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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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PAIRS OF INDECOMPOSABLE CONTINUA WHOSE PRODUCT IS MUTUALLY APOSYNDETIC

Alejandro Illanes

Abstract

In this paper we prove that if p and q are relative prime positive integers and S_p, S_q are the respective p-adic and q-adic solenoids, the their topological product $S_p \times S_q$ is mutually aposyndetic. This answers a question by Charles L. Hagopian.

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Introduction

A *continuum* is a compact connected metric space. A *map* is a continuous function. A continuum X is said to be *mutually aposyndetic* provided that for any two distinct points x and y in X there exist two disjoint subcontinua L and M of X such that $x \in \text{int}_X(L)$ and $y \in \text{int}_X(M)$. A continuum X is said to be *strictly non-mutually aposyndetic* if each pair of subcontinua of X which have interiors intersect. Clearly, a nondegenerate mutually aposyndetic continuum is not strictly non-mutually aposyndetic.

The concept of mutual aposyndesis was introduced by Charles L. Hagopian in [1] where he proved that, the product of two chainable continua is strictly non-mutually aposyndetic if and only if each factor is indecomposable. He also asked the question ([1, p. 622]): Is the topological product of two indecomposable compact metric continua strictly non-mutually aposyndetic?

In this paper we answer Hagopian's question in the negative by showing that the product $S_p \times S_q$ is mutually aposyndetic, where for an integer $m \geq 2$, S_m is the m-adic solenoid and p and q are relative prime.

For a discussion on the relationship between aposyndesis and products we refer the interested reader to the paper by Leland E. Rogers ([3]).

1 $S_p \times S_q$ is mutually aposyndetic.

For each $p = 2, 3, ...$, let $f^p : S^1 \to S^1$ be given by $f^p(z) = z^p$ for each $z \in S^1$ (where S^1 is the unit circle in the plane, and z^p denotes the pth power of z using complex multiplication). For a given p, let

$$S_p = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}, \text{where each} X_n = S^1 \text{and each} f_n = f^p.$$

As usual S_p is called the *p-adic solenoid.*

Theorem *If p, $q \geq 2$ are relative prime integers, then $S_p \times S_q$ is mutually aposyndetic.*

Proof: We consider S_p with the usual group structure, where the product of two elements $u = (u_1, u_2, ...)$ and $v = (v_1, v_2, ...)$ in S_p is defined by $u * v = (u_1 v_1, u_2 v_2, ...)$ and $u_n v_n$ is the product of u_n and v_n as complex numbers.

We consider the exponential map $e : E^1 \to S^1$ given by $e(t) = (\cos(t), \sin(t))$. We also consider in S^1 the metric D defined by $D(z, w) =$ the length of the shortest subarc of S^1 which joins z and w. Given $z \in S^1$ and $\epsilon > 0$, define $N(\epsilon, z) = \{w \in S^1 : D(z, w) \leq \epsilon\}$.

Define $g_p : E^1 \to S_p$ by:

$$g_p(t) = (e(t), e(\frac{t}{p}), e(\frac{t}{p^2}), e(\frac{t}{p^3}), ...)$$

Given two points $z, w \in S^1$, define:

$$T(z, w) = \{(a * g_p(t), b * g_q(t)) \in S_p \times S_q : t \in E^1, a \in \rho_1^{-1}(z) \text{ and } b \in r_1^{-1}(w)\},$$

where, for each n, ρ_n (resp., r_n) is the n-th projection from the solenoid S_p (resp., S_q) into S^1.

We will prove that:

$$T(z, w) \text{ is a subcontinuum of } S_p \times S_q ...(1)$$

In order to prove that $T(z, w)$ is compact it is enough to show that $T(z, w)$ is the image $\mathcal{A}$ of the compact set $\rho_1^{-1}(z) \times r_1^{-1}(w) \times [0, 2\pi]$ under the continuous function $F((a, b, t)) = (a * g_p(t), b * g_q(t))$.

It is clear that $\mathcal{A} \subset T(z, w)$. For proving the other inclusion, take an element $\alpha = (a * g_p(t), b * g_q(t)) \in T(z, w)$, with

$t \in E^1$, $\rho_1(a) = z$ and $r_1(b) = w$. Let k be an integer and let $s \in [0, 2\pi)$ be such that $t = s+2k\pi$. Then $g_p(t) = g_p(s+2k\pi) = g_p(s) * g_p(2k\pi)$ and $g_q(t) = g_q(s+2k\pi) = g_q(s) * g_q(2k\pi)$, Since $\rho_1(a * g_p(2k\pi)) = z$ and $r_1(b * g_q(2k\pi)) = w$, we conclude that $\alpha \in \mathcal{A}$. This completes the proof of the compactness of $T(z, w)$.

Now, we will prove that $T(z, w)$ is connected.

Let $s_0, t_0 \in [0, 2\pi)$ be real numbers such that $e(s_0) = z$ and $e(t_0) = w$. Then the set $G = \{(g_p(s_0) * g_p(t), g_q(t_0) * g_q(t)) \in T(z, w) : t \in E^1\}$ is a connected subset of $T(z, w)$. Then, in order to show that $T(z, w)$ is connected, it will be enough to prove that G is dense in $T(z, w)$.

Take an element $\alpha = (a * g_p(t), b * g_q(t)) \in T(z, w)$, with $t \in E^1$, $a = (a_1, a_2, ...) \in \rho_1^{-1}(z)$ and $b = (b_1, b_2, ...) \in r_1^{-1}(w)$ and take a basic open subset $W = [(U_1 \times ... \times U_m \times S^1 \times ...) \cap S_p] \times [(V_1 \times ... \times V_m \times S^1 \times ...) \cap S_q]$ of $S_p \times S_q$ containing the point α, where $m \geq 2$ and $a_n e(\frac{t}{p^{n-1}}) \in U_n$ and $b_n e(\frac{t}{q^{n-1}}) \in V_n$ for each $1 \leq n \leq m$.

Since $a \in S_p$, $a_{m+1}^{p^m} = a_1 = z$, so a_{m+1} is a p^m-th root of z. On the other hand, $e(\frac{s_0}{p^m})$ is another p^m-th root of z. Then there is a p^m-th root x of 1 such that $a_{m+1} = e(\frac{s_0}{p^m})x$. Thus there exists $i \in \{1, ..., p^m\}$, such that $a_{m+1} = e(\frac{s_0}{p^m})e(\frac{2\pi i}{p^m})$. Similarly, there exists $j \in \{1, ..., q^m\}$ such that $b_{m+1} = e(\frac{t_0}{q^m})e(\frac{2\pi j}{q^m})$.

Since p^m and q^m are relative prime, there exists integers i_1 and j_1 such that $i-j = i_1 p^m + j_1 q^m$. Let $k = i - i_1 p^m = j + j_1 q^m$. Define $\beta = (g_p(s_0) * g_p(t + 2\pi k), g_q(t_0) * g_q(t + 2\pi k)) \in G$.

For each $1 \leq n \leq m$,

$$\begin{aligned} a_n e(\frac{t}{p^{n-1}}) &= a_{m+1}^{p^{m-n+1}} e(\frac{t}{p^{n-1}}) \\ &= e(\frac{s_0 p^{m-n+1}}{p^m}) e(\frac{2\pi i p^{m-n+1}}{p^m}) e(\frac{t}{p^{n-1}}) \\ &= e(\frac{s_0}{p^{n-1}}) e(\frac{t + 2\pi i}{p^{n-1}}). \end{aligned}$$

On the other hand,

$$\begin{aligned} e(\frac{s_0}{p^{n-1}})e(\frac{t+2\pi k}{p^{n-1}}) &= e(\frac{s_0}{p^{n-1}})e(\frac{t+2\pi i}{p^{n-1}})e(\frac{-2\pi i_1 p^m}{p^{n-1}}) \\ &= a_n e(\frac{t}{p^{n-1}}). \end{aligned}$$

Thus $e(\frac{s_0}{p^{n-1}})e(\frac{t+2\pi k}{p^{n-1}}) = a_n e(\frac{t}{p^{n-1}})$.

Similarly, for each $1 \leq n \leq m$, $e(\frac{t_0}{q^{n-1}})e(\frac{t+2\pi k}{q^{n-1}}) = b_n e(\frac{t}{q^{n-1}})$.

This proves that $\beta \in W \cap G$. Thus G is dense in $T(z,w)$. Therefore, $T(z,w)$ is connected

Hence, $T(z,w)$ is a subcontinuum of $S_p \times S_q$.

Now, we will show that there is a homeomorphism $h : S_p \to S_p$ such that, for each $b \in S_p - \rho_1^{-1}(1)$, $\rho_1(b) \neq \rho_1(h(b))$.

Fix a homeomorphism $\gamma : S^1 \to S^1$ such that $\gamma(1) = 1$, $z \neq \gamma(z)$ for each $z \in S^1 - \{1\}$ and $D(z, \gamma(z)) < \pi$ for every $z \in S^1$. Consider a continuous fold $\delta : S^1 - \{-1\} \to S^1 - \{-1\}$ of the p-th root function.

Define $h : S_p \to S_p$ by:

$$h(b) = (\gamma(b_1), b_2\delta(\frac{\gamma(b_1)}{b_1}), b_3\delta(\delta(\frac{\gamma(b_1)}{b_1})), b_4\delta(\delta(\delta(\frac{\gamma(b_1)}{b_1}))), \ldots),$$

where $b_n = \rho_n(b)$.

Clearly, h has the desired properties.

We are ready to prove that $S_p \times S_q$ is mutually aposyndetic. Let (a,b) and (c,d) be two distinct points of $S_p \times S_q$. Since S_p and S_q are topological groups, applying a translation if necessary, we may assume that $a = (1,1,\ldots) \in S_p$ and $c = (1,1,\ldots) \in S_p$. Since $b \neq a$ or $c \neq d$, we may also assume that $c \neq d$. Then there exists $n \geq 1$ such that $c_n \neq d_n$. Since S_q is homeomorphic to $\{(u_n, u_{n+1}, \ldots) \in S_q : (u_1, u_2, \ldots) \in S_q\}$, we may assume that $c_1 \neq d_1$, that is $d_1 \neq 1$. Finally, if $b_1 = d_1$, applying the homeomorphism constructed in the paragraph above, we may assume that $b_1 \neq d_1$. Let $\epsilon = D(d_1, b_1)/3$.

Define

$$L = [\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,b_1))] \cup T(1,b_1)\text{and}$$

$$M = [\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,d_1))] \cup T(1,d_1).$$

Clearly L and M are closed subsets of $S_p \times S_q$, $(a,b) \in Int(L)$ and $(c,d) \in Int(M)$.

Since $N(\epsilon,b_1) \cap N(\epsilon,d_1) = \emptyset$, we have $[\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,b_1))]$ does not intersect $[\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,d_1))]$.

If there is a point $(u * g_p(s), v * g_q(s))$ in $T(1,b_1) \cap [\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,d_1))]$, with $s \in E^1$, $\rho_1(u) = 1$ and $r_1(v) = b_1$, then $e(s) = \rho_1(u * g_p(s)) \in N(\epsilon,1)$ and $b_1 e(s) = r_1(v * g_q(s)) \in N(\epsilon,d_1)$. Since $D(e(s),1) \leq \epsilon$, then $D(b_1 e(s), b_1) \leq \epsilon$. Thus $b_1 e(s) \in N(\epsilon,b_1) \cap N(\epsilon,d_1)$, which contradicts the choice of ϵ. Therefore, $T(1,b_1)$ does not intersect $[\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,d_1))]$.

Similarly, $T(1,d_1)$ does not intersect $[\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,b_1))]$.

Finally, if there is a point $(u * g_p(s), v * g_q(s)) = (x * g_p(t), y * g_q(t))$ in $T(1,b_1) \cap T(1,d_1)$, where $s,t \in E^1$, $\rho_1(u) = 1 = \rho_1(x)$, $r_1(v) = b_1$ and $r_1(y) = d_1$, then $e(s) = \rho_1(u * g_p(s)) = \rho_1(x * g_p(t)) = e(t)$. Thus $b_1 e(s) = r_1(v * g_q(s)) = r_1(y * g_q(t)) = d_1 e(t)$. This implies that $b_1 = d_1$. This contradiction proves that $T(1,b_1) \cap T(1,d_1) = \emptyset$.

Therefore, $L \cap M = \emptyset$.

In order to prove that L is connected, take any point (u,v) in $\rho_1^{-1}(N(\epsilon,1)) \times r_1^{-1}(N(\epsilon,b_1))$. Then $D(u_1,1) \leq \epsilon$ and $D(v_1,b_1) \leq \epsilon$, where $u_1 = \rho_1(u)$ and $v_1 = r_1(v)$. Let $\lambda, \eta : [0,1] \to S^1$ be maps such that $\lambda(0) = u_1$, $\eta(0) = v_1$, $\lambda(1) = 1$,

$\eta(1) = b_1$ and $D(\lambda(t), 1)$, $D(\lambda(t), u_1)$, $D(\eta(t), b_1)$, and $D(\eta(t), v_1) \leq \epsilon$ for every $t \in [0,1]$. Consider the continuous fold

$$\delta_p : S^1 - \{-1\} \to S^1 - \{-1\} (\text{resp.}, \delta_q : S^1 - \{-1\} \to S^1 - \{-1\})$$

of the p-th root (resp., q-th root) function such that $\delta_p(1) = 1$ (resp., $\delta_q(1) = 1$).

Define $\sigma : [0,1] \to S_p \times S_q$ by: $\sigma(t) =$

$$[(\lambda(t), u_2\delta_p(\tfrac{\lambda(t)}{u_1}), u_3\delta_p(\delta_p(\tfrac{\lambda(t)}{u_1})), \ldots), \\ (\eta(t), v_2\delta_q(\tfrac{\eta(t)}{v_1}), v_3\delta_q(\delta_q(\tfrac{\eta(t)}{v_1})), \ldots)].$$

Then σ is continuous, $\sigma(t) \in \rho_1^{-1}(N(\epsilon, 1)) \times r_1^{-1}(N(\epsilon, b_1)) \subset L$ for every $t \in [0,1]$, $\sigma(0) = (u, v)$ and $\sigma(1) \in T(1, b_1)$.

Hence (u, v) can be connected with $T(1, b_1)$ by a connected subset of L. Since $T(1, b_1)$ is connected, we conclude that L is connected.

Similarly, M is connected.

Therefore, $S_p \times S_q$ is mutually aposyndetic.

Questions

QUESTION 1. (C. L. Hagopian) Are there two tree-like indecomposable continua X and Y such that $X \times Y$ is mutually aposyndetic?

QUESTION 2. ([2, p. 87]) If M is an indecomposable plane continuum, must the product $M \times M$ be strictly non-mutually aposyndetic?

QUESTION 3. Is there an indecomposable continuum X such that $X \times X$ is mutually aposyndetic?

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Instituto de Matemáticas
UNAM, Cd. Universitaria
México, 04510
D.F., MEXICO
e-mail address: illanes@gauss.matem.unam.mx