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# UNIVERSALITY OF WEAKLY ARC-PRESERVING MAPPINGS

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## Abstract

We investigate relationships between confluent, semi-confluent, weakly confluent, weakly arc-preserving and universal mappings between continua, especially onto trees and dendrites.

A *continuum* means a compact, connected metric space, and a *mapping* means a continuous transformation. A mapping  $f : X \rightarrow Y$  between topological spaces  $X$  and  $Y$  is said to be *universal* provided that it has a coincidence with every

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mapping from  $X$  into  $Y$ , or — more precisely — provided that for every mapping  $g : X \rightarrow Y$  there exists a point  $x \in X$  such that  $f(x) = g(x)$ . Obviously any universal mapping must be surjective. The concept of a universal mapping has been introduced in [16, p. 603] by W. Holsztyński.

Questions concerning universal mappings are related to fixed point questions. If there exists a universal mapping from  $X$  onto  $Y$ , then for any mapping  $h : Y \rightarrow Y$  defining  $g : X \rightarrow Y$  by  $g = h \circ f$  we get, by the universality of  $f$ , a point  $x \in X$  such that  $g(x) = h(f(x)) = f(x)$ . Putting  $y = f(x) \in Y$  we get  $h(y) = y$ , so the mapping  $h$  has a fixed point. Thus we have the following (well known) result.

**1. Statement** *A universal mapping from  $X$  onto  $Y$  can exist only if  $Y$  has the fixed point property.*

Holsztyński has proved that each mapping from a connected space onto a chainable continuum is universal [17, Theorem 3, p. 437]. Chainability is essential in this result: one can easily define a mapping from an arc onto a simple triod which is not universal.

An arc and a simple triod are the simplest examples of trees. By a *tree* is meant a linear graph (i.e., the union of a finite collection of arcs) containing no simple closed curve.

A. D. Wallace has shown that any *monotone* mapping (i.e., a mapping with connected point-inverses) from a continuum onto a tree is universal [33, (D), p. 759]. One can ask for what (larger) classes of mappings this result is true. Recall that a mapping  $f : X \rightarrow Y$  between continua is said to be *weakly monotone* provided that for every subcontinuum  $Q$  of  $Y$  such that  $\text{int } Q \neq \emptyset$  and for every component  $C$  of  $f^{-1}(Q)$  the equality  $f(C) = Q$  holds. H. Schirmer (who renamed universal mappings coincidence producing ones) has generalized Wallace's result in [31, Theorem 1, p. 418] proving that if a mapping  $f : X \rightarrow Y$  from a continuum  $X$  onto a tree  $Y$  is weakly monotone, then  $f$  is universal.

Recall that if the domain space  $X$  is a locally connected continuum, then weakly monotone mappings onto  $Y$  coincide with quasi-monotone ones as well as with OM-mappings (i.e., compositions of monotone and open ones), see e.g. [22, (6.2), p. 51]. A wider class of mappings is that of confluent ones. A mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  is said to be *confluent* provided that for every subcontinuum  $Q$  of  $Y$  and for every component  $C$  of  $f^{-1}(Q)$  the equality  $f(C) = Q$  holds. Since for trees  $Y$  the only subcontinua with the empty interior are singletons, weakly monotone mappings onto trees coincide with confluent ones. So, Schirmer's result can be reformulated as follows.

**2. Theorem** (Schirmer) *If a mapping  $f : X \rightarrow Y$  from a continuum  $X$  onto a tree  $Y$  is confluent, then  $f$  is universal.*

One can try to generalize Theorem 2 extending the class of continua to which the range space belongs. Assuming that  $Y$  is one-dimensional, one can consider the following. A continuum  $Y$  is said to be *tree-like* provided that it is the inverse limit of trees. It is known that each tree-like continuum is *hereditarily unicoherent*, i.e., the intersection of every two of its subcontinua is connected. A hereditarily unicoherent and hereditarily decomposable continuum is called a  $\lambda$ -*dendroid*; if it is additionally arcwise connected, then it is called a *dendroid*. A locally connected continuum containing no simple closed curve is named a *dendrite*. Equivalently, a dendrite is a locally connected dendroid. If a dendrite has finitely many end points, then it is a tree. A continuum is said to be *hereditarily arcwise connected* provided that each of its subcontinua is arcwise connected.

**3. Questions** Let  $f : X \rightarrow Y$  be a surjective mapping between continua. Under what conditions about  $f$  and about  $Y$  the mapping  $f$  is universal? In particular, is  $f$  universal if  $f$  satisfies some conditions related to confluence and  $Y$  is a) a dendrite, b) a dendroid, c) a  $\lambda$ -dendroid, d) a tree-like

continuum having the fixed point property?

Note that each  $\lambda$ -dendroid has the fixed point property [23], while tree-like continua are known without this property [1]. Thus the restriction in Question 3 d) to tree-like continua  $Y$  having the fixed point property is necessary by Statement 1.

Relationships between universality of the mapping and various conditions expressed in terms associated to confluence of the mapping are known from the literature. Some of them are recalled in the paper. For new results concerning the implications mentioned in Questions 3 see below Fact 5, Theorems 25, 35, 39 and 43, Corollaries 36, 37 and 44, and Examples 15, 33 and 41.

Classes of mappings that contain confluent mappings are the classes of semi-confluent and of weakly confluent mappings.

A mapping  $f : X \rightarrow Y$  between continua is said to be:

- *semi-confluent* provided that for every subcontinuum  $Q$  of  $Y$  and for every two components  $C_1$  and  $C_2$  of  $f^{-1}(Q)$  either  $f(C_1) \subset f(C_2)$  or  $f(C_2) \subset f(C_1)$ ;
- *weakly confluent (pseudo-confluent)* provided that for every (irreducible) subcontinuum  $Q$  of  $Y$  there is a continuum  $C$  in  $X$  such that  $f(C) = Q$ .

Thus each confluent mapping is semi-confluent, each semi-confluent is weakly confluent (see [22, Theorem (3.8), p. 13]), and each weakly confluent is pseudo-confluent. Properties of weakly confluent mappings between continua (on graphs, in particular) were investigated e.g. in [10], [26], [32], and in many other articles. Relations between universal mappings and weakly confluent ones, especially for mappings between trees, were studied in [27], [8], [24], [25], and in some other papers. A characterization of universal mappings between trees has been obtained by Eberhart and Fugate in [9].

Among other results the following theorem is known [27, Corollary 2.6, p. 225].

**4. Theorem (Nadler)** *Each universal mapping from a contin-*

um onto a locally connected continuum is weakly confluent.

Local connectedness of the range space is essential in the result: examples are constructed in [27, (2.14), (2.16) and Remark (2.17), p. 227-229] of universal mappings between plane curves that are not pseudo-confluent even. Further, the result cannot be strengthened to get semi-confluence in the conclusion, as the next fact shows.

**5. Fact** *There is a universal mapping  $f : [0, 1] \rightarrow [0, 1]$  which is not semi-confluent.*

**Proof:** The mapping  $f$  defined by

$$f(x) = \begin{cases} 1/4 - x, & \text{for } x \in [0, 1/4], \\ 2x - 1/2, & \text{for } x \in (1/4, 3/4), \\ 7/4 - x, & \text{for } x \in [3/4, 1] \end{cases}$$

has the needed properties.

Concerning the opposite implication, from weak confluence to universality of a mapping onto a locally connected continuum, Nadler constructs in [27, Example 3.3, p. 231] an example of a monotone mapping between 3-cubes that is not universal, and gives some conditions under which a weakly confluent mapping onto an  $n$ -cube is universal. Further, for mappings between trees (i.e., when both domain and range are trees), recall the following result, which was announced in [8, Theorem 3, p. 212], and which has been proved in [25, Theorem 4.9, p. 808].

**6. Theorem** (Eberhart and Fugate, Marsh) *Each weakly confluent self-mapping on a tree is universal.*

Theorems 4 and 6 imply the following result.

**7. Corollary** *Let  $X$  be a tree. A self-mapping  $f : X \rightarrow X$  is universal if and only if it is weakly confluent.*

An example is constructed in [8, Example 2, p. 213] of a weakly confluent mapping between different trees which is not universal (see also [24, Example 2, p. 385]. We repeat its construction here, because we also will use it later, to describe other examples.

**8. Example** (Eberhart and Fugate, Marsh) *There exists a weakly confluent mapping between different trees which is not universal.*

**Proof:** Denote by  $xy$  the straight line segment in the plane with end points  $x$  and  $y$ . Take a straight line segment  $pq$ , and let segments  $ab$  and  $cd$  be perpendicular to  $pq$  so that  $p \in ab \setminus \{a, b\}$  and  $q \in cd \setminus \{c, d\}$ . Put  $H = pq \cup ab \cup cd$ . Let  $T$  be a triod with the vertex  $z$  whose arms are straight line segments  $zu$ ,  $zv$  and  $zw$ . To describe the needed mappings pick up points  $r$ ,  $s$  and  $t$  in the segment  $pq \subset H$  ordered so that  $p < r < s < t < q$ , and a point  $y \in zw \setminus \{z, w\} \subset T$ . Define  $f : H \rightarrow T$  and  $g : H \rightarrow T$  as piecewise linear mappings determined by their values on the mentioned points of  $H$  as follows.

$$\begin{aligned} f(a) &= w, & f(b) &= f(c) = u, & f(d) &= f(r) = y, \\ f(p) &= f(q) = f(s) = z, & f(t) &= v; \\ g(a) &= g(b) = g(p) = v, & g(c) &= g(d) = g(q) = g(t) = w, \\ g(s) &= u, & g(r) &= z. \end{aligned}$$

The reader can verify that  $f$  is weakly confluent, and that  $f$  and  $g$  have no coincidence point.

Let us mention that, according to the remark in [8, Example 2, p. 213], one can modify the above example in such a way that both mappings between some trees are weakly confluent, and still not have a coincidence point. The mapping in Example 8 is not semi-confluent. A stronger example is shown below, in Example 15: there is a semi-confluent mapping between trees which is not universal.

So, as it is observed in [8, p. 213], in the light of Examples 8 and 15 it is clear that if we wish to conclude that a mapping between trees is universal, we need a stronger property than weak confluence (ever stronger than semi-confluence). A sufficiently strong condition is given by the notion of a weakly arc-preserving mapping, [8, p. 213].

A mapping  $f : X \rightarrow Y$  between trees is said to be *arc-preserving* provided that it is surjective, and for each arc  $A \subset X$  its image  $f(A)$  is either an arc or a point; it is *weakly arc-preserving* provided that there is a subtree  $X'$  of  $X$  such that the restriction  $f|X' : X' \rightarrow Y$  is arc-preserving.

It is announced in [8, Theorem 5 (a), p. 213] that every surjective confluent mapping between trees is weakly arc-preserving. The result is a consequence of some more general one: the same conclusion holds for wider classes of spaces. The next theorem show this. However, to formulate it, we have to extend the concepts of an arc-preserving and a weakly arc-preserving mapping to the case when both the domain and the range spaces belong to wider classes than one of trees. The former extension is known, because the concept of an arc-preserving mapping has already been considered on arbitrary (mainly locally connected) continua, not necessarily on trees, see [34], [13], [14], and [15] for example. Thus, as it is used in the above quoted papers, a mapping between arcwise connected continua is arc-preserving if it is surjective, and for each arc in the domain its image is either an arc or a point. Along the same lines of ideas the definition of a weakly arc-preserving mapping can be extended to be applied to arbitrary continua. So, let us accept the following definition. A mapping  $f : X \rightarrow Y$  between continua is said to be:

- *arc-preserving* provided that it is surjective, and for each arc  $A \subset X$  its image  $f(A)$  is either an arc or a point;
- *weakly arc-preserving* provided that there is an arcwise connected subcontinuum  $X'$  of  $X$  such that the restriction  $f|X' : X' \rightarrow Y$  is arc-preserving.



Note that a mapping can be arc-preserving and not weakly arc-preserving: see an example in Remark 17 b) below.

The following statement is a consequence of the definitions.

**9. Fact** *If  $X$  is an arcwise connected continuum, then each arc-preserving mapping defined on  $X$  is weakly arc-preserving.*

A mapping  $f : X \rightarrow Y$  is called an *OM-mapping* provided that it is the composition of a monotone and an open mapping, i.e., there exist a space  $Z$  and mappings  $f_1 : X \rightarrow Z$  and  $f_2 : Z \rightarrow Y$  such that  $f_1$  is monotone,  $f_2$  is open, and  $f = f_2 \circ f_1$ . Note that since each monotone as well as each open mapping between compact spaces is confluent, and since composition of two confluent mappings is confluent (see [5, V, VI and III, p. 214]), it follows that each OM-mapping is confluent. The inverse implication holds if the range  $Y$  is a locally connected continuum, [20, Corollary 5.2, p. 109].

To show the mentioned theorem we need one more result. The result, besides its application in the proof of Theorem 11 below, is interesting by itself. It resembles a mapping characterization of hereditarily unicoherent continua saying that a continuum is hereditarily unicoherent if and only if every monotone mapping defined on it is hereditarily monotone, see [22, (6.10), p. 53].

**10. Theorem** *Every monotone mapping defined on a hereditarily arcwise connected continuum onto a continuum containing no simple closed curve is hereditarily monotone.*

**Proof:** Let  $f : X \rightarrow Y$  be a monotone mapping from a hereditarily arcwise connected continuum onto a continuum  $Y$  containing no simple closed curve. Suppose on the contrary that  $f$  is not hereditarily monotone, i.e., that there exists a subcontinuum  $C$  of  $X$  such that  $f|C : C \rightarrow f(C) \subset Y$  is not monotone. Thus there is a point  $q \in f(C)$  such that  $C \cap f^{-1}(q)$  is not

connected. Since  $C$  is arcwise connected, one can find an arc  $A \subset C$  with end points  $a, b$  such that  $A \cap C \cap f^{-1}(q) = \{a, b\}$  and  $a, b$  belong to two different components of  $C \cap f^{-1}(q)$ . Fix a point  $p \in A \setminus \{a, b\}$  and observe the following.

CLAIM. There exists a sequence of points  $\{q_n\}$  in  $Y \setminus \{q\}$  such that  $\lim q_n = q$  and

$$ap \cap f^{-1}(q_n) \neq \emptyset \neq bp \cap f^{-1}(q_n) \quad \text{for each } n \in \mathbb{N},$$

where  $ap$  and  $bp$  are subarcs of  $A$  with the respective end points.

Indeed, otherwise one could find points  $r_1 \in ap$  and  $r_2 \in bp$  such that  $f(r_1) = f(r_2)$  and that for each pair of points  $x_1$  and  $x_2$  with  $x_1 \in ar_1 \setminus \{a, r_1\} \subset A$  and  $x_2 \in br_2 \setminus \{b, r_2\} \subset A$  we have  $f(x_1) \neq f(x_2)$ . Hence the set  $f(ar_1) \cup f(br_2)$  would contain a simple closed curve, a contradiction. Thus the claim is shown.

Let the symbol  $\preceq$  stand for the ordering of the arc  $ap$  from  $p$  to  $a$  and for the ordering of the arc  $bp$  from  $p$  to  $b$ . Further, for each  $n \in \mathbb{N}$ , let  $a_n$  be the first ( $a'_n$  be the last) point of  $ap \cap f^{-1}(q_n)$  with respect to this ordering on  $ap$ . Similarly, let  $b_n$  be the first ( $b'_n$  be the last) point of  $bp \cap f^{-1}(q_n)$  with respect to  $\preceq$  on  $bp$ . We obviously have  $\lim a_n = \lim a'_n = a$  and  $\lim b_n = \lim b'_n = b$ . Replacing the sequence  $\{q_n\}$  by some of its subsequences we can additionally assume that

$$p \preceq a_1 \preceq a'_1 \preceq a_2 \preceq a'_2 \preceq \dots, \quad p \preceq b_1 \preceq b'_1 \preceq b_2 \preceq b'_2 \preceq \dots,$$

$$\text{and } q_i \neq q_j \quad \text{for } i \neq j.$$

For each  $n \in \mathbb{N}$  let  $A_n \subset f^{-1}(q_n)$  be an arc irreducible with respect to intersecting both  $a_n a'_n \cap f^{-1}(q_n)$  and  $b_n b'_n \cap f^{-1}(q_n)$ , and denote by  $a''_n$  and  $b''_n$  the respective end points of  $A_n$ . Then the union

$$U = A_1 \cup b''_1 b''_2 \cup A_2 \cup a''_2 a''_3 \cup A_3 \cup b''_3 b''_4 \cup A_4 \cup a''_4 a''_5 \cup \dots$$

is homeomorphic to the closed half line  $[0, \infty)$ , and its compactification  $\text{cl } U$  has the nondegenerate remainder  $\text{Ls } A_n \subset f^{-1}(q)$

disjoint with  $U$ . Hence  $\text{cl}U \subset X$  is not arcwise connected, a contradiction. The proof is complete.

Now we are able to formulate and prove the above mentioned result.

**11. Theorem** *Every confluent mapping from a hereditarily arcwise connected continuum onto a dendrite is weakly arc-preserving.*

**Proof:** Let a continuum  $X$  be hereditarily arcwise connected,  $Y$  be a dendrite, and a mapping  $f : X \rightarrow Y$  be a confluent surjection. Since each confluent mapping onto a locally connected continuum is an OM-mapping, [20, Corollary 5.2, p. 109], there exists a continuum  $Z$  and mappings  $f_1 : X \rightarrow Z$  and  $f_2 : Z \rightarrow Y$  such that  $f_1$  is monotone,  $f_2$  is open, and  $f = f_2 \circ f_1$ . Note that (by Whyburn's monotone-light factorization theorem, [35, (4.2), p. 143]) the open mapping  $f_2$  is light. Recall that for every dendrite  $D$ , for every compact space  $Z$  and for every light open mapping  $f_2 : Z \rightarrow f_2(Z) = Y$  with  $D \subset Y$  there exists a homeomorphic copy  $Z'$  of  $D$  in  $Z$  such that the restriction  $f_2|_{Z'} : Z' \rightarrow f(Z') = D$  is a homeomorphism, [35, (2.4), p. 188]. Taking  $D = Y$  we find a subcontinuum  $Z'$  of  $Z$  such that the restriction  $f_2|_{Z'} : Z' \rightarrow Y$  is a homeomorphism. Let  $X' = f_1^{-1}(Z')$ . By monotonicity of  $f_1$  the set  $X'$  is a subcontinuum of  $X$ , and since  $X$  is hereditarily arcwise connected,  $X'$  is arcwise connected. By Theorem 10 the restriction  $f_1|_{X'} : X' \rightarrow f_1(X') = Z'$  is monotone. Therefore  $f|_{X'} = (f_2|_{Z'}) \circ (f_1|_{X'})$ ; thus the restriction  $f|_{X'}$  is the composition of a monotone mapping and a homeomorphism, so it is monotone. Further, it is a surjection and, by Theorem 10, it is hereditarily monotone. Since the monotone image of an arc is an arc, [35, (1.1), p. 165], the restriction  $f|_{X'}$  is arc-preserving, and thus  $f$  is weakly arc-preserving, as needed. The proof is complete.

**12. Corollary** *Every confluent mapping from a dendroid onto*

*a locally connected continuum is weakly arc-preserving.*

**Proof:** Let  $X$  be a dendroid,  $Y$  be a locally connected continuum, and a mapping  $f : X \rightarrow Y$  be a confluent surjection. Since each subcontinuum of a dendroid is a dendroid, [3, (49), p. 240], each dendroid is hereditarily arcwise connected. Since each confluent image of a dendroid is a dendroid, [5, Corollary 1, p. 219], and since  $Y$  is locally connected, we infer that  $Y$  is a dendrite. Now the conclusion follows from Theorem 11.

The heredity of the arcwise connectedness of the domain is an essential assumption in Theorem 11, as is shown in the next example. In its formulation the concept of an *arc of pseudo-arcs* is used as defined in [2, p. 173], where this continuum is named “a continuous snake-like arc of pseudo-arcs”.

**13. Example** *There exists a monotone and not weakly arc-preserving mapping from the cone over an arc of pseudo-arcs onto a simple triod.*

**Proof:** Let  $P$  stand for an arc of pseudo-arcs, and let  $\pi : P \rightarrow [0, 1]$  be the natural projection. Thus  $\pi$  is a monotone and open mapping whose point-inverses are pseudo-arcs. Consider the mapping

$$\text{cone}(\pi) : \text{cone}(P) \rightarrow \text{cone}([0, 1]).$$

Put  $X = \text{cone}(P)$ ,  $T = \text{cone}([0, 1])$ ,  $f_1 = \text{cone}(\pi)$ , and note that the continuum  $X$  is uniquely arcwise connected (since  $P$  does not contain any arc) and unicoherent (since each cone is contractible, and thus unicoherent, see [19, §54, VI, Theorem 3, p. 375; §57, I, Theorem 9, p. 435 and II, Theorem 2, p. 437]). Note further that  $T$  is a triangular disk with the base  $[0, 1]$  (we denote the vertex of  $T$  by  $v$ ), and that the mapping  $f_1 : X \rightarrow T$  is monotone by its definition. Further, denote by  $Y$  the simple triod contained in  $T$  being the union of the base  $[0, 1]$  of  $T$  and the straight line segment joining the vertex  $v$  of  $T$  with the middle point  $1/2$  of the base. Let  $f_2 : T \rightarrow Y$  be a

monotone retraction such that  $f_2^{-1}(0) = 0$  and  $f_2^{-1}(1) = 1$  (i.e., the point-inverses of the two end points of  $Y$  distinct from  $v$  are singletons). Then the composition  $f = f_2 \circ f_1 : X \rightarrow Y$  is monotone. We will show that it is not weakly arc-preserving.

To this goal suppose that there is an arcwise connected subcontinuum  $X'$  of  $X$  such that  $f(X') = Y$  and that  $f|X'$  is arc-preserving. Choose points  $a \in X' \cap f^{-1}(0)$  and  $b \in X' \cap f^{-1}(1)$ , and observe that the only arc  $A$  joining  $a$  with  $b$  in  $X$  is the union of two segments: from the vertex of  $X$  to  $a$  and to  $b$ , respectively. By its uniqueness,  $A$  is contained in  $X'$ . However,  $f(A) = Y$  by the definition of  $f$ , so  $f|X'$  is not arc-preserving. The argument is complete.

**14. Example** *There are dendroids  $X$  and  $Y$  and an open (thus confluent) surjective mapping  $f : X \rightarrow Y$  which is not weakly arc-preserving.*

**Proof:** The dendroid  $X$  will be constructed in the plane  $\mathbb{R}^2$ . Let the set  $B = \text{cl}\{(2, 1/n) : n \in \mathbb{N}\}$  stand for the base of the cone  $C$  with the vertex  $v = (-1, 0)$ , let  $A$  denote the straight line segment with end points  $(-2, 0)$  and  $(-1, 0)$ , and put  $X^+ = A \cup C$ . Thus  $X^+$  is a dendroid lying in the upper (closed) half-plane. Let  $s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the central symmetry with respect to the origin  $c = (0, 0)$ , i.e.,  $s((x, y)) = (-x, -y)$ . Define  $X = X^+ \cup s(X^+)$ . Consider an equivalence relation  $*$  on  $X$  that identifies each point  $p \in X$  with its image  $s(p) \in X$ . Let  $Y$  stand for the quotient space  $X/*$  and let  $f : X \rightarrow Y$  be the quotient mapping. Then  $f$  is open, thus confluent, [5, VI, p. 214], whence it follows that  $Y$  is a dendroid, [5, Corollary 1, p. 219]. Note that  $Y$  has the arc from  $f((2, 0))$  to  $f((0, 0))$  as a continuum of convergence. Thus  $Y$  is not locally connected. To verify that  $f$  is not weakly arc-preserving take an arbitrary subcontinuum  $X'$  of  $X$  with  $f(X') = Y$ . Thus  $X'$  contains either infinitely many points of  $B$  or infinitely many points of  $s(B)$ . Without loss of generality we can assume the former possibility. Then the limit point  $a = (2, 0)$  of  $B$  is in  $X'$ , and

taking a point  $b \in B \setminus \{a\}$  we see that the arc  $ab$  is mapped under  $f$  onto the triod with vertex  $f(v)$  whose arms are the arcs  $f(v)f(a)$ ,  $f(v)f(b)$  and  $f(v)f(c)$ . Therefore there is no subcontinuum  $X'$  of  $X$  with  $f(X') = Y$  and such that the restriction  $f|_{X'}$  is arc-preserving.

Example 8 shows that weak confluence of a mapping between trees does not imply its universality. However, an example can be constructed that shows much more: the universality of a mapping between trees cannot be achieved even if the mapping is semi-confluent. The needed mapping is described in [24, Example 2, p. 385], where it is shown that it is weakly confluent and not universal. We will show that it is semi-confluent and not weakly arc-preserving. To this end we start with recalling the definition of the mapping.

**15. Example (Marsh)** *There exists a semi-confluent mapping between trees which is neither universal nor weakly arc-preserving.*

**Proof:** As in the proof of Example 8, denote by  $xy$  the straight line segment in the plane with end points  $x$  and  $y$ . Take a straight line segment  $pq$ , choose points  $p_1$  and  $q_1$  in  $pq$  such that  $p < p_1 < q_1 < q$  in an order  $<$  of  $pq$  from  $p$  to  $q$ , and let segments  $ab$  and  $cd$  be perpendicular to  $pq$  at  $p_1$  and  $q_1$ , respectively, so that  $p_1 \in ab \setminus \{a, b\}$  and  $q_1 \in cd \setminus \{c, d\}$ . Put  $H = pq \cup ab \cup cd$ . Let  $T$  be a simple triod with the vertex  $z$  whose arms are straight line segments  $zu$ ,  $zv$  and  $zw$ . To describe the needed mapping pick up points  $r$ ,  $s$  and  $t$  in the segment  $p_1q_1 \subset pq \subset H$  ordered so that  $p_1 < r < s < t < q_1$ , and points  $v_1 \in zv \setminus \{z, v\}$  and  $w_1 \in zw \setminus \{z, w\}$ . Define  $f : H \rightarrow T$  as a piecewise linear mapping with respect to the triangulation of  $H$  described above, determined by its values on the mentioned points of  $H$  as follows.

$$\begin{aligned} f(a) = f(c) = u, \quad f(b) = f(t) = w_1, \quad f(d) = f(r) = v_1, \\ f(p) = v, \quad f(q) = w, \quad f(p_1) = f(s) = f(q_1) = z. \end{aligned}$$

It is shown in [24, Example 2, p. 385] that  $f$  is not universal. We will show that it is semi-confluent. So, let  $Q$  be a subcontinuum of  $Y$ . Consider four cases.

- 1) If  $\{v_1, w_1\} \subset Q$ , then  $f^{-1}(Q)$  is connected.
- 2) If  $z \in Q$  and exactly one of the two points  $v_1, w_1$  is in  $Q$ , then  $f^{-1}(Q)$  has two components, one of which is mapped onto  $Q$ .
- 3) If  $z \in Q$  and neither  $v_1$  nor  $w_1$  is in  $Q$ , then  $f^{-1}(Q)$  has three components, two of which are mapped onto  $Q$ .
- 4) If  $z \notin Q$ , then  $f^{-1}(Q)$  has at most four components. At least one of them is mapped onto  $Q$ , and images of any other two are equal.

Therefore, in any case, the condition of semi-confluence of  $f$  is satisfied. To see that  $f$  is not weakly arc-preserving suppose the contrary. Thus there exists an arcwise connected subset  $X'$  of  $X$  such that  $f|X' : X' \rightarrow Y$  is an arc-preserving surjection. Since  $f^{-1}(v) = \{p\}$  and  $f^{-1}(w) = \{q\}$ , we infer that  $p, q \in X'$ , whence  $pq \subset X'$ . Further  $f^{-1}(u) = \{a, c\}$ , so either  $a \in X'$  or  $c \in X'$  (or both). If  $a \in X'$ , then the arc  $ap_1 \cup p_1q$  is contained in  $X'$ ; however, its image  $f(ap_1 \cup p_1q)$  is the triod in  $Y$  with the end points  $u, v_1$  and  $w$ . If  $c \in X'$ , then the arc  $cq_1 \cup q_1p$  lies in  $X'$ , while its image is the triod with the end points  $u, v$  and  $w_1$ . Therefore in both cases  $f|X'$  is not arc-preserving. The argument is complete.

**16. Remark** The statement that the mapping  $f$  of Example 15 is not weakly arc-preserving can also be derived from its non-universality. Namely it is known that each weakly arc-preserving mapping between trees is universal, [24, Theorems 1 and 3, p. 376 and 383].

**17. Remarks** a) Confluence of the considered mapping in Theorem 11 and Corollary 12 is an essential assumption, and it cannot be weakened to semi-confluence, even if both the domain and the range are trees. This is shown in Example 15.

b) To see that arcwise connectedness of the domain is in-

dispensable in Corollary 12 take as  $X$  an arc of pseudo-arcs. It is well known that  $X$  is a hereditarily unicoherent and not arcwise connected continuum. The natural projection  $f$  of  $X$  onto an arc is a monotone and open (thus confluent) mapping which is not weakly arc-preserving.

c) Hereditary unicoherence of the domain is necessary in Corollary 12. In fact, take as  $X$  the unit circle in the complex plane, and define  $f : X \rightarrow X$  by  $f(z) = z^2$ . Then  $f$  is an open (thus confluent) mapping which is not weakly arc-preserving.

d) Example 14 shows that local connectedness of the range space is an essential assumption in Theorem 11 and Corollary 12.

e) The inverse implications to that of Theorem 11 and Corollary 12 are not true, even under an additional assumption that both domain and range are trees. Moreover, not only confluence, but even semi-confluence does not follow if the mapping between trees is weakly arc-preserving. Indeed, consider the following example. In the plane  $\mathbb{R}^2$  denote by  $pq$  the straight line segment with end points  $p$  and  $q$ . Put

$$v = (0, 0), \quad a = (-1, 0), \quad b = (1, 0), \quad c = (0, -1), \quad d = (0, 1),$$

$$v' = (0, 2), \quad b' = (1, 2), \quad c' = (0, 3),$$

$$v'' = (0, 4), \quad a'' = (-1, 4), \quad d'' = (0, 5).$$

Let

$$X = ab \cup cd'' \cup v'b' \cup v''a'' \quad \text{and} \quad Y = ab \cup cd.$$

Thus both  $X$  and  $Y$  are trees, and  $Y \subset X$ . Define  $f : X \rightarrow Y$  as a piecewise linear retraction determined by  $f(v') = f(v'') = v$ ,  $f(b') = b$ ,  $f(c') = c$ ,  $f(a'') = a$  and  $f(d'') = d$ . Then  $f$  is weakly arc-preserving (since  $f|Y$  is the identity), but it is not semi-confluent.

f) Neither Theorem 11 nor Corollary 12 can be sharpened to obtain "arc-preserving" in the conclusion, even if openness of



the mapping is additionally assumed. Indeed, keeping notation of the previous example, let  $X = cc' \cup vb \cup v'b'$  and  $Y = cd \cup vb \subset X$ . Define  $f : X \rightarrow Y$  as a piecewise linear open retraction determined by  $f(v') = v$ ,  $f(b') = b$  and  $f(c') = c$ . Then the arc from  $b$  to  $c'$  in  $X$  is mapped onto the whole  $Y$ .

In connection with the example of Remark 17 c) it would be interesting to know if hereditary unicoherence of the domain continuum  $X$  in Corollary 12 can be replaced by its unicoherence and one-dimensionality. In other words we have the following question.

**18. Question** Does there exist an arcwise connected, unicoherent and one-dimensional continuum  $X$  and a confluent mapping from  $X$  onto a locally connected continuum  $Y$  which is not weakly arc-preserving?

Observe that if such a continuum  $X$  does exist, then it cannot be locally connected, because every locally connected, unicoherent and one-dimensional continuum is a dendrite, [19, §57, III, Corollary 8, p. 442]. If we neglect the condition  $\dim X = 1$  in Question 18, then Example 13 shows even a monotone mapping from a two-dimensional continuum  $X$  onto a simple triod that is not weakly arc-preserving.

In connection with Example 14 and Remark 17 d) let us comment that, in Corollary 12, under an additional assumption concerning the domain  $X$  the local connectedness of the range space  $Y$  can be omitted. To formulate and prove this result we need two definitions. A dendroid  $X$  is said to be *smooth* provided there is a point  $p \in X$  such that for each point  $x \in X$  and for each sequence of points  $\{x_n\}$  converging to  $x$  the sequence of arcs  $\{px_n\}$  converges to the arc  $px$ . A mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  is said to be *monotone relative to a point*  $p \in X$  provided that for each subcontinuum  $Q$  of  $Y$  such that  $f(p) \in Q$  the preimage  $f^{-1}(Q)$  is connected.

**19. Theorem** *Every confluent mapping from a smooth dendroid is weakly arc-preserving.*

**Proof:** Let  $f : X \rightarrow Y$  be a confluent surjection defined on a smooth dendroid  $X$ . By [21, Proposition 3.1, p. 722] there exists a subcontinuum  $X'$  of  $X$  such that  $f|X'$  is a confluent mapping of  $X'$  onto  $Y$  and  $X'$  is minimal with respect to this property. Note that  $X'$ , as a subcontinuum of a dendroid, is arcwise connected. By [21, Theorem 3.2, p. 722] the mapping  $f|X' : X' \rightarrow Y$  is monotone relative to a point  $p \in X'$ . Then every arc  $ab$  in  $X'$  that does not contain  $p$  is the union of two arcs:  $ca \cup cb$ , where  $c \in ab$  is such that  $pc \cap ab = \{c\}$ . Thus the partial mappings  $f|ca$  and  $f|cb$  are monotone, and therefore  $f(ab)$  is the union of two arcs  $f(ca)$  and  $f(cb)$ . Assume on the contrary that  $f(ab)$  is not an arc. Then it is a triod. Denote its vertex by  $v$ , and observe that  $f(a)$ ,  $f(b)$  and  $f(c)$  are the end points of the triod. Consider the arc  $f(a)f(b) = vf(a) \cup vf(b)$  in  $Y$  and note that the component of its preimage  $f^{-1}(f(a)f(b))$  that contains the point  $a$  is mapped under  $f$  onto  $vf(a)$  only, contrary to confluence of  $f$ . This finishes the proof.

**20. Remarks** a) Smoothness of the domain dendroid is essential in Theorem 19 by Example 14, where  $X$  is not smooth.

b) Example 15 shows that Theorem 19 cannot be generalized to semi-confluent mappings.

Now the following result, [8, Theorem 5 (a), p. 213], is a consequence of either Theorem 11, or Corollary 12, or Theorem 19.

**21. Corollary** (Eberhart and Fugate) *Every surjective confluent mapping between trees is weakly arc-preserving.*

Another result that was announced in [8, p. 213] as Theorem 5 (b) concerns the converse implication, i.e. from weak arc-preservation to weak confluence, for mappings between trees. Again the result can be drawn from a stronger statement.

**22. Theorem** *Let  $f : X \rightarrow Y$  be a weakly arc-preserving mapping from a continuum  $X$  onto a tree  $Y$ . Then for each subcontinuum  $Q$  of  $Y$  there exists a tree  $T \subset X$  such that the restriction  $f|_T : T \rightarrow Q$  is arc-preserving, and  $f(T) = Q$ . Consequently,  $f$  is weakly confluent.*

**Proof:** Since  $f$  is weakly arc-preserving, there is an arcwise connected subcontinuum  $X'$  of  $X$  such that the restriction  $f|_{X'} : X' \rightarrow Y$  is arc-preserving. Let  $Q$  be a subcontinuum of  $Y$ . We will find a tree  $T \subset X'$  such that  $f(T) = Q$ . Since  $Q$  is a tree, it is the union of a finite number, say  $n$ , of arcs. We apply the induction with respect to  $n$ .

If  $n = 1$ , then  $Q$  is an arc  $pq$ . Choose  $a \in f^{-1}(p) \cap X'$  and  $b \in f^{-1}(q) \cap X'$ . Then there is an arc  $ab \subset X'$ , and since  $f|_{X'}$  is arc-preserving, the image  $f(ab)$  is an arc in  $Y$  containing both  $p$  and  $q$ . Then the restriction  $f|_{ab} : ab \rightarrow f(ab)$  is a weakly confluent mapping as a mapping onto an arc, [30, Lemma, p. 236], and thus there exists a subarc  $T$  of  $ab$  with  $f(T) = pq = Q$ .

Now let  $Q$  be the union of some  $n + 1$  arcs (which is not an arc). Then there are a tree  $P$  and an arc  $pq$  such that  $P$  is the union of some  $n$  arcs,  $Q = P \cup pq$  and  $P \cap pq = \{p\}$ . By the inductive assumption there is a tree  $A \subset X'$  such that  $f(A) = P$ . Let  $a \in A \cap f^{-1}(p)$ . Take  $b \in X' \cap f^{-1}(q)$  and choose an arc  $ab \subset X'$ . Since  $f|_{X'}$  is arc-preserving, we infer that  $f(ab)$  is an arc. Order the arc  $ab$  from  $a$  to  $b$ . Let  $a'$  be the last point of  $ab$  such that  $a' \in A$ , and let  $b'$  be the first one in  $f^{-1}(q)$ . Take the arc  $a'b' \subset ab$ . Then  $A \cup a'b'$  is a tree. We will show that  $f(a'b') = pq$ . To this aim suppose on the contrary that there is a point  $x \in a'b'$  such that  $f(x) \notin pq$ . Since  $f(a'b')$  is an arc, the points  $f(x)$ ,  $p$  and  $q$  lie in one arc, and then  $f(x) \notin pq$  implies that either  $q \in pf(x)$ , or  $p \in f(x)q$ . Since  $f(a') \in f(A) = P$ , we see that  $f^{-1}(q) \cap a'x \neq \emptyset$ , and thus the former possibility contradicts to the choice of  $b'$ . So  $f(x)q$  is an arc to which  $p$  belongs. Since by assumption  $Q = P \cup pq$  is not

an arc, there is a point  $y \in P \setminus f(x)q$ . Choose  $x' \in A \cap f^{-1}(y)$ . By the definition of  $a'$  the union  $x'a' \cup a'b'$  is an arc in  $X'$  whose image contains the triod  $yp \cup f(x)q$  (having  $p$  as its vertex), contrary to the assumption that  $f|X'$  is arc-preserving. So, we have proved that  $f(a'b') = pq$ . Putting  $T = A \cup a'b'$  we see that  $T$  is a tree such that  $f(T) = f(A \cup a'b') = P \cup pq = Q$  as needed. This finishes the proof.

The following example, which is well known, will be used several times in the sequel. We exhibit some additional properties of the mapping described in this example. Recall that the *Cantor fan* is the cone over the Cantor ternary set  $C$  in  $[0, 1]$ .

**23. Example** *There exists an arc-preserving and not weakly confluent (consequently not universal) mapping from the Cantor fan onto the two-cell.*

**Proof:** Let  $C$  be the Cantor ternary set lying in the standard way in  $[0, 1]$ , and let  $g : C \rightarrow [0, 1]$  be the well known Cantor-Lebesgue step function (see e.g. [18, §16, II, (8), p. 150]; compare [35, Chapter II, §4, p. 35]). Consider the Cantor fan  $X$  as the cone over  $C$  with the vertex  $v$ , and the two-cell  $Y$  as the cone over  $[0, 1]$  with the vertex  $v'$ . For each point  $c \in C$  map linearly the segment  $vc$  onto the segment  $v'g(c)$ , and let  $f : X \rightarrow Y$  be the resulting mapping. Observe that any arc  $A$  in  $X$  is contained either in one segment of the form  $vc$  (where  $c \in C$ ) or in the union of two such segments, and therefore, by the definitions of  $g$  and  $f$ , its image  $f(A)$  is a segment in  $Y$ . Thus  $f$  is arc-preserving. It is not weakly confluent since components of the preimage of the base segment  $[0, 1]$  of  $Y$  are singletons in  $C$  which form the base of  $X$ . Applying Theorem 4 we see that  $f$  is not universal. The argument is complete.

**24. Remarks** a) The example in Remark 17 e) shows that Theorem 22 cannot be sharpened to get the semi-confluence of  $f$  in the conclusion.

b) For a related result, with a dendrite as the range space  $Y$  see below Theorem 25 and Corollary 26.

c) For a related result, with a dendroid as the range space  $Y$  see below Theorem 30.

d) Example 23 shows that even if the mapping  $f$  is arc-preserving and  $Y$  is assumed to be a locally connected continuum, (instead of being a tree as in Theorem 22), the weak confluence of  $f$  cannot be achieved.

It is announced in [8, Theorem 6, p. 214] that each weakly arc-preserving mapping between trees is universal. This can be shown with help of some results from [24] in the following way. Let a weakly arc-preserving mapping  $f : X \rightarrow Y$  between trees  $X$  and  $Y$  be given. Then there exists a subtree  $X'$  of  $X$  such that the restriction  $f|_{X'} : X' \rightarrow Y$  is a surjective  $u$ -mapping (see [24, Theorem 3, p. 383]; since the definition of a  $u$ -mapping consists of five rather technical conditions, each of which needs some additionally defined auxiliary concepts, the reader is referred to the source paper [24, p. 374] to see the details). Further, Theorem 1 of [24, p. 376] says that if there exists a subtree  $X'$  of  $X$  having the above mentioned property, then  $f$  is universal. So, the argument is complete.

However, the above mentioned statement (i.e., Theorem 6 of [8, p. 214]) is again a direct consequence of a more general result, formulated below. Namely the assumption concerning the domain (of being a tree) has been deleted, and the assumption concerning the range (also of being a tree) has been replaced by a less restrictive one, of being a dendrite.

**25. Theorem** *Each weakly arc-preserving mapping from a continuum onto a dendrite is universal.*

**Proof:** Let a mapping  $f : X \rightarrow Y$  from a continuum  $X$  onto a dendrite  $Y$  be weakly arc-preserving, and suppose on the contrary that it is not universal. Then there exists a mapping  $g : X \rightarrow Y$  such that  $f(x) \neq g(x)$  for each  $x \in X$ . Then by compactness of  $X$  there exists an  $\varepsilon > 0$  with  $d(f(x), g(x)) > \varepsilon$

(here  $d$  stands for the metric on  $Y$ ). Further, by compactness of  $Y$ , there exists a finite  $\varepsilon$ -net  $S$  in  $Y$ . Denote by  $Y_\varepsilon$  the smallest continuum in  $Y$  containing  $S$ . Then  $Y_\varepsilon$  is a tree, and there exists the unique monotone retraction  $r : Y \rightarrow Y_\varepsilon$  (see [12, Theorem, p. 157]). This retraction is an  $\varepsilon$ -mapping (i.e., the diameters of point-inverses are less than  $\varepsilon$ ). Further, the composition  $r \circ f : X \rightarrow Y_\varepsilon$  is weakly arc-preserving. Applying Theorem 22 to this mapping we infer that there exists a tree  $T \subset X'$  such that  $(r \circ f)|_T : T \rightarrow Y_\varepsilon$  is an arc-preserving surjection. Since by [24, Theorems 1 and 3, p. 376 and 383] each arc-preserving mapping between trees is universal, we conclude that  $(r \circ f)|_T$  is universal. Thus there exists a point  $x \in T$  such that  $(r \circ f)(x) = (r \circ g)(x)$ . Since  $r$  is an  $\varepsilon$ -mapping, we have  $d(f(x), g(x)) < \varepsilon$ , contrary to the definition of  $\varepsilon$ .

The next result is a consequence of Theorems 25 and 4.

**26. Corollary** *Each weakly arc-preserving mapping from a continuum onto a dendrite is weakly confluent.*

Corollary 26 cannot be sharpened to get semi-confluence in the conclusion even if the mapping is arc-preserving (see Fact 5 and compare Remark 17 e).

As a consequence of either Theorem 22 or Corollary 26 we get the above mentioned result formulated in [8, p. 213] as Theorem 5 (b).

**27. Corollary** (Eberhart and Fugate) *Each weakly arc-preserving mapping between trees is weakly confluent.*

It follows from Corollaries 21 and 27 that for mappings between trees weakly arc-preserving mappings form a class that is intermediate between confluent and weakly confluent mappings. The next question is related to this statement.

**28. Questions** For what continua  $X$  and  $Y$  a) is each confluent mapping  $f : X \rightarrow Y$  weakly arc-preserving? b) is each weakly arc-preserving mapping  $f : X \rightarrow Y$  weakly confluent?

Some contributions to these questions will be presented in the sequel.

**29. Remarks** a) The assumption concerning the mapping (of being weakly arc-preserving) is necessary in Theorem 25 by Example 15.

b) The assumption that the range space is a dendrite is essential in Theorem 25 because of Example 23.

c) An example is constructed in [24, Example 1, p. 384] of a  $u$ -mapping between trees (which is universal according to [24, Theorem 1, p. 376]) that is not weakly arc-preserving. Therefore the converse to Theorem 25 is not true.

In connection with Theorem 22, and as an application of Theorem 25, we have the following result.

**30. Theorem** *Each weakly arc-preserving mapping from a continuum onto a dendroid is weakly confluent.*

**Proof:** Let  $f : X \rightarrow Y$  be such a mapping, and let  $Q$  be a subcontinuum of  $Y$ . Choose a countable dense subset  $\{d_1, d_2, \dots, d_n, \dots\}$  of  $Q$ , and let  $Q_n$  be the unique subcontinuum of  $Q$  irreducible with respect to containing the set  $\{d_1, d_2, \dots, d_n\}$  (i.e., such that  $Q_n$  contains this set and no proper subcontinuum of  $Q_n$  contains it; see [4, T1, p. 187] for existence and uniqueness in any hereditarily unicoherent continuum). Then  $Q_n \subset Q_{n+1}$  for each  $n \in \mathbb{N}$ , and  $Q = \text{Lim} Q_n$ . Let  $X'$  be a subset of  $X$  as in the definition of a weakly arc-preserving mapping. For each  $i \in \mathbb{N}$  choose a point  $e_i \in X' \cap f^{-1}(d_i)$  and let  $A_i$  be an arc in  $X'$  joining  $e_1$  with  $e_i$ . Define  $L_n = \bigcup \{A_i : i \in \{1, \dots, n\}\}$  for each  $n \in \mathbb{N}$ . Then  $L_n$  is a locally connected (so arcwise connected) continuum contained in  $X'$ , and thus the restriction  $f|_{L_n} : L_n \rightarrow f(L_n) \supset Q_n$  is arc-preserving by the definition of  $X'$ . Hence  $f(L_n)$  is a locally connected subcontinuum of the dendroid  $Y$ , so it is a dendrite. Therefore  $f|_{L_n} : L_n \rightarrow f(L_n)$  is an arc-preserving mapping onto a dendrite, whence it follows by Theorem 25 that

it is universal, and consequently weakly confluent by Theorem 4. Thus for each  $n \in \mathbb{N}$  there exists a subcontinuum  $K_n$  of  $L_n$  such that  $f(K_n) = Q_n$ . Choosing a convergent subsequence of the sequence  $\{K_n\}$  if necessary, we can consider a continuum  $K = \text{Lim} K_n$ . Then  $f(K) = Q$  by continuity of  $f$ . So,  $f$  is weakly confluent, and the proof is finished.

**31. Remarks** a) The assumption that the range  $Y$  is a dendroid is indispensable in Theorem 30 by Example 23.

b) Example 15 shows that the converse implication to that of Theorem 30 does not hold, even if the assumption is strengthened to semi-confluence of the mapping.

One can ask if, in Theorem 25, the assumption concerning  $Y$  of being a dendrite can be relaxed to being a dendroid. In other words, we have the following question, which is related to Remark 29 c) and to Theorem 30.

**32. Question** Is every weakly arc-preserving mapping from a continuum onto a dendroid universal?

The next example shows that Theorem 25 cannot be sharpened replacing weakly arc-preserving mappings by weakly confluent ones. To this aim a definition is needed. By the *Gehman dendrite*  $G$  we mean a dendrite having the set  $E(G)$  of its end points homeomorphic to the Cantor set, and such that all its ramification points are of order 3 (see [11, the example, p. 42]; also [28, p. 422-423] for a detailed description, and [29, Fig. 1, p. 203] for a picture).

**33. Example** *There exists a weakly confluent surjective self-mapping on the Gehman dendrite which is not universal.*

**Proof:** First we consider two trees,  $H$  and  $T$ , as defined in Example 8. To the tree  $H$  add an arc  $tt'$  so that  $H \cap tt' = \{t\}$ , and put  $H' = H \cup tt'$ . Thus  $H'$  is a tree with  $E(H') = \{a, b, c, d, t'\}$ . In the Gehman dendrite  $G$  we fix a point  $e'$  such that  $\text{ord}(e', G) = 2$ . With each end point  $e$  of  $H'$  we associate



a (pointed) copy  $G_e$  of the Gehman dendrite  $G$  in such a way that the copies are pairwise disjoint and in each of them we identify  $e \in E(H')$  with  $e' \in G_e$  so that  $H \cap G_e = \{e\}$  (i.e.,  $H$  and  $G_e$  have one point  $e = e'$  in common only). We do the same with the tree  $T$ . Recall that  $E(T) = \{u, v, w\}$ . Then we define

$$X = H' \cup \bigcup \{G_e : e \in E(H')\} \text{ and } Y = T \cup \bigcup \{G_e : e \in E(T)\}.$$

Note that both  $X$  and  $Y$  have ramification points of order 3 only, and that  $E(X)$  and  $E(Y)$  are homeomorphic to the Cantor set. Thus  $X$  and  $Y$  are homeomorphic to the Gehman dendrite.

We extend the mapping  $f : H \rightarrow T$  (defined in Example 8) to a mapping  $f^* : X \rightarrow Y$  so that the restrictions  $f^*|_{tt'} : tt' \rightarrow \{v\}$  and  $f^*|_{G_d} : G_d \rightarrow \{y\}$  are constant mappings, and  $f^*|_{G_e} : G_e \rightarrow G_{f(e)}$  are homeomorphisms for  $e \in \{a, b, c, t'\}$ . Thus  $f^*$  is defined. The reader can verify, considering all possible kinds of subcontinua of  $Y$ , that  $f^*$  is weakly confluent. To see that  $f^*$  is not universal, recall that  $f : H \rightarrow T$  is not universal, i.e., there is a mapping  $g : H \rightarrow T$  (defined in Example 8) such that  $f(x) \neq g(x)$  for each  $x \in H$ . To extend  $g$  to a mapping  $g^* : X \rightarrow Y$  such that  $f^*(x) \neq g^*(x)$  for each  $x \in X$  we consider the (unique) monotone retraction  $m : X \rightarrow H$  and put  $g^* = g \circ m$ . Then  $g(X) = T$ . Consider four cases. First we see that if  $x \in H$ , then since  $f^*|_H = f|_H$  and  $g^*|_H = g|_H$ , and since  $f$  and  $g$  have no coincidence point on  $H$ , we get  $f^*(x) \neq g^*(x)$ . Second, if  $x \in \bigcup \{G_e : e \in \{a, b, c, t'\}\} \setminus H$ , then  $f^*(x) \notin T$  while  $g^*(x) \in T$ . Third, if  $x \in G_d$ , then  $f^*(x) = f(d) \neq g(d) = g^*(d)$ . Fourth, if  $x \in tt'$ , then  $f^*(x) = f(t) \neq g(t) = g^*(t)$ . The proof is complete.

**34. Remark** Example 33 shows that Theorem 6 cannot be generalized to self-mappings between dendrites.

Our next result generalizes Theorem 2 of Schirmer.

**35. Theorem** *Let a continuum  $Y$  be the inverse limit of an inverse sequence of trees  $Y_n$  with confluent bonding mappings. Then each confluent mapping  $f : X \rightarrow Y$  from a continuum  $X$  onto  $Y$  is universal.*

**Proof:** For each  $n \in \mathbb{N}$  let  $\pi_n : Y \rightarrow Y_n$  denote the projection from the inverse limit space  $Y$  into the  $n$ -th factor space  $Y_n$ . Since the bonding mappings are confluent, the projections  $\pi_n$  also are confluent, [6, Corollary 7, p. 5].

Suppose on the contrary that there exists a mapping  $g : X \rightarrow Y$  that misses  $f$ . Then by compactness of  $X$  there is an  $\varepsilon > 0$  such that  $d(f(x), g(x)) > \varepsilon$  for each point  $x \in X$  (here  $d$  stands for the metric in  $Y$ ). Let  $n$  be great enough so that the projection  $\pi_n$  is an  $\varepsilon$ -mapping. The composition  $\pi_n \circ f : X \rightarrow Y_n$  is a confluent mapping onto a tree, so it is universal by Theorem 2. Thus there is a point  $x_0 \in X$  such that  $\pi_n(f(x_0)) = \pi_n(g(x_0))$ . Consequently the set  $\pi_n^{-1}(\pi_n(f(x_0)))$  contains both  $f(x_0)$  and  $g(x_0)$  whose distance is greater than  $\varepsilon$ , contrary to the choice of  $n$ . The proof is complete.

Since every dendrite  $Y$  can be represented as the inverse limit of an inverse (increasing) sequence of trees  $Y_n \subset Y$  with monotone (thus confluent) retractions as bonding mappings, we get the following corollary.

**36. Corollary** *Each confluent mapping from a continuum onto a dendrite is universal.*

To formulate the next corollary to Theorem 35 recall that a continuum  $Y$  is said to have the *property of Kelley* provided that for each point  $y \in Y$ , for each sequence of points  $y_n$  converging to  $y$  and for each subcontinuum  $K$  of  $Y$  containing the point  $y$  there is a sequence of subcontinua  $K_n$  with  $y_n \in K_n$  that has  $K$  as its limit. A dendroid is called a *fan* if it has only one ramification point. A fan is said to be *finite* provided it has finitely many end points. Thus each finite fan is a tree.

**37. Corollary** *Each confluent mapping from a continuum*

*onto a fan having the property of Kelley is universal.*

**Proof:** Since a fan has the property of Kelley if and only if it is the inverse limit of an inverse sequence of finite fans with confluent bonding mappings, [7, Theorem 3, equivalence of conditions (a) and (d), p. 75], the conclusion follows from Theorem 35.

**38. Questions** Is any confluent mapping from a continuum (from a dendroid) onto a dendroid universal?

As it has been recalled in Theorem 4, each universal mapping from a continuum onto a locally connected continuum is weakly confluent. Thus arc-preserving mappings, and (more general)  $u$ -mappings between trees (see [24, p. 374] for the definition) are weakly confluent (apply [24, Theorem 1, p. 376; compare a remark on p. 380]), but not conversely. To find more close relations between universal mappings and mappings related to (or defined by) some confluence conditions, let us recall the concepts on  $n$ -weakly confluent and of inductively weakly confluent mappings (see [26, p. 236]).

Let  $f : X \rightarrow Y$  be a mapping between continua  $X$  and  $Y$ . We apply an inductive definition. The mapping  $f$  is said to be *0-weakly confluent* provided that it is a surjection. Let a nonnegative integer  $n$  be fixed. We say that  $f$  is  $(n + 1)$ -*weakly confluent* provided that for each subcontinuum  $Q$  of  $Y$  there exists a component of  $f^{-1}(Q)$  such that the restriction  $f|K : K \rightarrow Q$  is  $n$ -weakly confluent. The mapping  $f$  is said to be *inductively weakly confluent* provided that it is  $n$ -weakly confluent for each nonnegative integer  $n$ . Each semi-confluent mapping between continua is inductively weakly confluent, [26, Theorem 1, p. 236].

The next result is related to Nadler's Theorem 4 above.

**39. Theorem** *Each universal mapping from a continuum onto a hereditarily locally connected continuum is inductively weakly confluent.*

**Proof:** We will use induction to prove that each universal mapping from a continuum onto a hereditarily locally connected continuum is  $n$ -weakly confluent for each nonnegative integer  $n$ . For  $n = 0$  the assertion holds by assumption. Assume that every mapping from a continuum onto a hereditarily locally connected continuum is  $n$ -weakly confluent for some  $n \geq 0$ . Let  $X$  and  $Y$  be continua, with  $Y$  hereditarily locally connected, and let a mapping  $f : X \rightarrow Y$  be universal. Consider a subcontinuum  $Q$  of  $Y$ . By Theorem 4 of Nadler there is a component  $C$  of  $f^{-1}(Q)$  such that  $f(C) = Q$ . Then the restriction  $f|C : C \rightarrow Q$  is  $n$ -weakly confluent by the assumption. This proves  $(n + 1)$ -weak confluence of  $f$ , and so finishes the proof.

Note that Theorem 39 cannot be strengthened to get semi-confluence in the conclusion as it follows from Fact 5.

**40. Remark** As the reader already observed, semi-confluent mappings as well as universal mappings between trees are situated between confluent and inductively weakly confluent mappings: the implication from confluence to semi-confluence is obvious, from semi-confluence to inductive weak confluence is proved in Theorem 1 of [26, p. 236]; the two corresponding implications for universal mappings are given by Theorems 2 and 39, respectively. So, it is natural to ask if there is any implication between classes of semi-confluent and of universal mappings (onto trees). The answer is negative for both cases. Fact 5 shows that universality does not imply semi-confluence even for mappings between arcs, while the opposite implication does not hold by Example 15.

A mapping  $f : X \rightarrow Y$  between continua  $X$  and  $Y$  is said to be *hereditarily weakly confluent* provided that for each subcontinuum  $K$  of  $X$  the restriction  $f|K : K \rightarrow f(K) \subset Y$  is weakly confluent. Obviously each hereditarily weakly confluent is inductively weakly confluent, [26, Theorem 2, p. 237]. The next example shows that Theorem 39 cannot be sharpened by

replacing “inductively” by “hereditarily” in the conclusion.

**41. Example** *There is a universal mapping between simple triods which is not hereditarily weakly confluent.*

**Proof:** Denote by  $pq$  the straight line segment in the plane with end points  $p$  and  $q$ . Put  $v = (0, 0)$ ,  $a = (1, 0)$ ,  $b = (0, -1)$ ,  $c = (0, 1)$ ,  $d = (0, 2)$  and  $e = (0, 3)$ . Let  $X = be \cup va$  and  $Y = bc \cup va$ . Thus  $Y \subset X$ . Define  $f : X \rightarrow Y$  as a piecewise linear retraction determined by  $f(d) = v$  and  $f(e) = a$ . Then  $f$  is universal. Consider the restriction  $f|_{be} : be \rightarrow Y$ , and observe that if  $Q$  is a small connected neighborhood of  $v$ , then  $(f|_{be})^{-1}(Q)$  has two components, no one of which is mapped onto  $Q$ . Thus  $f|_{be}$  is not weakly confluent, and consequently  $f$  is not hereditarily weakly confluent.

**42. Theorem** *Each hereditarily weakly confluent mapping is arc-preserving.*

**Proof:** Let a mapping  $f : X \rightarrow Y$  between continua be hereditarily weakly confluent. Then for each arc  $A \subset X$  the restriction  $f|_A : A \rightarrow f(A) \subset Y$  is hereditarily weakly confluent, so  $f(A)$  is an arc by [22, (8.21), p. 74]. The argument is complete.

**43. Theorem** *Each hereditarily weakly confluent mapping from an arcwise connected continuum onto a dendrite is universal.*

**Proof:** Let  $f : X \rightarrow Y$  be such a mapping. Then it is arc-preserving by Theorem 42, so it is weakly arc-preserving by Fact 9, and the conclusion follows from Theorem 25.

**44. Corollary** *Each hereditarily weakly confluent mapping between trees is universal.*

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