO-spaces at the end of §1.

We now give a proposition and two corollaries concerning the linearly ordered d-extension $L(X)$ of a GO-space X.

Proposition 3.9 *Let* $(X,\mathcal{U},\leqslant)$ *be a GO-uniform space. Then we have that* $(\widetilde{X},\widetilde{\mathcal{U}},\widetilde{\leqslant}) = L(X)$ *if and only if*

(i) *every gap is a* $\mathcal{U}$*-gap and*

(ii) *every pseudo-gap is not a* $\mathcal{U}$*-pseudo-gap.*

Proof: This follows from Remarks 3.6 and 3.8. □

We note that this can only be in the case that every gap is a Q-gap.

Corollary 3.10 *Let* $(X,\mathcal{U},\leqslant)$ *be a GO-uniform space, then the following are equivalent:*

(i) $(\widetilde{X},\widetilde{\leqslant})$ *is a LOTS;*

(ii) $\widetilde{X} \supset L(X)$*;*

(iii) *every pseudo-gap is not a* $\mathcal{U}$*-pseudo-gap.*

Corollary 3.11 *Let* $(X,\tau,\leqslant)$ *be a GO-space. If there exists a GO-uniformity* $\mathcal{U}$*, compatible with* τ*, such that* $(\widetilde{X},\widetilde{\mathcal{U}},\widetilde{\leqslant}) = L(X)$ *then every gap of* X *is a Q-gap.*

We finish this section with a proposition concerning the completeness of a GO-uniform space.

Proposition 3.12 *Let* $(X,\mathcal{U},\leqslant)$ *be a GO-uniform space, then we have that* $(X,\mathcal{U})$ *is complete if and only if*

(i) *every gap is a* $\mathcal{U}$*-gap and*

(ii) *every pseudo-gap is a* $\mathcal{U}$*-pseudo-gap.*

Proof: From *Remarks* 3.5, 3.6 and 3.8 one can see that a gap (respectively, pseudo-gap) is a $\mathcal{U}$-gap (respectively, $\mathcal{U}$-pseudo-gap) if and only if there is no minimal Cauchy filter converging to the gap (A,B) (respectively, there is only one minimal Cauchy filter corresponding to the pseudo-gap (A,B), the one converging to the point defining the pseudo-gap).

4 GO-Spaces With Unique Compatible GO-Uniformity

Let $(X, \tau, \leqslant)$ be a GO-space. The set of linearly ordered compactifications of X is the same as that of $L(X)$, since different compactifications depend only on how the internal gaps are filled, either by one point or two points. These ordered compactifications are in 1–1 correspondence with ordered proximities, where an ordered proximity δ is a Efremovič proximity with the extra properties:

(i) $x, y \in L(X),\ x < y \Rightarrow\ ]\leftarrow, x]\ \overline{\delta}\ [y, \rightarrow[$;

(ii) $A, B \subset L(X),\ A\overline{\delta}B \Rightarrow\ \exists$ a finite number of open convex sets $O_i \subset L(X), i = 1, \ldots, k$ such that $A \subset \cup_{i=1}^{k} O_i \subset L(X) - B$

(see Fedorčuk, [7]).

Fedorčuk also proved that such an ordered proximity has one and only one uniformity compatible with it, this uniformity can easily be seen to be a GO-uniformity on $L(X)$ which induces a GO-uniformity on X compatible with τ and whose completion is the corresponding compactification.

We now prove a characterization of GO-spaces which have a unique compatible GO-uniformity. As in §2, by a uniformity class $E(\mathcal{U})$, we mean the class of all GO-uniformities that have the same minimal Cauchy filters as $\mathcal{U}$.

Theorem 4.1 *Let $(X, \tau, \leqslant)$ be a GO-space, then the following are equivalent:*

(i) *On X there exists only one compatible GO-uniformity class;*

(ii) *On X there exists only one compatible GO-uniformity;*

(iii) *X has no internal gaps, no Q-endgaps and no Q-pseudo-gaps.*

Proof: (i) $\Rightarrow$ (ii). If there exists only one compatible GO-uniformity class, the completion with respect to this class must be the Dedekind compactification X^+ of X. As stated above, this implies that the class consists of only one GO-uniformity.

(ii) $\Rightarrow$ (iii). Say X has only one compatible GO-uniformity. Then X has only one completion and, in particular, one linearly ordered compactification, X^+. This implies that X cannot have any internal gaps, as an internal gap can be either filled with one point or with two points, turning it into a jump, thus leading to two different compactifications. Let (A, B) be a Q-pseudo-gap of $(X, \tau, \leqslant)$. Consider the GO-extension of X obtained by filling every gap with a point and also every pseudo-gap, except (A, B). The constructed extension $\widehat{X}$ is paracompact, and hence the universal uniformity $\widehat{\mathcal{U}}$ is a complete GO-uniformity. Say (A, B) corresponds to a point $a_0 \in X$ with $]\leftarrow, a_0] \in \tau - \lambda(\leqslant)$. Take the open cover $\{\ ]\leftarrow, a_0],]\,\widehat{a}, \widehat{b}\,[\ : \text{for all } \widehat{a}, \widehat{b} \in \widehat{X},\ a_0 < \widehat{a} < \widehat{b}\ \}$ of $\widehat{X}$. This is a normal cover and the intersection of this cover with X is a normal open cover of X and consists of convex sets. Thus it belongs to the universal uniformity of X. Hence the pseudo-gap is not filled in the completion with respect to the universal uniformity of X, which is the only uniformity compatible with τ. This contradicts the fact that the completion is the Dedekind compactification. Hence there cannot be any Q-pseudo-gaps. The same argument shows that there cannot be any Q-endgaps.

(iii) $\Rightarrow$ (i). Let $(X, \tau, \leqslant)$ be such that there are no internal gaps, no Q-pseudo-gaps and no Q-endgaps. Say that there are two compatible GO- uniformity classes on X. These give two different completions of the GO- space X and they can differ only on an internal gap, or Q-pseudo-gap, or Q-endgap of X. Thus there can only be one compatible GO-uniformity class (see Remark 3.8). □

As Examples 5.5, 5.6 and 5.7 show, no two of the requirments listed in (iii) of Theorem 4.1 are enough for a unique compatible GO-uniformity on $(X, \tau, \leqslant)$.

With respect to what was said in the first paragraph of this chapter we now prove that in a GO-space $(X, \tau, \leqslant)$ there is a 1–1 correspondence between GO-paracompactifications and GO-uniformity classes. Let pX be a paracompact GO-d-extension (i.e. a GO-paracompactification) of X. The universal uniformity on pX is a complete GO-uniformity, $p\mathcal{U}$. This uniformity induces on X a GO-uniformity $\mathcal{U}_E$ compatible with τ. It is not difficult to see that this uniformity is preuniversal and that $(\widetilde{X}, \widetilde{\mathcal{U}}_E) = (pX, p\mathcal{U})$. We say that pX induces on X the GO-uniformity class $E(\mathcal{U})$, of which $\mathcal{U}_E$ is the E-leader.

Theorem 4.2 *Let $(X, \tau, \leqslant)$ be a GO-space. For every GO-uniformity class $E(\mathcal{U})$, there exists one and only one paracompact GO-d-extension pX, which induces on X the class $E(\mathcal{U})$.*

Proof: Let $E(\mathcal{U})$ be a GO-uniformity class on $(X, \tau, \leqslant)$. Then by Theorem 2.9, the completion of the E-leader of the class, $\widetilde{\mathcal{U}}_E$, defines a GO-paracompactification $(\widetilde{X}, \widetilde{\mathcal{U}}_E, \widetilde{\leqslant})$. It is not difficult to see that this GO-paracompactification induces on X the initial GO-uniformity class $E(\mathcal{U})$. It is also not difficult to see that two different GO-paracompactifications induce different GO-uniformity classes on X (cf. 3).

5 Examples

Example 5.1 Let $(X, \tau, \leqslant)$ be an arbitrary GO-space. The universal uniformity is a GO-uniformity. This uniformity gives rise to the smallest paracompact GO-d-extension. Every non Q-gap (A, B) is filled in with an element $c = (A, B)$ such that

$a < c < b$ for every $a \in A, b \in B$. Every non Q-pseudo-gap $(\,]\leftarrow, c],]c, \rightarrow[\,)$ gives rise to a point $c^+ \in \widetilde{X} - X$ such that $c < c^+ < b$ for every $b \in]c, \rightarrow[$ and every non Q-pseudo-gap $(\,]\leftarrow, c[, [c, \rightarrow[\,)$ gives rise to a point $c^- \in \widetilde{X} - X$ such that $a < c^- < c$ for every $a \in]\leftarrow, c[$.

Example 5.2 The GO-space $[0, \omega_1[$, where ω_1 is the first uncountable ordinal, has only one paracompact GO-d-extension (cf. Theorem 4.1). Thus there is only one completion, which is the Čech-Stone compactification $[0, \omega_1]$. Notice that the space is not paracompact, the only gap in $[0, \omega_1[$, which is an endgap, is not a Q-gap. Let us now construct a subbase for the unique uniformity $\mathcal{U}$ on $[0, \omega_1[$. If α is a non-limit ordinal $< \omega_1$ and $\neq 0$ then let $\mathcal{U}_\alpha = \{\ [0, \alpha[, \{\alpha\},]\alpha, \omega_1[\ \}$. If $\alpha = 0$ then let $\mathcal{U}_0 = \{\ \{0\},]0, \omega_1[\ \}$. If α is a limit ordinal $< \omega_1$ let $\mathcal{U}_\alpha^i = \{\ [0, \alpha_i],]\alpha_i, \alpha],]\alpha, \omega_1[\ \}$, where $\alpha_i < \alpha$ for every $i \in \mathbb{N}$ and $\lim \alpha_i = \alpha$.

The above covers form a subbase for a compatible GO-uniformity on $[0, \omega_1[$, which is precompact and whose completion is the Čech-Stone compactification. A consequence of Remark 3.8 is that the cover $\{\ \{\alpha\},]\gamma, \beta] :\ \alpha$ ranges over all non-limit ordinals $< \omega_1$, and β ranges over all limit ordinals $< \omega_1$, with γ being any ordinal $< \beta\ \}$ is not a normal cover.

Example 5.3 Let $(M, \tau, \leqslant)$ be the Michael line and let $\mathcal{I}$ be the irrational numbers. Let $\mathcal{B}_n = \{\ B(x, \frac{1}{n}) :\ x \in M\ \}$, where $B(x, \varepsilon)$ is the usual ε-nbd ball in $\mathbb{R}$ with the standard metric, and let $\mathcal{S}_1 = \{\ \mathcal{B}_n :\ n \in \mathbb{N}\ \}$. Also, let $\mathcal{S}_2 = \{\ \{\ ]\leftarrow, p[, \{p\},]p, \rightarrow[\ \} :\ p \in \mathcal{I}\ \}$. Then $\mathcal{S}_1 \cup \mathcal{S}_2$ give a subbase for a compatible GO-uniformity. It can easily be seen that the completion with respect to this uniformity gives the Dedekind completion of the Michael line, i.e. the Dedekind compactification M^+ with its end points removed. In this case, the completion is $L(M)$ (cf. Proposition 3.9).

On the other hand the Michael line, being paracompact, has the universal uniformity, which is a complete GO-uniformity consisting of all open covers. The completion of M in this case coincides with M itself.

Example 5.4 Another important GO-space is the Sorgenfrey line $(S, \tau, \leqslant)$. Let $\mathcal{S}_1$ be as in Example 5.3 (changing M to S), and let $\mathcal{S}_2 = \{ \{] \leftarrow, x[, [x, \rightarrow [\} : x \in S \}$. Then $\mathcal{S}_1 \cup \mathcal{S}_2$ give a subbase for a compatible GO-uniformity. Again, the completion of S with respect to this uniformity is the Dedekind completion of the Sorgenfrey line, and again it is $L(S)$.

As in Example 5.3, since S is paracompact, the universal uniformity, which is a complete GO-uniformity, consists of all open covers. The completion of S in this case is S itself.

Example 5.5 Let $X = [0, 1]$ with the usual order and with the Sorgenfrey topology, i.e. $[0, 1]$ taken as a subspace of $(S, \tau, \leqslant)$. Then X has no internal gaps and no (Q-)endgaps, but has Q-pseudo-gaps. It can easily be seen that X has more than one compatible GO-uniformity class (cf. Theorem 4.1).

Example 5.6 Let $X = [-1, 0[\cup]0, 1]$ with usual order and topology of the real line, i.e. X is taken as a subspace of $(\mathbb{R}, \lambda(\leqslant), \leqslant)$. Then X has no (Q-)pseudo-gaps, no (Q-)endgaps but has an internal gap. It can easily be seen that X has more than one compatible GO-uniformity class (cf. Theorem 4.1).

Example 5.7 Let $X = (\mathbb{R}, \lambda(\leqslant), \leqslant)$. Then $\mathbb{R}$ has no internal gaps, no (Q-)pseudo-gaps but has two Q-endgaps. Being paracompact, the universal uniformity is a compatible GO-uniformity which is complete, hence the completion, which is the Dedekind completion, in this case is $\mathbb{R}$ itself. On the other hand, the Dedekind compactification of $\mathbb{R}$, that is filling in the two endgaps (which is homeomorphic to $[0, 1] \subset \mathbb{R}$) gives rise to a GO-uniformity whose completion is $\mathbb{R}^+$. Thus there is more than one compatible GO-uniformity class on X (cf. Theorem 4.1).

We note that the universal uniformity on $\mathbb{R}$ is the uniformity induced by the metric $\rho(x, y) = |x - y|$. Let us look at two other metrics on $\mathbb{R}$.

Take $\rho_e(x,y) = |e^x - e^y|$. Then the uniformity induced by this metric, which is a compatible GO-uniformity, is not in the same uniformity class as the universal uniformity. The completion with respect to this uniformity is obtained by filling in the left endgap, i.e. it is homeomorphic to $[0,1[\subset \mathbb{R}$. If we take $\rho_3(x,y) = |x^3 - y^3|$, then this metric is not uniformly equivalent to ρ but it is also a complete metric and thus we have that the uniformities $\mathcal{U}(\rho)$ and $\mathcal{U}(\rho_3)$, induced by ρ and ρ_3 respectively, are not the same (in fact we have $\mathcal{U}(\rho_3) \subsetneqq \mathcal{U}(\rho)$) but are in the same GO-uniformity class, that is they give the same completion, $\mathbb{R}$.

Example 5.8 Let X be the set $\{x_1, x_2, x_3\}$ and let the linear order $\leqslant$ on X be the following: $x_1 < x_2 < x_3$. Also, let the topology τ on X be $\{\emptyset, X, \{x_1, x_2\}, \{x_2, x_3\}, \{x_2\}\}$. Then τ is a T_0, convex topology on $(X, \leqslant)$ for which $\lambda(\leqslant) \not\subseteq \tau$.

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