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A NORMAL SPACE Z WITH INDZ = 1 NO COMPACTIFICATION OF WHICH HAS TRANSFINITE DIMENSION

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Abstract

We construct a first countable, countably paracompact and collectionwise normal space Z with ind$Z = 1$ such that every Lindelöf (or even strongly paracompact) extension of Z has small transfinite inductive dimension, trind, equal to ∞.

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1 Introduction

It is well known that a separable metrizable space X has a metrizable compactification Y with ind$Y$ = ind$X$. However, for more general spaces there is no compactification theorem for ind : van Mill and Przymuniński [5] constructed a perfectly normal first countable space X^* such that ind$X^* = 1$ while ind$Y = \infty$ for every Lindelöf extension Y of X^*. While there is no compactification theorem for trind in the class of separable metrizable spaces, it is known that a separable metrizable space X with trind$X < \infty$ has a metrizable compactification Y with trind$Y < \infty$ [3, problems 7.2.H and 7.2.I]. Recently, however, Kimura [4] constructed a Tychonoff, but not normal, space X with ind$X = 1$ no compactification of which has trind, and asked whether a normal space with similar properties exist. The space Z constructed here answers the question.

Both Kimura's space and ours are obtained by an appropriate modification of the construction of [5]. The construction of the space Z with the properties announced in the abstract, which is carried out in section 3, requires the space described in section 2.

All spaces in this paper are Tychonoff, I denotes the unit interval [0,1], N the natural numbers, $\omega = \{0\} \cup N$, $|X|$ is the cardinality of a set X, $c = |I|$ and $\omega(c)$ is the first ordinal of cardinality c.

2 A zero-dimensional, first countable, collectionwise normal and countably paracompact space Y which is strongly infinite-dimensional

An example of such a space is the one constructed in [1], which has the additional property that it is the limit of an inverse sequence of strongly zero-dimensional Lindelöf spaces. For the convenience of the reader, we describe here a direct construction of such a space. Both here and in [1], we apply van Douwen's technique of assigning limit points to appropriately selected sequences.

Let $\{S_\alpha = (S_{\alpha 1}, S_{\alpha 2}, \ldots) : \alpha < \omega(c)\}$ be an enumeration of all sequences $S = (S_1, S_2, \ldots)$, where each S_i is a countable subset of I^ω with $|\bigcap_{i\in N} \mathrm{cl} S_i| = c$. Let $\prec$ be a well-ordering on I^ω of the same type as $\omega(c)$. For each $\alpha < \omega(c)$, using transfinite induction, fix a point x_α in $\bigcap_{i\in N} \mathrm{cl} S_{\alpha i}$ and a sequence $\{x_{\alpha n}\}$ consisting of infinitely many points from each $S_{\alpha i}$ such that $d(x_\alpha, x_{\alpha n}) \leq 1/n$ and $y \prec x_\alpha$ for $y \in \{x_\beta : \beta < \alpha\} \cup \bigcup\{S_{\beta i} : \beta \leq \alpha, i \in N\}$. We then define for each x in I^ω basic open neighbourhoods $B_n(x)$, $n \in N$, of a new and finer topology on I^ω, so that $d(x, y) \leq 1/n$ for each point y of $B_n(x)$. This is done as follows. For x not of the form x_α, we set $B_n(x) = \{x\}$. For points of the form x_α the definition is by transfinite induction: Assuming this has been done for all $x \prec x_\alpha$, we set $B_n(x_\alpha) = \{x_\alpha\} \cup \bigcup_{k\geq 2n} B_k(x_{ak})$.

Y denotes I^ω with its new topology. Evidently, Y is Hausdorff and first countable, and one readily checks that each $B_n(x)$ is countable and compact, and hence closed in Y. Hence Y is zero-dimensional and therefore Tychonoff. It is also a fact that $x_\alpha = \lim x_{\alpha n}$ and if $\bigcap_{i\in N} E_i = \emptyset$, where each E_i is closed in Y, then $\bigcap_{i\in N} \mathrm{cl} E_i$ is countable, where cl denotes closure in I^ω. One can then prove, as in [1], that Y is normal, countably paracompact and collectionwise normal.

For each i in N, let $f_i : I^\omega \to I$ denote the i^{th} canonical projection. Let $A_i = f_i^{-1}[0, 1/4]$ and $B_i = f_i^{-1}[3/4, 1]$ and suppose M_i is a partition between A_i and B_i in Y. Then $M_i = E_i \cap F_i$ for some closed sets E_i, F_i of Y such that $E_i \cap A_i = F_i \cap B_i = \emptyset$ and $Y = E_i \cup F_i$. Consequently, $\mathrm{cl} E_i \cap \mathrm{cl} F_i$ is a partition in I^ω between $f_i^{-1}(0)$ and $f_i^{-1}(1)$, which implies that $|\bigcap_{i\in N} \mathrm{cl} E_i \cap \mathrm{cl} F_i| = c$. It follows that $\bigcap_{i\in N} M_i \neq \emptyset$ and $Y$is strongly infinite-dimensional.

3 The space Z

In the sequel, the symbols Y, A_i, B_i and f_i will have the meaning they were given in section 2.

Let $\{S_\alpha : \alpha < \omega(c)\}$ be an enumeration of all countable subsets of the Cantor set C with $|\mathrm{cl}S_\alpha| = c$. Let $\prec$ be a well-ordering on C of the same type as $\omega(c)$. For each $\alpha < \omega(c)$ and each $i \in N$, using transfinite induction, pick a point $x_{\alpha i}$ of $\mathrm{cl}S_\alpha$ and a subsequence $S_{\alpha i} = \{x_{\alpha i n} : n \in N\}$ of S_α such that $d(x_{\alpha i}, x_{\alpha i n}) \leq 1/n$ and $y \prec x_{\alpha i}$ for $y \in \{x_{\beta j} : \beta < \alpha, j \in N\} \cup \{x_{\alpha j} : j < i\} \cup \bigcup\{S_\beta : \beta \leq \alpha\}$.

Let $Z = C \times (Y \cup \{\theta\})$, where θ is a point outside Y. For $x \in C$ and $n \in N$, let

$$U(x,n) = B(x, 1/n) \times (Y \cup \{\theta\}) - \{x\} \times Y$$

where $B(x, 1/n) = \{y \in C : d(x,y) < 1/n\}$. The topology of Z is defined as follows. All sets of the form $\{x\} \times G$, where G is open in Y, are open in Z. Points (x, θ), where x is not of the form $x_{\alpha i}$, have as basic open neighbourhoods the sets $U(x,n)$. Finally, points (x_{ai}, θ) have as basic open neighbourhoods the sets of the form

$$V(x_{\alpha i}, n) = U(x_{\alpha i}, n) - S_{\alpha i} \times f_i^{-1}[1/4 + 1/n, 1].$$

One readily sees that Z is first countable and Tychonoff and $C \times \{\theta\}$ is homeomorphic with C. Also, points of Z have open neighbourhoods with boundary in $C \times Y$, which is a direct sum of copies of the zero-dimensional space Y. Hence $\mathrm{ind}Z \leq 1$. That, in fact, $\mathrm{ind}Z = 1$ will follow from proposition 2 and the fact that every zero-dimensional space has a zero-dimensional compactification.

Proposition 1. *Z is collectionwise normal and countably paracompact.*

Proof: Let $\{F_s : s \in S\}$ be a discrete collection of closed sets of Z. Put $T = \{s \in S : F_s \cap (C \times \{\theta\}) \neq \emptyset\}$. As $C \times \{\theta\}$ is compact, for each t in T, we can construct an open neighbourhood G_t of $F_t \cap (C \times \{\theta\})$ in Z such that $F_s \cap \mathrm{cl}G_t = \emptyset$ for $s \neq t$. Note that, because $C \times \{\theta\}$ is compact and $\{F_s : s \in S\}$ is discrete, T is finite. For $s \notin T$, we set $G_s = \emptyset$. Let $\{U_s : s \in S\}$ be a collection of mutually disjoint open sets of the collectionwise normal space $C \times Y$ such that $F_s \cap (C \times Y) \subset U_s$. Finally, put

$$V_s = U_s \cup G_s - \bigcup \{\mathrm{cl}G_t : t \in S, t \neq s\}.$$

Then, for each s in S, V_s is an open neighbourhood of F_s in Z such that $V_s \cap V_t = \emptyset$ for $s \neq t$. This shows that Z is collectionwise normal.

Let $\{F_n : n \in N\}$ be a decreasing sequence of closed sets of Z such that $\bigcap_{n \in N} F_n = \emptyset$. To prove that the normal space Z is countably paracompact, it suffices to find, for each n in N, an open neighbourhood G_n of F_n in Z such that $\bigcap_{n \in N} G_n = \emptyset$ [2, corollary 5.2.2]. As $C \times \{\theta\}$ is compact, there is k in N such that, for $n \geq k, F_n$ is contained in the countably paracompact and normal space $C \times Y$. Consequently, for $n \geq k$, an open neighbourhood G_n of F_n in Z can be chosen so that $\bigcap_{n \geq k} G_n = \emptyset$. To complete the proof, it suffices to set $G_n = Z$ for $n < k$. □

Proposition 2. *Let X be a strongly paracompact space containing Z. Then trindX* $= \infty$.

Proof: Following [5], we say that a point x of C *separates*A_i, B_i in X if

$$\mathrm{cl}(\{x\} \times A_i) \cap \mathrm{cl}(\{x\} \times B_i) = \emptyset,$$

where cl denotes closure in X. Let D_i be the set of all points of C that do not separate A_i, B_i.

For each $\alpha < \omega(c)$ and each $i \in N, S_{\alpha i} \times B_i$ is a closed set of Z that does not contain $(x_{\alpha i}, \theta)$. Hence, for some n in N, $\text{cl}V(x_{\alpha i}, n) \cap \text{cl}(S_{\alpha i} \times B_i) = \emptyset$. This implies that only a finite number of members of $S_{\alpha i}$ belong to D_i. We claim that D_i is countable: if it is uncountable, for some $\alpha < \omega(c)$, it contains S_α as a dense subset, and hence $S_{\alpha i} \subset D_i$. It follows that there is at least one point x of C that separates A_i, B_i for each i in N.

Suppose M_i is a partition between $\text{cl}(\{x\} \times A_i)$ and $\text{cl}(\{x\} \times B_i)$ in X for each i in N. Then, $M_i \cap (\{x\} \times Y)$ is a partition in $\{x\} \times Y$ between $\{x\} \times A_i$ and $\{x\} \times B_i$. It follows from section 2 that $\bigcap_{i \in N} M_i \neq \emptyset$ and X is strongly infinite-dimensional. Because X is strongly paracompact, this implies $\text{trind}X = \infty$ [3, problem 7.1.F]. □

Added in proof: Modifying slightly the construction of section 3 and using instead of the space of section 2 the direct sum Y of perfectly normal spaces Y_n with $\text{ind}Y_n = 0$ and dim $Y_n = n$ (E. Pol and R. Pol, A hereditarily normal strongly zero-dimensional space containing subspaces of abitrarily large dimension, Fund. Math. 102 (1979), 137-142), Elzbieta Pol has contructed a perfectly normal space X with $\text{ind}X = 1$ such that $\text{trind}S = \infty$ for every compactification S of X.

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