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# AN ISOCOMPACT TYCHONOFF SPACE WHOSE SQUARE IS NOT ISOCOMPACT

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**Abstract**

A space $X$ is called isocompact if every countably compact closed subspace of $X$ is compact, and $X$ is called *cl*-isocompact if the closure of every countably compact subspace is compact. We construct a subspace $X$ of $\beta(\omega)$ which contains a dense countably compact dense subspace and whose every countably compact closed subset is finite. That is, $X$ is a nice isocompact Tychonoff space that is not *cl*-isocompact. We also find two isocompact subspaces $Y$ and $Z$ of $\beta(\omega)$

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such that $Y \times Z$ is not isocompact and hence $Y \oplus Z$ is an isocompact Tychonoff space whose square is not isocompact.

**Definition 1** (P. Bacon [Ba]) *A space $X$ is called isocompact if every countably compact closed subspace of $X$ is compact.*

**Definition 2** (M. Sakai [Sa]) *A space $X$ is called cl-isocompact if the closure of every countably compact subspace is compact.*

The Isocompact spaces exist in abundance. In fact, every space that satisfies one of the following properties is isocompact:

(1) ([Ba]) almost realcompact,
(2) ([WW1]) weakly $\delta\theta$-refinable,
(3) ([Sa]) weakly Borel complete.

In particular, if $X$ is either realcompact, paracompact, metacompact, meta-Lindelöf or quasi-developable of non-measurable cardinality, then $X$ is isocompact. We should notice that all the spaces that have one of the properties listed above are *cl*-isocompact. It is evident that a *cl*-isocompact space is isocompact. The authors did not find, in the literature, an example of an isocompact Tychonoff space that is not *cl*-isocompact. One of the aims of this article is to give a nice example of such a space. For some other properties of isocompact spaces not considered here, the reader is referred to the papers quoted above and [WW2], [WW3] and [Hi].

As we pointed out in the previous paragraph, every *cl*-isocompact space is isocompact. A property that is needed for an isocompact space to be *cl*-isocompact is the following (this property was introduced in [GS]): A space $X$ has *property CC* if, whenever $D$ is a countable closed discrete subset of $X$ and $C$ is a countably compact subspace of $X$ such that $D \cap C = \emptyset$, then there is a confinite subset $E$ of $D$ that is completely separated from $cl_X(C)$. Observe that every *cl*-isocompact has property $\mathcal{CC}$, and it is shown in [GS] that a space $X$ is *cl*-isocompact if and only if $X$ is an isocompact space with property $\mathcal{CC}$. In [Du], the author defines a space to be weakly normal if disjoint closed sets, one of which is countable, can be separated by disjoint open sets. We should note that every weakly normal space has property $\mathcal{CC}$ (see [Ha, Th. 13]) and hence a weakly normal isocompact space is *cl*-isocompact.

Next we shall construct our space. First, we state and prove some preliminary results.

We omit the proof of the following Lemma (se [Fr] and [Co, Lemma 8.2]).

**Notation** *If $D$ is a countable infinite discrete subset of $X$, then we write $D^* = cl_X(D) - D$.*

**Lemma 3** *Let $X$ be a space in which every countable discrete subspace is $C^*$-embedded and let $A$ and $B$ be countable discrete subspaces of $X$. If $\mathrm{cl}(A) \cap B$ and $A \cap \mathrm{cl}(B)$ are finite, then $A^* \cap B^* = \emptyset$. In addition, if $\mathrm{cl}(A) \cap B = \emptyset = A \cap \mathrm{cl}(B)$, then $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \emptyset$.*

It is known that the spaces $\beta(\omega)$ and $\omega^*$ have the property stated in Lemma 3.

**Lemma 4** *Let $Y$ be a subset of $\omega^*$ of cardinality strictly less than $2^c$. Then there is a set $\{F_\xi : \xi < 2^c\}$ of countable infinite discrete subsets of $\omega^*$ such that $\mathrm{cl}(F_\xi) \cap \mathrm{cl}(D) = \emptyset$ for every countable infinite discrete subset $D$ of $Y$ and for every $\xi < 2^c$, and such that $\{\mathrm{cl}(F_\xi) : \xi < 2^c\}$ is pairwise disjoint.*

**Proof:** We need the assumption of the existence of $2^c$-many weak-$P$-points in $\omega^*$ (for a proof of this fact see [Si]). Enumerate the set of weak-$P$-points of $\omega^*$ as $\{p_{(\xi,n)} : n < \omega, \xi < 2^c\}$. Set $F_\xi = \{p_{(\xi,n)} : n < \omega\}$ for each $\xi < 2^c$. Since $\mathrm{cl}(F_\xi) \cap F_\zeta = \emptyset = F_\xi \cap \mathrm{cl}(F_\zeta)$ for every $\xi < \zeta < 2^c$, by Lemma

3, $\mathrm{cl}(F_\xi)\cap\mathrm{cl}(F_\zeta)=\emptyset$ provided that $\xi<\zeta<2^c$. Since $|Y|<2^c$, the set $\mathcal{E}=\{\xi<2^c : Y\cap\mathrm{cl}(F_\xi)=\emptyset\}$ has size $2^c$. Now, let $D$ be a countable infinite discrete subset of $Y$ and fix $\xi\in\mathcal{E}$. Since $F_\xi$ consists of weak-$P$-points, we obtain that $F_\xi\cap\mathrm{cl}(D)=\emptyset$ and since $\mathrm{cl}(F_\xi)\cap D=\emptyset$, by Lemma 3, $\mathrm{cl}(F_\xi)\cap\mathrm{cl}(D)=\emptyset$. Thus, $\{F_\xi : \xi\in\mathcal{E}\}$ is the required set. □

We should remark that if $D$ is a countable infinite discrete subset of $\omega^*$, then $D^*$ is homeomorphic to $\omega^*$.

**Lemma 5** *Let $X$ be a subset of $\omega^*$ with $|X|\leq c$ and let $\{E_\xi : \xi<c\}$ be a set of discrete subsets of $\omega^*$ such that $|E_\xi|\leq\omega$ for every $\xi<c$ and*

$$\mathrm{cl}(E_\xi)\cap X=\emptyset \text{ for every } \xi<c.$$

*If $\kappa<2^c$ is a cardinal number and $\{D_\theta : \theta<c\}$ is a set of countable inifinite discrete subsets of $X$ such that $|cl(D_\theta)\cap E_\xi|<\omega$ for every $(\theta,\xi)\in c\times c$, then there is a set $\{F_{\theta,\eta} : (\theta,\eta)\in c\times\kappa\}$ of countable infinite discrete subsets of $\omega^*$ such that :*

*(1) $F_{\theta,\eta}\subseteq D^*_\theta$ for every $\theta<c$ and for every $\eta<\kappa$; and*

*(2) $\mathrm{cl}(F_{\theta,\eta})\cap(X\cup\bigcup_{\xi<c}\mathrm{cl}(E_\xi)\cup \bigcup\{\mathrm{cl}(F_{\upsilon,\iota}) : (\nu,\iota)\in c\times\kappa,\ (\upsilon,\iota)\neq(\theta,\eta)\})=\emptyset$ for every $(\theta,\eta)\in c\times\kappa$.*

**Proof:** First, observe from Lemma 3 that

$$(*)\qquad E^*_\xi\cap D^*_\theta=\emptyset \text{ for every } (\theta,\xi)\in c\times c.$$

Set $Y=X\cup\bigcup_{\xi<c}E_\xi$. Since $|\mathrm{cl}(D_0)\cap Y|\leq c$, by Lemma 4, there is a set $\{F_{0,\eta} : \eta<\kappa\}$ of countable infinite discrete subsets of $D^*_0$ satisfying that $\{\mathrm{cl}(F_{0,\eta}) : \eta<\kappa\}$ is pairwise disjoint and $\mathrm{cl}(F_{0,\eta})\cap\mathrm{cl}(D)=\emptyset$ for every countable infinite discrete subset $D$ of $D^*_0\cap Y$ and for $\eta<\kappa$. It is clear that $\mathrm{cl}(F_{0,\eta})\cap X=\emptyset$ for $\eta<\kappa$. Suppose that $\mathrm{cl}(F_{0,\eta})\cap\mathrm{cl}(E_\xi)\neq\emptyset$

for some $\xi < c$ and for some $\eta < \kappa$. By Lemma 3, we have that either $\mathrm{cl}(F_{0,\eta}) \cap E_\xi \neq \emptyset$ or $F_{0,\eta} \cap \mathrm{cl}(E_\xi) \neq \emptyset$. Since $\mathrm{cl}(F_{0,\eta}) \cap E_\xi \subseteq D_0^* \cap Y$ and $E_\xi$ is a countable discrete set, we must have that $\mathrm{cl}(F_{0,\eta}) \cap E_\xi \subseteq \mathrm{cl}(F_{0,\eta}) \cap E_\xi \cap D_0^* \cap Y = \emptyset$. So, $\emptyset \neq F_{0,\eta} \cap \mathrm{cl}(E_\xi) \subseteq D_0^* \cap \mathrm{cl}(E_\xi)$. Since $D_0^* \cap E_\xi^* = \emptyset$, we must have that $\emptyset \neq F_{0,\eta} \cap \mathrm{cl}(E_\xi) \subseteq F_{0,\eta} \cap D_0^* \cap E_\xi$, but this is a contradiction. Thus,

$$\mathrm{cl}(F_{0,\eta}) \cap (X \cup \bigcup_{\xi<c} \mathrm{cl}(E_\xi) \cup \bigcup_{\eta\neq\iota} \mathrm{cl}(F_{0,\iota})) = \emptyset \text{ for every } \eta < \kappa.$$

We shall proceed by transfinite induction. Assume that for every $\theta < \lambda < c$, we have defined a set $\{F_{\theta,\eta} : \eta < \kappa\}$ of countable infinite discrete subsets of $D_\theta^*$ such that, for every $\eta < \kappa$.

$$(**)\mathrm{cl}(F_{\theta,\eta}) \cap (X \cup \bigcup_{\xi<c} \mathrm{cl}(E_\xi) \cup \bigcup\{\mathrm{cl}(F_{\upsilon,\iota}) : (\upsilon,\iota) \in (\theta+1)\times\kappa, (\upsilon,\iota) \neq (\theta,\eta)\} = \emptyset.$$

Put $Z = X \cup \bigcup_{\xi<c} E_\xi \cup \bigcup_{\theta<\lambda}\bigcup_{\eta<\kappa} F_{\theta,\eta}$. According to Lemma 4, we can find a set $\{F_{\lambda,\eta} : \eta < \kappa\}$ of countable infinite discrete subsets of $D_\lambda^*$ such that $\{\mathrm{cl}(F_{\lambda,\eta}) : \eta < \kappa\}$ is pairwise disjoint and $\mathrm{cl}(F_{\lambda.\eta}) \cap \mathrm{cl}(D) = \emptyset$ for every countable infinite discrete subset $D$ of $D_\lambda^* \cap Z$ and for every $\eta < \kappa$. Fix $\eta < \kappa$. Then, $\mathrm{cl}(F_{\lambda,\eta}) \cap X = \emptyset$ and, by arguing as above, we have that $\mathrm{cl}(F_{\lambda,\eta}) \cap (\cup_{\xi<c}\mathrm{cl}(E_\xi)) = \emptyset$. Suppose now that $\mathrm{cl}(F_{\lambda,\eta})\cap\mathrm{cl}(F_{\theta,\iota}) \neq \emptyset$ for some $(\emptyset,\iota) \in \lambda\times\kappa$. Since $D_\lambda \cap \mathrm{cl}(F_{\theta,\iota}) = \emptyset$ and $(F_{\theta,\iota} - D_\lambda^*)\cap\mathrm{cl}(D_\lambda) = \emptyset$, by Lemma 3, we obtain that $\mathrm{cl}(D_\lambda) \cap \mathrm{cl}(F_{\theta,\iota} - D_\lambda^*) = \emptyset$. Hence, $\emptyset \neq \mathrm{cl}(F_{\lambda,\eta}) \cap \mathrm{cl}(F_{\theta,\iota}) \subseteq \mathrm{cl}(F_{\lambda,\eta}) \cap \mathrm{cl}(F_{\theta,\iota} \cap D_\lambda^*)$ which is a contradiction since $F_{\theta,\iota} \cap D_\lambda^*$ is a discrete subset of $D_\lambda^* \cap Z$. Thus, the set $F_\lambda, \eta$ satisfies $({}^*,{}^*)$ for every $\eta < \kappa$. consider the set $\{F_{\theta,\eta} : (\theta,\eta) \in c\times\kappa\}$. It follows from $({}^*\,{}^*)$ that

$$\mathrm{cl}(F_{\theta,\eta}) \cap (X \cup \bigcup_{\xi<c} \mathrm{cl}(E_\xi) \cup \bigcup\{\mathrm{cl}(F_{\upsilon,\iota}) : (\upsilon,\iota) \in c\times\kappa,\ (\upsilon,\iota) \neq (\theta,\eta)\}) = \emptyset$$

for every $(\theta,\eta) \in< c\times\kappa$. □

**Theorem 6** *Every subspace $X$ of $\beta(\omega)$ with $|X\cap\omega^*| \leq c$ can be* *extended to a subspace $\Xi(X)$ of $\beta(\omega)$ such that every countably compact closed subset of $\Xi(X)$ is finite.*

**Proof:** Our space $\Xi(X)$ will be constructed by transfinite induction. We may assumet that $|X| = c$. Indeed, enumerate all the infinite countable discrete subsets of $X \cap \omega^*$ as $\{D(0)_\theta : \theta < c\}$. Using Lemma 5, when $\kappa = 1$ and $E_\xi = \theta$ for each $\xi < c$, we can find a set $\{F(0)_\theta : \theta < c\}$ of infinite countable discrete subssets of $\omega^*$ such that

$(^*) F(0)_\theta \subseteq D(0)^*_\theta$ for every $\theta < c$; and

$(^{*}\,^{*}) \mathrm{cl}(F(0)_\theta) \cap (X \cup \bigcup\{\mathrm{cl}(F(0)_\iota) : \iota \neq \theta, \iota < c\}) = \emptyset$ for every $\theta < c$.

Put $X_0 = X \cup \bigcup_{\theta<c} F(0)_\theta$. Now, we proceed by transfinite induction. Suppose that for each $\upsilon < \lambda < \omega_1$ we have defined a subspace $X_\upsilon$ of $\beta(\omega)$ and two sets $\{F(\upsilon)_\theta : \theta < c\}$ and $\{D(\upsilon)_\theta : \theta < c\}$ of countable infinite discrete subsets of $\omega^*$ such that :

(1) $X_\upsilon = X \cup \bigcup_{\mu<\upsilon} \bigcup_{\theta<c} F(\mu)_\theta$;

(2) $\{D(\upsilon)_\theta : \theta < c\}$ is an enumeration of $\{D : D$ is a countable infinite discrete subset of $X_\upsilon \cap \omega^*$, $|\mathrm{cl}(D) \cap F(\mu)_\iota| < \omega$ for each $(\mu, \iota) \in \upsilon \times c\}$ for every $1 \leq \upsilon < \lambda$;

(3) $F(\upsilon)_\theta \subseteq D(\upsilon)^*_\theta$ for every $\upsilon < \lambda$ and for every $\theta < c$;

(4) $\mathrm{cl}(F(\upsilon)_\theta) \cap (X \cup \bigcup\{\mathrm{cl}(F(\mu)_\iota) : (\mu, \iota) \neq (\upsilon, \theta), (\mu, \iota) \in (\upsilon + 1) \times c\}) = \emptyset$ for every $\upsilon < \lambda$ and for every $\theta < c$.

Set $X_\lambda = X \cup \bigcup_{\upsilon<\lambda} \bigcup_{\theta<c} F(\upsilon)_\theta$. Now, enumerate the set

$$\{D : D \text{ is a countable infinite discrete subset of } X_\lambda \cap \omega^*, \\ |\mathrm{cl}(D) \cap F(\upsilon)_\theta| < \omega \text{ for every } (\upsilon, \theta) \in \lambda \times c\}$$

as $\{D(\lambda)_\theta : \theta < c\}$. By applying Lemma 5, for $\kappa = 1$ and for the set $\{F(\upsilon)_\theta : (\upsilon, \theta) \in \lambda \times c\}$, we can find a set $\{F(\lambda)_\theta :$

$\theta < c\}$ of countable infinite discrete subsets of $\omega^*$ such that $\{\mathrm{cl}(F(\lambda)_\theta) : \theta < c\}$ is pairwise disjoint,

(a) $F(\lambda)_\theta \subseteq D(\lambda)^*_\theta$ for every $\theta < c$; and

(b) $\mathrm{cl}(F(\lambda)_\theta) \cap (X \cup \bigcup_{\upsilon<\lambda} \bigcup_{\iota<c} \mathrm{cl}(F(\upsilon)_\iota)) = \emptyset$ for every $\theta < c$.

We claim that $\Xi(X) = \bigcup_{\lambda<\omega_1} X_\lambda$ satisfies the conditions. Observe from clause (4) that $\mathrm{cl}(F(\lambda)_\theta) \cap \mathrm{cl}(F(\mu)_\iota) = \emptyset$ for every $(\lambda, \theta), (\mu, \iota) \in \omega_1 \times c$ with $(\lambda, \theta) \neq (\mu, \iota)$. Hence, we have that $F(\lambda)_\theta$ is a closed discrete subset of $\Xi(X)$ for every $(\lambda, \theta) \in \omega_1 \times c$. Let $C$ be a countably compact closed subset of $\Xi(X)$. Without loss of generality we may suppose that $C = \mathrm{cl}_{\Xi(X)}(D)$, where $D$ is a countable infinite discrete subset of $\Xi(X) \cap \omega^*$. Then there is $\lambda < \omega_1$ for which $D \subseteq X_\lambda$. Since $\mathrm{cl}_{\Xi(X)}(D)$ is countably compact, then $\mathrm{cl}(D) \cap F(\upsilon)_\theta$ is finite for every $(\upsilon, \theta) \in \lambda \times c$. Then, $D = D(\lambda)_\iota$ for some $\iota < c$. By clause (3) we obtain that $F(\lambda)_\iota \subseteq D^* \cap \Xi(X) \subseteq \mathrm{cl}_{\Xi(X)}(D)$, but this is a contradiction. □

**Example 7** Let $X$ be a countably compact subspace of $\beta(\omega)$ such that $|X| = c$ and $\omega \subseteq X$ (for the existence of a space with these properties see [Va 2.13]). Then, $\Xi(X)$ is an isocompact (Tychonoff) space which is not *cl*-isocompact.

It is noticed in [Sa] that the product of a set of *cl*-isocompact spaces is *cl*-isocompact; M. Sakai [Sa] proved that if $X$ is *cl*-isocompact and $Y$ is isocompact, then $X \times Y$ is isocompact; and P. Bacon [Ba] showed that if $X$ is isocompact and $Y$ is a locally compact isocompact space, then $X \times Y$ is isocompact: some improvements of these three results are available in [GS]. But, this is not the case for isocompactness, using the method developed here it is possible to define two isocompact Tychonoff spaces $Y$ and $Z$ such that $Y \times Z$ is not isocompact.

**Example 8** Let $X$ be a countably compact subspace of $\beta(\omega)$ such that $\omega \subseteq X$ and $|X| = c$. In the proof of Theorem 6 we used Lemma 5 for $\kappa = 1$. We should remark that, by applying Lemma 5 for any $\kappa < 2^c$, it is possible to define, for every $\eta < \kappa$, a set $\{F(v)_{(\theta,\eta)} : \nu < \omega_1, \ \theta < c\}$ of countably infinite discrete subsets of $\omega^*$ such that

$$(\bigcup_{\nu<\omega_1}\bigcup_{\theta<c} F(\nu)_{(\theta,\eta)}) \cap (\bigcup_{\nu<\omega_1}\bigcup_{\theta<c} F(\nu)_{(\theta,\iota)}) = \emptyset,$$

for every $\eta < \iota < \kappa$, and the corresponding $\Xi$-expansion $\Xi_\eta(X) = X \cup \bigcup_{\nu<\omega_1}\bigcup_{\theta<c} F(\nu)_{(\theta,\eta)}$ of $X$ have the property that every countably compact closed subset is finite, for every $\eta < \kappa$. So, $\Xi_\eta(X)$ is isocompact for every $\eta < \kappa$, but if $\eta < \iota < \kappa$, then $\Xi_\eta(X) \times \Xi_\iota(X)$ cannot be isocompact since it contains a closed copy of $X$. Since isocompactness is preserved under finite topological sums and closed-hereditary, we have that the topological sum $\Xi_\eta(X) \oplus \Xi_\iota(X)$ is a Tychonoff space whose square is not isocompact provided that $\eta < \iota < \kappa$.

**Question 9** Does there exist for every $2 \leq n < \omega$ an isocompact (Tychonoff) space $X$ such that $X^n$ is isocompact and $X^{n+1}$ is not isocompact?

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