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REMARKS ON CLOSED RELATIONS AND A THEOREM OF HUREWICZ

J. Chaber and R. Pol

Abstract

Two topics concerning closed symmetric relations on metrizable spaces are discussed. Firstly, a refinement of a theorem of Hurewicz is given and some of its applications presented. Among them is an analogue for metrizable spaces of a dichotomy of Feng for analytic sets. Secondly, closed relations on the Baire space $B(\aleph_1)$ are discussed. As an application a dichotomy involving Lusin-Sierpiński indices on coanalytic sets is provided.

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1 Introduction.

We shall consider in this note only metrizable spaces. Our terminology follows [Ku66]. In particular, by a perfect set in X we mean a nonempty closed subset of X without relatively isolated points. Therefore, a perfect set in X may be countable, i.e., a closed copy of the rationals.

A closed relation R on X is a closed set $R \subset X \times X$. The symbol Δ will denote the diagonal of the square. We say that $A \subset X$ is R-homogeneous if $A \times A \subset R \cup \Delta$, and that A is R-independent if $A \times A \cap R \subset \Delta$.

The following is one of the main results discussed in this note.

Theorem 1.1 *Let $f : X \to Y$ be a continuous mapping between metrizable spaces and let $R \subset Y \times Y$ be a closed symmetric relation. Then either*

(i) *X is the union of a σ-discrete collection of sets with R-homogeneous images,*
or else

(ii) *X contains a perfect set P such that f embeds P homeomorphically into Y and the closure $\overline{f(P)}$ is R-independent.*

The theorem is a refinement of some results of Hurewicz [Hu34], cf. [Ku66, §36, V, Remark]. Our proof is in fact a modification of the original Hurewicz's arguments. We have replaced, however, Hurewicz's "Häufungssysteme" by certain dyadic systems which we found more handy, cf. also [vD87].

An immediate consequence of Theorem 1.1 is a dichotomy of Feng [Fe93] that for any closed relation R on an analytic space X, either X is a countable union of R-homogeneous sets, or else X contains an R-independent Cantor set.

Another consequence is a result about sets of monotonicity of arbitrary real-valued functions, which generalizes a theorem of Filipczak [Fi66] concerning real-valued continuous functions on perfect sets in the real line.

Incidentally, Proposition 2.1 on which Theorem 1.1 is based yields readily another classical result of Hurewicz that a first category metrizable space contains a closed copy of the rationals.

We shall discuss these applications in Section 3.

Section 4 is concerned with closed relations on the Baire space $B(\aleph_1)$. We provide in this case a variation of Theorem 1.1 involving a natural layer structure of $B(\aleph_1)$. The proof is based on somewhat different ideas. We give also an example indicating limitations of possible further generalizations of our results.

The theorem concerning $B(\aleph_1)$ is then applied to closed relations on coanalytic sets of irrationals. Feng [Fe93] demonstrated that the validity for coanalytic sets of his dichotomy for analytic sets depends on axioms for set theory. A dichotomy we shall consider is essentially related to stratifications of coanalytic sets into Lusin's constituents. It is based on a link between the constituents and the layer structure of $B(\aleph_1)$, discussed by G. Gruenhage and the authors in [ChGP95].

2 A refinement of a theorem of Hurewicz.

Given $R \subset Y \times Y$, we shall denote by R_y the vertical section $\{y' \in Y : (y, y') \in R\}$ of R.

Proposition 2.1 *Let $f : X \to Y$ be a continuous mapping between metrizable spaces and let $R \subset Y \times Y$ be a closed symmetric relation containing the diagonal. Assume that there exists a dense subset D of X such that*

(1)

$$\textit{no neighbourhood of } x \textit{ is contained in } f^{-1}(R_{f(x)}) \textit{ for } x \in D.$$

Then X contains a perfect set P such that f embeds P homeomorphically into Y and the closure $\overline{f(P)}$ is R-independent.

Proof: We define, by induction on $n \geq 0$, finite sets $A_0 \subset A_1 \subset \ldots \subset D$ with A_n of cardinality 2^n, and collections $\{U_n(x) : x \in A_n\}$ of pairwise disjoint open sets in X such that, for $n \geq 1$,

$$\text{(2)} \qquad x \in U_n(x), \ \operatorname{diam}(U_n(x)) < 1/n \text{ and } \operatorname{diam}(f(U_n(x))) < 1/n \text{ for } x \in A_n.$$

We start the construction by fixing an arbitrary $x \in D$ and putting $A_0 = \{x\}$, $U_0(x) = X$. The set A_{n+1} is constructed by adding for each $x \in A_n$ a point $\widetilde{x}$ satisfying

$$\text{(3)} \qquad \widetilde{x} \in D \cap U_n(x) \setminus f^{-1}(R_{f(x)}).$$

Since D is dense in X, condition (1) and the fact that R is closed assure that this can be done. Furthermore, as $(f(x), f(\widetilde{x})) \notin R$, we can choose open neighbourhoods of x and $\widetilde{x}$ so that (2) and the following two conditions hold true:

$$\text{(4)} \qquad \overline{f(U_{n+1}(x))} \times \overline{f(U_{n+1}(\widetilde{x}))} \cap R = \emptyset,$$

$$\text{(5)} \qquad \overline{U_{n+1}(x)} \cup \overline{U_{n+1}(\widetilde{x})} \subset U_n(x).$$

Observe that (4) and (5) imply

$$\text{(6)} \quad \overline{f(U_n(x))} \times \overline{f(U_n(x'))} \cap R = \emptyset \text{ for } x \neq x' \text{ in } A_n, \ n \geq 1.$$

In particular, since R contains the diagonal of Y,

$$\text{(7)} \quad \overline{f(U_n(x))} \cap \overline{f(U_n(x'))} = \emptyset \text{ for } x \neq x' \text{ in } A_n, \ n \geq 1.$$

Let $P = \bigcap_{n \geq 1} \bigcup \{U_n(x) : x \in A_n\}$. By (5) P is closed in X. Condition (2) implies that $\bigcup_{n \geq 0} A_n$ is dense in P and (5) assures that each $x \in A_n$ is the limit of the sequence of its duplicates $\widetilde{x}_i \in A_{i+1} \setminus A_i$, $i \geq n$. It follows that P is perfect.

Since $f(P) \subset \bigcap_{n \geq 1} \bigcup \{f(U_n(x)) : x \in A_n\}$, by (7), the restriction of f to P is a homeomorphism onto $f(P)$. Finally, by (2), any two different points $y, y' \in \overline{f(P)}$ with $\operatorname{dist}(y, y') >$

$1/n$ belong to disjoint sets $\overline{f(U_n(x))}$ and $\overline{f(U_n(x'))}$, and hence (6) implies $(y, y') \notin R$. Consequently $\overline{f(P)}$ is R-independent and the proof is completed. □

Proof of Theorem 1.1. We can assume, without loss of generality, that $\Delta \subset R$. Let $\mathcal{I}$ be the collection of all subsets X' of X such that the restriction of f to X' satisfies condition (i) in Theorem 1.1. We shall find a decomposition

$$X = X_1 \cup X_2 \tag{8}$$

with X_1 in $\mathcal{I}$, and X_2 closed, containing a dense set D satisfying (1) in Proposition 2.1 for the map being the restriction of f to X_2. This will complete the proof, because then the assumption that $X_2 \neq \emptyset$ yields, by Proposition 2.1, condition (ii) in Theorem 1.1.

To get the decomposition (8) we apply a standard transfinite exhaustion procedure, cf. [Hu34, footnote 1]. At each stage we remove, from what remains, the union of all relatively open nonempty subsets belonging to $\mathcal{I}$. When the procedure terminates, we are left with a closed set X_2 such that

(9) no nonempty open subset of X_2 is in $\mathcal{I}$

and $X_1 = X \setminus X_2 \in \mathcal{I}$, cf. [St63, Theorem 4'].

Consider the case $X_2 \neq \emptyset$. To simplify the notation, assume that $X_2 = X$. We shall find a dense in X set D satisfying (1).

For $n > 0$ put $F_n = \{x \in X : \mathrm{dist}(x, \overline{X \setminus f^{-1}(R_{f(x)})}) \geq 1/n\}$. Since $D = X \setminus \bigcup_{n>0} F_n$ clearly satisfies (1), it remains to prove that this set is dense in X. By (9) it suffices to show that $\bigcup_{n>0} F_n$ is in $\mathcal{I}$. In fact, since $\mathcal{I}$ is closed with respect to the unions of σ-discrete collections, it is enough to show that each F_n is locally in $\mathcal{I}$.

To this end, observe that $x \in F_n$ and $\mathrm{dist}(x, x') < 1/n$ imply that $(f(x), f(x')) \in R$. Hence $f(A \cap F_n)$ is R-homogeneous provided $\mathrm{diam}(A) < 1/n$ and the proof is completed. □

The perfect set P constructed in the proof of Proposition 2.1 is separable (in fact, each perfect set in X contains a separable perfect set). Example 4.2 shows that we can not demand the perfect set P in Proposition 2.1 (or in Theorem 1.1) to be non-separable, even if the space X is not separable.

3 Some applications of Theorem 1.1.

We call a space X hereditarily Baire if all closed subspaces of X are Baire. The following statement generalizes Feng's dichotomy for analytic sets, cited in the introduction.

Corollary 3.1 *Let Y be a metrizable space which is the projection of a hereditarily Baire space $X \subset Y \times \mathbf{N}^{\mathbf{N}}$. Then for any closed symmetric relation R on Y, either*

(i) *Y is the union of a σ-discrete collection of closed R-homogeneous sets,*

or else

(ii) *Y contains an uncountable perfect R-independent set.*

Proof: Let us apply Theorem 1.1 with $f : X \to Y$ being the projection parallel to the irrationals. Since f takes σ-discrete collections to collections with σ-discrete refinements, and closures of R-homogeneous sets are R-homogeneous, the first possibility in Theorem 1.1 gives (i). The second possibility produces an R-independent set $\overline{f(P)}$ in Y with P being a perfect subset of X and f being one-to-one on P. Since X is a hereditarily Baire space, P is uncountable and $\overline{f(P)}$ is a perfect set witnessing (ii). □

In the next application of Theorem 1.1 we follow closely Feng's application of his dichotomy to sets of monotonicity of real-valued functions, cf. [Fe93, Sec. 5].

Let us consider an antisymmetric total relation L on a space E, i.e., $L \cap L^{-1} = \Delta$, and $L \cup L^{-1} = E \times E$. A real-valued function $\varphi : E \to \mathbf{R}$ is L-increasing (L-decreasing) on $A \subset E$ if for any s, t in A, sLt implies $\varphi(s) \leq \varphi(t)$ ($\varphi(s) \geq \varphi(t)$), and we say that φ is strictly L-increasing, or decreasing, on A if the inequalities are sharp, whenever $s \neq t$.

Corollary 3.2 *Let $L \subset E \times E$ be a closed antisymmetric and total relation on a metrizable space E and let $\varphi : E \to \mathbf{R}$ be a real-valued function. Then either*

(i) *E is the union of a σ-discrete collection of sets on which φ is L-decreasing,*

or else

(ii) *there exists a perfect subspace P of the graph of φ such that φ is strictly L-increasing on the projection of P onto E.*

Proof: Let $X = \{(s, \varphi(s)) : s \in E\}$ be the graph of φ and let $p(s, \varphi(s)) = s$, $q(s, \varphi(s)) = \varphi(s)$ be the projections of X onto E and $\mathbf{R}$, respectively. Both p and q are continuous, and therefore $F = \{(x, x') : p(x)Lp(x') \text{ and } q(x) \geq q(x')\}$ is a closed relation on X.

Define $R = F \cup F^{-1}$. Observe that if $A \subset X$ is R-homogeneous then φ is L-decreasing on $p(A)$ and if A is R-independent then φ is strictly L-increasing on $p(A)$.

Now we apply Theorem 1.1, with f being the identity, to the relation R. Since the projection p parallel to the real line takes σ-discrete collections to collections with σ-discrete refinements, the assertion of Theorem 1.1 translates readily to the dichotomy in Corollary 3.2. □

Remark 3.3 Condition (ii) in Corollary 3.2 can not be strengthened by demanding that φ is strictly increasing on a perfect subset of E. To see this let us split the real line into Bernstein sets A, B, cf. [Ku66, §40, I], and let $\varphi : \mathbf{R} \to \mathbf{R}$ be defined by $\varphi(x) = x$ for $x \in A$ and $\varphi(x) = -x$ for $x \in B$.

Remark 3.4 We have included enough of Hurewicz's arguments in Proposition 2.1 to recover from its assertion yet another celebrated Hurewicz's result: metrizable spaces of first category contain closed copies of the rationals, cf. [Hu28, p. 88].

To see this, let us consider $X = \bigcup_{i>0} F_i$ such that each F_i is a closed subset of X with empty interior. Represent each F_i as the union of a locally finite collection $\mathcal{A}_i$ of closed sets with the diameter not greater than $1/i$. The relation $R = \bigcup\{A \times A : A \in \bigcup_{i>0} \mathcal{A}_i\}$ is closed and symmetric. Applying Proposition 2.1, with f being the identity (and $D = X$), one gets a perfect R-independent set P in X. One can assume, without loss of generality, that P is separable, and then P hits only countably many elements of the collection $\bigcup_i \mathcal{A}_i$. Since P is R-independent, each intersection contains exactly one point. It follows that P is a closed copy of the rationals in X.

A slightly more direct approach, resembling the one from [Hu28, p. 88], is to modify the proof of Proposition 2.1. If F_i are closed boundary sets in X, then in the proof of Proposition 2.1, one can choose the points of $A_{n+1} \setminus A_n$ and the corresponding neighbourhoods $U_{n+1}(\widetilde{x})$ outside of $\bigcup_{i\leq n} F_i$. This construction can be used to show that any regular space of first category which satisfies the first axiom of countability contains a closed copy of the rationals, cf. [vD87] and [De88].

4 Closed relations in the Baire space $B(\aleph_1)$ and Lusin's constituents.

The Baire space $B(\aleph_1)$ is the countable product of the discrete space of cardinality $\aleph_1$. We shall consider points of $B(\aleph_1)$ as functions $x : \mathbf{N} \to \omega_1$. The restriction of $x \in B(\aleph_1)$ to the set $\{0, \dots, n-1\}$ will be denoted by $x|n$.

We define

$$\kappa(x) = \min\{\alpha : x(\mathbf{N}) \subset [0, \alpha)\}. \tag{1}$$

The function $\kappa : B(\aleph_1) \to \omega_1$ determines a stratification of

$B(\aleph_1)$ into layers

$$B_\xi = \kappa^{-1}(\{\xi\}), \tag{2}$$

cf. [ChGP98, Sec. 2] for a discussion of the stratification.

Let us recall that a set of countable ordinals is stationary if it intersects each closed unbounded set in ω_1.

Theorem 4.1 *Let M be a closed subset of $B(\aleph_1)$ and let R be a closed symmetric relation on M. Then either*

(i) *M is the union of a σ-discrete collection of sets which are either R-homogeneous or separable,*
or else

(ii) *for all but non-stationary many ξ the trace $B_\xi \cap M$ contains an R-independent Cantor set.*

Proof: let $\mathcal{I}$ be the collection of all subsets of M which are either R-homogeneous or separable. By the transfinite exhaustion procedure similar to that described in the proof of Theorem 1.1, one can write $M = J \cup K$, where J is the union of a σ-discrete subcollection of $\mathcal{I}$, and K is a closed set with no nonempty relatively open subset in $\mathcal{I}$.

If $K = \emptyset$, we get (i). Assume otherwise. Then K is non-separable at each point and by a characterization of Stone [St62, Sec. 2], there exists a homeomorphism h of $B(\aleph_1)$ onto K. By [ChGP98, Lemma 2.4],

$$h(B_\xi) \subset B_\xi \text{ for all but non-stationary many } \xi. \tag{3}$$

Let $\widetilde{R} = (h \times h)^{-1}(R \cup \Delta)$. Then no nonempty open set in $B(\aleph_1)$ is $\widetilde{R}$-homogeneous, i.e., no point of the diagonal is in the interior of $\widetilde{R}$.

From [ChGP98, Lemma 2.8] one infers that for all but non-stationary many ξ, no point of the diagonal of B_ξ is in the interior of $\widetilde{R} \cap (B_\xi \times B_\xi)$ relatively to $B_\xi \times B_\xi$. Therefore, by a modification of Mycielski's theorem [My64, Theorem 1] indicated in [NP96], cf. [ChGP98, Lemma2.6], for each such ξ, there exists an $\widetilde{R}$-independent Cantor set $C_\xi \subset B_\xi$. This, together with (3), implies (ii). $\square$

If possibility (i) in Theorem 4.1 fails, (ii) provides us with a large collection of separable R-independent sets. We can not expect, however, to get a non-separable independent set, even if (i) is violated in a strong way. This is illustrated by the following example, which bears some resemblance to the example described in [SS??, Theorem 4.3].

Example 4.2 *There exists a closed symmetric relation R on $B(\aleph_1)$ such that*

(*) each R-homogeneous set intersects every layer B_ξ in at most a singleton,

and

(**) each R-independent set is separable.

To see this let, cf. (1),

$$(4) \quad F = \{(x,y) : \text{ for some } n, \quad x|n = y|n \text{ and } \kappa(x) \leq y(n)\}.$$

Let us check that

$$(5) \qquad \Delta \cup F \text{ is closed in } B(\aleph_1) \times B(\aleph_1).$$

To this end choose $(x,y) \notin \Delta \cup F$. Let n be the first number with $x(n) \neq y(n)$. By (4) $\kappa(x) > y(n)$, so there exists an $m \geq n$ such that $x(m) \geq y(n)$. Thus the neighbourhood of (x,y) determined by $x|(m+1)$ and $y|(n+1)$ misses $\Delta \cup F$.

Put

$$(6) \qquad R = \Delta \cup F \cup F^{-1}$$

By (5) R is a closed relation. If $x \neq y$ are in B_ξ for some ξ, then (4) and (2) show that $\{x,y\}$ is R-independent, hence $(*)$ is satisfied. It remains to check $(**)$.

Let us consider a non-separable set P in $B(\aleph_1)$. We claim that there exist an $n \geq 0$ and a point x in P such that

$$\sup\{y(n) : y \in P \text{ and } x|n = y|n\} = \omega_1. \tag{7}$$

Otherwise, one could pick inductively $\alpha_0, \alpha_1, \ldots$ so that $x|n$ is in $[0,\alpha_0] \times [0,\alpha_1] \times \ldots [0,\alpha_{n-1}]$ for all x in P. But then κ is bounded on P, contradicting non-separability of P.

Let x in P be as in (7). Pick y in P with $x|n = y|n$ and $y(n) > \kappa(x)$. Then (x, y) is in F. □

Now, as promised in the introduction, we shall apply Theorem 4.1 to closed relations on coanalytic sets, or more precisely, to stratifications of coanalytic sets into Lusin's constituents.

Let $\mathbf{Q}$ be the set of rational numbers, let $2^{\mathbf{Q}}$ be the Cantor space of all subsets of $\mathbf{Q}$ with the topology of pointwise convergence and let

$$WO = \{A \in 2^{\mathbf{Q}} : A \text{ is well-ordered}\}.$$

Let $E \subset \mathbf{N}^{\mathbf{N}}$ be a coanalytic set of irrationals. Each Borel map $\phi : \mathbf{N}^{\mathbf{N}} \to 2^{\mathbf{Q}}$ such that $\phi^{-1}(WO) = E$ (which can be identified with a Borel Lusin sieve through which $\mathbf{N}^{\mathbf{N}} \setminus E$ is sifted, cf. [ChGP95, 6.1]) determines a Lusin-Sierpiński index $\delta : E \to \omega_1$, where $\delta(x)$ is the order type of $\phi(x)$, and

$$E_\xi = \delta^{-1}(\{\xi\})$$

is the ξth constituent of E corresponding to the index.

Our application will be based on a link between the constituents and the stratification of the Baire space $B(\aleph_1)$ into layers (2). The link is provided by the following Proposition which can be derived, with some minor adjustments, from [ChGP95], Lemma 2.1, Remark 5.1 and Comment 6.1.

Proposition 4.3 *Let E be a coanalytic subset of $\mathbf{N}^{\mathbf{N}}$ and let E_ξ be the constituents of E corresponding to a Lusin-Sierpiński* *index on E. Then there exists a closed set M in $B(\aleph_1)$ and a continuous map $\pi : M \to E$ such that*

$$\pi^{-1}(E_\xi) = B_\xi \cap M \text{ for } \xi \geq \omega. \tag{8}$$

We shall say that a pairwise disjoint collection $\mathcal{E}$ of subsets of E is *δ-dissipated* if each selector for $\mathcal{E}$ intersects only non-stationary many constituents, each in at most countable set.

Any selector of a σ-discrete collection $\mathcal{A}$ of subsets of $B(\aleph_1)$ intersects only non-stationary many layers B_ξ, each in at most countable set, cf. [ChGP98, Lemma 2.1], [Po77, Theorem 1]. Thus the map π from Proposition 4.3 transfers σ-discrete collections of subsets of $B(\aleph_1)$ to δ-dissipated collections of subsets of E (one has to shrink the images to make them pairwise disjoint, and this can be done without altering the union of the collection).

Corollary 4.4 *Let R be a closed symmetric relation on a co-analytic set $E \subset \mathbf{N}^{\mathbf{N}}$. Then for any Lusin-Sierpiński index δ on E, either*

(i) *there is a δ-dissipated collection of R-homogeneous sets which covers all but non-stationary many constituents E_ξ,*
or else

(ii) *all but non-stationary many constituents E_ξ contain an R-independent Cantor set.*

Proof: Let π and M be as in Proposition 4.3. Define

$$\widetilde{R} = (\pi \times \pi)^{-1}(R \cup \Delta)$$

and let us apply Theorem 4.1 to the relation $\widetilde{R}$. Since π is one-to-one on any $\widetilde{R}$-independent set, condition (ii) in Theorem 4.1 implies (ii), cf. (8).

Assume that (i) in Theorem 4.1 is satisfied. Then there exists a σ-discrete collection $\mathcal{A}$ of $\widetilde{R}$-homogeneous sets which covers all but non-stationary many traces $B_\xi \cap M$, cf. [ChGP98, Lemma 2.2], [Po77, Theorem 1]. Thus the observation following the definition of δ-dissipated collections gives (i). $\square$

Remark 4.5 The Feng's result concerning analytic spaces, cited in the introduction, can be formally derived from Corollary 4.4.

Indeed, let $E = \mathbf{N}^{\mathbf{N}} \times WO$ and let p, q be the projections of E onto $\mathbf{N}^{\mathbf{N}}$ and WO, respectively. Define a Lusin-Sierpiński index δ on E by putting $\delta(x) = \text{type}(q(x))$ for $x \in E$.

If R is a closed symmetric relation on $\mathbf{N}^{\mathbf{N}}$, then Corollary 4.4 applied to the relation $\widetilde{R} = (p \times p)^{-1}(R \cup \Delta)$ on E gives Feng's dichotomy for R.

Remark 4.6 As pointed out by Feng [Fe93], in any model of set theory with an uncountable coanalytic set not containing any Cantor subset, his dichotomy fails for C and R being the diagonal. For any Lusin-Sierpiński index δ on such C, the range $\delta(C)$ is non-stationary, cf. [ChGP95, 6.1]. One can, however, use C to define $\widetilde{C}$ and a Lusin-Sierpiński index $\widetilde{\delta}$ on $\widetilde{C}$ so that $\widetilde{\delta}(\widetilde{C})$ is stationary, and still Feng's dichotomy is not true for $\widetilde{C}$.

To see this, let δ be a Lusin-Sierpiński index on C. Set $\widetilde{C} = C \times WO$ and notice that the rank $\widetilde{\delta} : \widetilde{C} \to \omega_1$ defined by $\widetilde{\delta}(x, A) = \delta(x) + \text{type}(A)$ is a Lusin-Sierpiński index on $\widetilde{C}$.

Let p denote the projection of $\widetilde{C}$ onto C. The relation $\widetilde{\Delta} = (p \times p)^{-1}(\Delta)$ on $\widetilde{C}$ witnesses the failure of Feng's dichotomy for $\widetilde{C}$.

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Warsaw University
Warsaw, Poland
email address: chaber@mimuw.edu.pl
pol@mimuw.edu.pl