Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 22, 1997

ORTHOCOMPACT SUBSPACES IN PRODUCTS OF TWO ORDINALS

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Abstract

It is known that, for subspaces A and B of an ordinal, normality, orthocompactness and weak suborthocompactness of $X = A \times B$ are equivalent. So it is natural to ask whether they are equivalent for *all* subspaces of the product of two ordinals. In this paper, we will characterize orthocompactness of such subspaces. As corollaries, we will show that, for such subspaces, orthocompactness and weakly suborthocompactness are equivalent, but orthocompactness and normality are not.

1991 *Mathematics Subject Classification.* 54B10, 54D20
Keywords and phrases. orthocompact, weakly suborthocompact, normal, product space

1 Introduction

It is well known that any ordinal with the order topology is hereditarily normal and hereditarily orthocompact. But, in general, products of two ordinals are not. In fact, $(\omega_1 + 1) \times \omega_1$ is neither normal nor orthocompact. In [KOT], it was proved that the normality, collectionwise normality and shrinking property of $A \times B$, where A and B are subspaces of ordinals, are equivalent. It was asked whether these properties are also equivalent for *all* subspaces of products of two ordinals [KOT, Problem (i)]. Recently, in [KNSY], this problem was solved affirmatively. On the other hand, in [KY], it was proved that the normality, orthocompactness and weak suborthocompactness of such $A \times B$'s are equivalent. So it is natural to ask whether these properties are also equivalent for *all* subspaces of products of two ordinals.

In this paper, orthocompactness of such subspaces will be characterized. As corollaries, it will be shown, in the realm of subspaces of products of two ordinals:

(1) Orthocompactness and weak suborthocompactness are equivalent.

(2) There is an orthocompact subspace of ω_1^2 which is not normal.

(3) Normal subspaces of ω_1^2 are orthocompact.

(4) There is a normal subspace of $(\omega_1 + 1)^2$ which is not orthocompact.

(5) If X is a subspace of $\omega_1 \times \omega_2$ such that $X \cap (\alpha + 1) \times \omega_2$ and $X \cap \omega_1 \times (\beta + 1)$ are orthocompact for each $\alpha < \omega_1$ and $\beta < \omega_2$, then X is orthocompact.

We recall basic definitions and introduce specific notation from [KNSY]. In our discussion, we always assume $X \subset (\lambda + 1)^2$ for some suitably large ordinal λ. Moreover, in general, the letters μ and ν stand for limit ordinals with $\mu \leq \lambda$ and $\nu \leq \lambda$. For each $A \subset \lambda + 1$ and $B \subset \lambda + 1$ put

$$X_A = A \times (\lambda + 1) \cap X, X^B = (\lambda + 1) \times B \cap X,$$

and

$$X_A^B = X_A \cap X^B.$$

For each $\alpha \leq \lambda$ and $\beta \leq \lambda$, put

$$V_\alpha(X) = \{\beta \leq \lambda : \langle \alpha, \beta \rangle \in X\},$$

$$H_\beta(X) = \{\alpha \leq \lambda : \langle \alpha, \beta \rangle \in X\}.$$

cfμ denotes the cofinality of the ordinal μ. When $\omega_1 \leq \mathrm{cf}\mu$, a subset S of μ is called *stationary in* μ if it intersects all cub (closed and unbounded) sets in μ. Moreover for each $A \subset \mu$, $\mathrm{Lim}_\mu(A)$ is the set $\{\alpha < \mu : \alpha = \sup(A \cap \alpha)\}$, in other words, the set of all cluster points of A in μ. Where for convenience, we consider $\sup\emptyset = -1$ and -1 is the immediate predecessor of the ordinal 0. Therefore $\mathrm{Lim}_\mu(A)$ is cub in μ whenever A is unbounded in μ. We will simply denote $\mathrm{Lim}_\mu(A)$ by $\mathrm{Lim}(A)$ if the situation is clear in its context. In particular, assume C is a cub set in μ with $\omega \leq \mathrm{cf}\mu$, then $\mathrm{Lim}(C) \subset C$. In this case, we define $\mathrm{Succ}(C) = C \setminus \mathrm{Lim}(C)$, and $p_C(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Then note that, for each $\alpha \in C$, $p_C(\alpha) \in C \cup \{-1\}$, and $p_C(\alpha) < \alpha$ iff $\alpha \in \mathrm{Succ}(C)$. So $p_C(\alpha)$ is the immediate predecessor of $\alpha \in \mathrm{Succ}(C)$ in $C \cup \{-1\}$. Moreover observe that $\mu \setminus C$ is covered by the pairwise disjoint collection $\{(p_C(\alpha), \alpha] : \alpha \in \mathrm{Succ}(C)\}$ of clopen intervals of μ.

A strictly increasing function $M : \mathrm{cf}\mu + 1 \to \mu + 1$ is said to be *normal* if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma \leq \mathrm{cf}\mu$ and $M(\mathrm{cf}\mu) = \mu$. Observe that, if $\omega_1 \leq \mathrm{cf}\mu$, then two normal functions on $\mathrm{cf}\mu + 1$ to $\mu + 1$ coincide on a cub set of $\mathrm{cf}\mu$. Note that a normal function on $\mathrm{cf}\mu + 1$ always exists if $\mathrm{cf}\mu \geq \omega$. So we always fix a normal function

$M : \mathrm{cf}\mu + 1 \to \mu + 1$ for each ordinal μ with $\mathrm{cf}\mu \geq \omega$. In particular, if μ is regular, i.e., $\mathrm{cf}\mu = \mu$, then we fix the identity map on $\mu+1$ as the normal function. For convenience, we define $M(-1) = -1$. Then M carries $\mathrm{cf}\mu + 1$ homeomorphically to the range $\mathrm{ran}M$ of M and $\mathrm{ran}M$ is closed in $\mu + 1$. Note that for all $S \subset \mu$ with $\omega_1 \leq \mathrm{cf}\mu$, S is stationary in μ if and only if $M^{-1}(S)$ is stationary in $\mathrm{cf}\mu$.

Let μ and ν be two limit ordinals with $\mu \leq \lambda$ and $\nu \leq \lambda$, moreover $M : \mathrm{cf}\mu + 1 \to \mu + 1$ and $N : \mathrm{cf}\nu + 1 \to \nu + 1$ be the fixed normal functions on $\mathrm{cf}\mu + 1$ and $\mathrm{cf}\nu + 1$ respectively. Furthermore assume $\langle \mu, \nu \rangle \notin X$ and $\omega_1 \leq \mathrm{cf}\mu = \mathrm{cf}\nu = \kappa$. Define

$$\triangle_{MN}(X) = \{\gamma < \kappa : \langle M(\gamma), N(\gamma) \rangle \in X\}.$$

Note that stationarity of $\triangle_{MN}(X)$ in κ does not depend on the choices of the normal functions M and N.

Let Y be a topological space. Subsets F and G of Y are said to be *separated* if there are disjoint open sets U and V containing F and G respectively, of course, separated sets are disjoint. A collection $\mathcal{V}$ of open sets of a space Y is said to be *interior preserving* if $\bigcap \mathcal{V}'$ is open for every $\mathcal{V}' \subset \mathcal{V}$. Observe that $\mathcal{V}$ is interior preserving iff $\bigcap(\mathcal{V})_y$ is a neighborhood of y for each $y \in \bigcup \mathcal{V}$, where $(\mathcal{V})_y = \{V \in \mathcal{V} : y \in V\}$. A space Y is *orthocompact* if, for every open cover $\mathcal{U}$ of Y, there is an interior preserving open refinement $\mathcal{V}$ of $\mathcal{U}$ which covers Y. Where $\mathcal{V}$ is said to be an *open refinement of* $\mathcal{U}$ if for each $V \in \mathcal{V}$, V is open and there is $U \in \mathcal{U}$ such that $V \subset U$. We do not require open refinements cover the space. Moreover Y is *weakly suborthocompact* if, for every open cover $\mathcal{U}$, there is an open refinement $\mathcal{V}$, which is represented as $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$, such that, for each $y \in Y$, there is $n \in \omega$ such that $y \in \bigcup \mathcal{V}_n$ and $\bigcap(\mathcal{V}_n)_y$ is a neighborhood of y. We call such a $\mathcal{V}$ as *weak ι-refinement* of $\mathcal{U}$.

2 Theorem and Lemmas

Using the notation described in section 1, we shall show:

Theorem *Assume $X \subset (\lambda+1)^2$. The following (1)-(3) are equivalent:*

(1) *X is orthocompact.*

(2) *X is weakly suborthocompact.*

(3) *For every $\langle \mu, \nu \rangle \in (\lambda+1)^2 \setminus X$ with $\omega_1 \leq \mathrm{cf}\mu = \mathrm{cf}\nu = \kappa$, the following (a)-(c) hold:*

  (a) *If $\triangle_{MN}(X)$ is not stationary in κ, then there is a cub set C in κ such that $X \cap M(C) \times N(C) = \emptyset$.*

  (b) *If $H_\nu(X) \cap \mu$ is stationary in μ, then there is $\mu' < \mu$ such that $\mathrm{Cl}_X X^\nu \cap X^{\{\nu\}}_{(\mu', \mu]} = \emptyset$, where $\mathrm{Cl}_X X^\nu$ denotes the closure of X^ν in X.*

  (c) *If $V_\mu(X) \cap \nu$ is stationary in ν, then there is $\nu' < \nu$ such that $\mathrm{Cl}_X X_\mu \cap X^{(\nu', \nu]}_{\{\mu\}} = \emptyset$.*

To prove the theorem, we need lemmas. First it is easy to show:

Lemma 1 *Assume, for each $\alpha \in A$, $\mathcal{V}_\alpha$ is an interior preserving collection of open sets in a space Y. If $\{\bigcup \mathcal{V}_\alpha : \alpha \in A\}$ is point finite, then $\mathcal{V} = \bigcup_{\alpha \in A} \mathcal{V}_\alpha$ is also interior preserving. In particular:*

(1) *If $\mathcal{Y}$ is a finite collection of orthocompact open subspaces of Y, then $\bigcup \mathcal{Y}$ is also orthocompact.*

(2) *If $\mathcal{Y}$ is a pairwise disjoint collection of orthocompact open subspaces of Y, then $\bigcup \mathcal{Y}$ is also orthocompact.*

Lemma 2 *Let κ be a regular uncountable cardinal and X a weakly suborthocompact subspace of $(\kappa+1)^2\setminus\{\langle\kappa,\kappa\rangle\}$. If $\triangle(X)=\{\gamma<\kappa:\langle\gamma,\gamma\rangle\in X\}$ is not stationary in κ, then there is a cub set C in κ such that $X\cap C^2=\emptyset$.*

Proof: Assume $\triangle(X)$ is not stationary in κ. Take a cub set D of κ which is disjoint from $\triangle(X)$. □

Claim 1 $\mathcal{X}=\{X_{(p_D(\alpha),\alpha]}^{(p_D(\alpha),\alpha]}:\alpha\in Succ(D)\}$ *covers* $X\cap\{\langle\gamma,\gamma\rangle:\gamma<\kappa\}$.

Proof: Let $\langle\gamma,\gamma\rangle\in X$. Then $\gamma\in\triangle(X)$. Take the minimal $\alpha\in D$ such that $\gamma\leq\alpha$. It follows from $D\cap\triangle(X)=\emptyset$ that $p_D(\alpha)<\gamma<\alpha$ and therefore $\alpha\in\mathrm{Succ}(D)$. So $\langle\gamma,\gamma\rangle\in X_{(p_D(\alpha),\alpha]}^{(p_D(\alpha),\alpha]}\in\mathcal{X}$.

Claim 2 $A=\{\alpha<\kappa:V_\alpha(X)\cap\kappa$ *is stationary in* $\kappa\}$ *is not stationary in* κ.

Proof: Assume A is stationary in κ. Put $Y=[\mathrm{X}\cap\{\langle\alpha,\beta\rangle:\alpha\leq\beta\leq\kappa\}]\setminus\bigcup\mathcal{X}$. Then Y is considered as the upper-left half of X. By Claim 1, Y is a closed subspace of X which is disjoint from $\{\langle\gamma,\gamma\rangle:\gamma<\kappa\}$. Observe that, for each $\alpha<\kappa$, $V_\alpha(X)\cap\kappa$ is stationary in κ iff so is $V_\alpha(Y)\cap\kappa$. For each $\alpha<\kappa$, put $U(\alpha)=Y_{[0,\alpha]}^{(\alpha,\kappa]}$. Then $\mathcal{U}=\{U(\alpha):\alpha<\kappa\}$ is an open cover of the weakly suborthocompact space Y. So there is a weak ι-refinement $\mathcal{V}=\bigcup_{n\in\omega}\mathcal{V}_n$ of $\mathcal{U}$. Fix $\alpha\in A$ and $\beta\in V_\alpha(Y)\cap\kappa$. Since $\langle\alpha,\beta\rangle\in\bigcup\mathcal{V}_n$ and $\bigcap(\mathcal{V}_n)_{\langle\alpha,\beta\rangle}$ is a neighborhood of $\langle\alpha,\beta\rangle$ for some $n\in\omega$, fix $n(\alpha,\beta)\in\omega$, $f(\alpha,\beta)<\alpha$ and $g(\alpha,\beta)<\beta$ such that

$$\langle\alpha,\beta\rangle\in\bigcup\mathcal{V}_{n(\alpha,\beta)}\text{ and }\langle\alpha,\beta\rangle\in Y_{(f(\alpha,\beta),\alpha]}^{(g(\alpha,\beta),\beta]}\subset\bigcap(\mathcal{V}_{n(\alpha,\beta)})_{\langle\alpha,\beta\rangle}.$$

It follows from $\langle\alpha,\beta\rangle\in Y$ that $\alpha<\beta$, so we may assume $\alpha\leq g(\alpha,\beta)$. Moreover, since $\mathcal{V}_{n(\alpha,\beta)}$ is a refinement of $\mathcal{U}$, we can fix $\gamma(\alpha,\beta)<\kappa$ such that

$$\bigcap(\mathcal{V}_{n(\alpha,\beta)})_{\langle\alpha,\beta\rangle}\subset U(\gamma(\alpha,\beta)).$$

Then it follows from $\langle\alpha,\beta\rangle \in U(\gamma(\alpha,\beta))$ that $\alpha \le \gamma(\alpha,\beta) < \beta$. Applying the PDL to $V_\alpha(Y)\cap\kappa$, we have a stationary set $T(\alpha) \subset V_\alpha(Y)\cap\kappa$, $n(\alpha)\in\omega$, $f(\alpha)<\alpha$, $g(\alpha)<\kappa$ and $\gamma(\alpha)<\kappa$ such that $n(\alpha,\beta)=n(\alpha)$, $f(\alpha,\beta)=f(\alpha)$, $g(\alpha,\beta)=g(\alpha)$ and $\gamma(\alpha,\beta)=\gamma(\alpha)$ for each $\beta\in T(\alpha)$. Here the condition "$f(\alpha)<\alpha$" is guaranteed by $|\alpha|<\kappa$. Next again applying the PDL to A, we find a stationary set $S\subset A$, $n_0\in\omega$ and $\alpha_0<\kappa$ such that $n(\alpha)=n_0$ and $f(\alpha)=\alpha_0$ for each $\alpha\in S$. Then for each $\beta\in T(\alpha)$ with $\alpha\in S$, we have

$$\langle\alpha,\beta\rangle \in \bigcup \mathcal{V}_{n_0} \text{ and } \langle\alpha,\beta\rangle \in Y^{(g(\alpha),\beta]}_{(\alpha_0,\alpha]} \subset \bigcap(\mathcal{V}_{n_0})_{\langle\alpha,\beta\rangle} \subset U(\gamma(\alpha)).$$

Take $\alpha_1\in S$ with $\alpha_0<\alpha_1$, and then take $\beta_1\in T(\alpha_1)$ with $g(\alpha_1)<\beta_1$. Moreover take $\alpha_2\in S$ with $\beta_1<\alpha_2$, and $\beta_2\in T(\alpha_1)$ with $g(\alpha_2)<\beta_2$. Finally take $\beta_3\in T(\alpha_2)$ with $\beta_2<\beta_3$. Then we have

$$\alpha_0<\alpha_1\le g(\alpha_1)<\beta_1<\alpha_2\le g(\alpha_2)<\beta_2<\beta_3.$$

It follows from

$$\langle\alpha_1,\beta_2\rangle \in Y^{(g(\alpha_2),\beta_3]}_{(\alpha_0,\alpha_2]} \subset \bigcap(\mathcal{V}_{n_0})_{\langle\alpha_2,\beta_3\rangle}$$

that

$$(\mathcal{V}_{n_0})_{\langle\alpha_2,\beta_3\rangle} \subset (\mathcal{V}_{n_0})_{\langle\alpha_1,\beta_2\rangle},$$

so

$$\bigcap(\mathcal{V}_{n_0})_{\langle\alpha_1,\beta_2\rangle} \subset \bigcap(\mathcal{V}_{n_0})_{\langle\alpha_2,\beta_3\rangle}.$$

Therefore we have

$$\langle\alpha_1,\beta_1\rangle \in Y^{(g(\alpha_1),\beta_2]}_{(\alpha_0,\alpha_1]} \subset \bigcap(\mathcal{V}_{n_0})_{\langle\alpha_1,\beta_2\rangle} \subset \bigcap(\mathcal{V}_{n_0})_{\langle\alpha_2,\beta_3\rangle} \subset U(\gamma(\alpha_2)).$$

It follows from the definition of $U(\gamma(\alpha_2))$ that $\gamma(\alpha_2)<\beta_1$. This contradicts $\beta_1<\alpha_2\le\gamma(\alpha_2,\beta_3)=\gamma(\alpha_2)$. Therefore A is not stationary in κ. $\square$

Considering the lower-right half of X, we can similarly show:

Claim 3 *$B = \{\beta < \kappa : H_\beta(X) \cap \kappa$ is stationary in $\kappa\}$ is not stationary in κ.*

To complete the proof of Lemma 2, let C' be a cub set which is disjoint from $A \cup B \cup \Delta(X)$. For each $\alpha \in C'$, since $\alpha \notin A \cup B$, take a cub set C_α in κ which is disjoint from $V_\alpha(X) \cup H_\alpha(X)$. By the method of [Ku, II 6.14], the diagonal intersection

$$\triangle_{\alpha \in C'} C_\alpha = \{\beta < \kappa : \forall \alpha \in C' \cap \beta (\beta \in C_\alpha)\}$$

is a cub set in κ. Then $C = C' \cap \triangle_{\alpha \in C'} C_\alpha$ is the desired cub set. To show this, assume $\langle \alpha, \beta \rangle \in X \cap C^2$. Since $C \subset C'$ and C' is disjoint from $\Delta(X)$, we have $\alpha \neq \beta$. So we may assume $\alpha < \beta$. Then it follows from $\alpha \in C \subset C'$ and $\beta \in C \subset \triangle_{\alpha \in C'} C_\alpha$ that $\beta \in C_\alpha$, therefore $\beta \notin V_\alpha(X)$. This contradicts $\langle \alpha, \beta \rangle \in X$. □

Lemma 3 *Assume X is a weakly suborthocompact subspace of $(\mu+1) \times (\nu+1) \setminus \{\langle \mu, \nu \rangle\}$ and $\omega_1 \leq \mathrm{cf}\mu = \mathrm{cf}\nu = \kappa$. If $H_\nu(X)$ is stationary in μ, then there is $\mu' < \mu$ such that $\mathrm{Cl}_X X^\nu \cap X^{\{\nu\}}_{(\mu', \mu]} = \emptyset$.*

Proof: Assume $\mathrm{Cl}_X X^\nu \cap X^{\{\nu\}}_{(\mu', \mu]} \neq \emptyset$ for each $\mu' < \mu$. Then $A = \{\alpha < \mu : \langle \alpha, \nu \rangle \in \mathrm{Cl}_X X^\nu\}$ is unbounded in μ and $A \subset H_\nu(X)$. For each $\gamma < \kappa$, put $U(\gamma) = X^{(N(\gamma), \nu]}_{[0, M(\gamma)]}$. Then $\mathcal{U} = \{U(\gamma) : \gamma < \kappa\} \cup \{X^\nu\}$ is an open cover of X. By the weak suborthocompactness of X, take a weak ι-refinement $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$ of $\mathcal{U}$. For each $\gamma \in M^{-1}(H_\nu(X)) \cap \mathrm{Lim}(\kappa)$, by $\langle M(\gamma), \nu \rangle \in X$, take $n(\gamma) \in \omega$, $f(\gamma) < \gamma$, $g(\gamma) < \kappa$ with $\gamma \leq g(\gamma)$, and $\delta(\gamma) < \kappa$ such that

$$\langle M(\gamma), \nu \rangle \in \bigcup \mathcal{V}_{n(\gamma)} \text{ and}$$

$$\langle M(\gamma), \nu \rangle \in X^{(N(g(\gamma)), \nu]}_{(M(f(\gamma)), M(\gamma)]} \subset \bigcap (\mathcal{V}_{n(\gamma)})_{\langle M(\gamma), \nu \rangle} \subset U(\delta(\gamma)).$$

It follows from $\langle M(\gamma), \nu \rangle \in U(\delta(\gamma))$ that $\gamma \leq \delta(\gamma)$. Applying the PDL, we find a stationary set $S \subset M^{-1}(H_\nu(X)) \cap \mathrm{Lim}(\kappa)$

in κ, $n_0 \in \omega$ and $\gamma_0 < \kappa$ such that $n(\gamma) = n_0$ and $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Note that $T = M(S) \cap \mathrm{Lim}_\mu(A)$ is stationary in μ. So fix $\gamma_1 \in M^{-1}(T)$ with $\gamma_0 < \gamma_1$. Since $M(\gamma_0) < M(\gamma_1) \in \mathrm{Lim}(A)$, we can find $\alpha' \in A$ with $M(\gamma_0) < \alpha' < M(\gamma_1)$. Then by $\langle \alpha', \nu \rangle \in \mathrm{Cl}_X X^\nu$, we can find ordinals α and β such that $\langle \alpha, \beta \rangle \in X^\nu \cap X^{(N(g(\gamma_1)),\nu]}_{(M(\gamma_0),\alpha']}$. Take $\gamma_2 \in S$ with $\gamma_1 < \gamma_2$ and $\beta < N(\gamma_2)$. Then note $\beta < N(\gamma_2) \le N(\delta(\gamma_2))$. It follows from

$$\langle M(\gamma_1), \nu \rangle \in X^{(N(g(\gamma_2)),\nu]}_{(M(\gamma_0),M(\gamma_2)]} \subset \bigcap (\mathcal{V}_{n_0})_{\langle M(\gamma_2),\nu \rangle}$$

that

$$\bigcap (\mathcal{V}_{n_0})_{\langle M(\gamma_1),\nu \rangle} \subset \bigcap (\mathcal{V}_{n_0})_{\langle M(\gamma_2),\nu \rangle}.$$

Then, since

$$\langle \alpha, \beta \rangle \in X^{(N(g(\gamma_1)),\nu]}_{(M(\gamma_0),M(\gamma_1)]} \subset \bigcap (\mathcal{V}_{n_0})_{\langle M(\gamma_1),\nu \rangle}$$

$$\subset \bigcap (\mathcal{V}_{n_0})_{\langle M(\gamma_2),\nu \rangle} \subset U(\delta(\gamma_2)) = X^{(N(\delta(\gamma_2)),\nu]}_{[0,M(\delta(\gamma_2))]},$$

we have $N(\delta(\gamma_2)) < \beta$. This is a contradiction. $\square$

Lemma 4 *Assume that $X \subset (\mu+1) \times (\nu+1)$ and that $X_{\mu'+1}$ and $X^{\nu'+1}$ are orthocompact for each $\mu' < \mu$ and $\nu' < \nu$. If one of $\mathrm{cf}\mu = 1$, $\mathrm{cf}\nu = 1$, $\langle \mu, \nu \rangle \in X$ or $\mathrm{cf}\mu \ne \mathrm{cf}\nu$ holds, then X is orthocompact.*

Proof: First assume $\mathrm{cf}\mu = 1$, that is, $\mu = \mu' + 1$ for some $\mu' < \mu$. Then since $X = X_{\mu'+1} \bigoplus X_{\{\mu\}}$ is the free union of two orthocompact subspaces, it is orthocompact. Similarly, if $\mathrm{cf}\nu = 1$, then X is orthocompact.

Next assume $\langle \mu, \nu \rangle \in X$ and $\mathcal{U}$ is an open cover of X. Fix $\mu' < \mu$, $\nu' < \nu$ and $U \in \mathcal{U}$ such that $X^{(\nu',\nu]}_{(\mu',\mu]} \subset U$. Since, by the assumption and Lemma 1(1), $X_{\mu'+1} \cup X^{\nu'+1}$ is orthocompact, take an interior preserving open refinement $\mathcal{V}$ of $\mathcal{U}$ whose union is $X_{\mu'+1} \cup X^{\nu'+1}$. Then $\mathcal{V} \cup \{U\}$ is an interior preserving open refinement of $\mathcal{U}$ covering X. So X is orthocompact.

Therefore we may assume $\omega \leq \mathrm{cf}\mu < \mathrm{cf}\nu$ and $\langle \mu, \nu \rangle \notin X$. To show X is orthocompact, let $\mathcal{U}$ be an open cover of X. The following claims will complete the proof.

Claim 1 *There is an interior preserving open refinement of $\mathcal{U}$ which covers $X_{\{\mu\}}$.*

Proof: Note that $\omega_1 \leq \mathrm{cf}\nu$. There are two cases.

Case 1. $V_\mu(X)$ is not stationary in ν.

Take a cub set D in $\mathrm{cf}\nu$ which is disjoint from $N^{-1}(V_\mu(X))$. Then

$$\{X^{(N(p_D(\delta)),N(\delta)]} : \delta \in \mathrm{Succ}(D)\}$$

is a pairwise disjoint collection of orthocompact clopen subspaces of X which covers $X_{\{\mu\}}$. Then by Lemma 1(2), we can find such an interior preserving open refinement of $\mathcal{U}$.

Case 2. $V_\mu(X)$ is stationary in ν.

For each $\delta \in N^{-1}(V_\mu(X)) \cap \mathrm{Lim}(\mathrm{cf}\nu)$, fix $U(\delta) \in \mathcal{U}$ with $\langle \mu, N(\delta) \rangle \in U(\delta)$. Then take $f(\delta) < \mathrm{cf}\mu$ and $g(\delta) < \delta$ such that $X^{(N(g(\delta)),N(\delta)]}_{(M(f(\delta)),\mu]} \subset U(\delta)$. By $g(\delta) < \delta$, applying the PDL, we find a stationary set $T' \subset N^{-1}(V_\mu(X)) \cap \mathrm{Lim}(\mathrm{cf}\nu)$ in $\mathrm{cf}\nu$ and $\delta_0 < \mathrm{cf}\nu$ such that $g(\delta) = \delta_0$ for each $\delta \in T'$. Next by $\mathrm{cf}\mu < \mathrm{cf}\nu$, again applying the PDL, we have a stationary set $T \subset T'$ in $\mathrm{cf}\nu$ and $\gamma_0 < \mathrm{cf}\mu$ such that $f(\delta) = \gamma_0$ for each $\delta \in T$. Then $\mathcal{V} = \{X^{(N(\delta_0),N(\delta)]}_{(M(\gamma_0),\mu]} : \delta \in T\}$ is an interior preserving open refinement of $\mathcal{U}$ whose union is $X^{(N(\delta_0),\nu)}_{(M(\gamma_0),\mu]}$, so it covers $X^{(N(\delta_0),\nu)}_{\{\mu\}}$. To show this $\mathcal{V}$ is interior preserving, let $\langle \alpha, \beta \rangle \in X^{(N(\delta_0),\nu)}_{(M(\gamma_0),\mu]}$. Then $X^{(N(\delta_0),\beta]}_{(M(\gamma_0),\alpha]}$ is an neighborhood of $\langle \alpha, \beta \rangle$ which is contained in $\bigcap(\mathcal{V})_{\langle \alpha,\beta \rangle}$.

Moreover, since, by the assumption, $X^{N(\delta_0)+1}$ is orthocompact, there is an interior preserving open refinement of $\mathcal{U}$ whose union is $X^{N(\delta_0)+1}$. Then the union of the both open refinements is an interior preserving open refinement of $\mathcal{U}$ which covers $X_{\{\mu\}}$. $\square$

Claim 2 *There is an interior preserving open refinement of $\mathcal{U}$ which covers $X^{\{\nu\}}$.*

Proof: If $\mathrm{cf}\mu = \omega$ holds, then $X^{\{\nu\}}$ is covered by the pairwise disjoint collection $\{X_{(M(n-1),M(n)]} : n \in \omega\}$ of orthocompact clopen subspaces. So we may assume $\omega_1 \leq \mathrm{cf}\mu$. There are two cases.

Case 1. $H_\nu(X)$ is not stationary in μ.
This case is similar to Claim 1-Case 1.

Case 2. $H_\nu(X)$ is stationary in μ.
This case is also similar to Claim 1-Case 2, but there is a technical difference. So we give its abstract proof. For each $\gamma \in M^{-1}(H_\nu(X)) \cap \mathrm{Lim}(\mathrm{cf}\mu)$, fix $U(\gamma) \in \mathcal{U}$, $f(\gamma) < \gamma$ and $g(\gamma) < \mathrm{cf}\nu$ such that $X^{(N(g(\gamma)),\nu]}_{(M(f(\gamma)),M(\gamma)]} \subset U(\gamma)$. By the PDL, we find a stationary set $S \subset M^{-1}(H_\nu(X)) \cap \mathrm{Lim}(\mathrm{cf}\mu)$ in $\mathrm{cf}\mu$ and $\gamma_0 < \mathrm{cf}\mu$ such that $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Put $\delta_0 = \sup\{g(\gamma) : \gamma \in S\}$. It follows from $\mathrm{cf}\mu < \mathrm{cf}\nu$ that $\delta_0 < \mathrm{cf}\nu$. Then $\{X^{(N(\delta_0),\nu]}_{(M(\gamma_0),M(\gamma)]} : \gamma \in S\}$ is an interior preserving open refinement of $\mathcal{U}$ which covers $X^{\{\nu\}}_{(M(\gamma_0),\mu)}$. Since $X_{M(\gamma_0)+1}$ is orthocompact, we can find an interior preserving open refinement of $\mathcal{U}$ which covers $X^{\{\nu\}}$. This completes the proof of Claim 2. □

Claim 3 *There is an interior preserving open refinement of $\mathcal{U}$ which covers X^ν_μ.*

Proof: If $\mathrm{cf}\mu = \omega$ holds, then X^ν_μ is covered by the pairwise disjoint collection $\{X_{(M(n-1),M(n)]} : n \in \omega\}$ of orthocompact clopen subspaces. So, as in Claim 2, we may assume $\omega_1 \leq \mathrm{cf}\mu$. Put $A = \{\alpha < \mu : V_\alpha(X) \cap \nu \text{ is stationary in} \nu\}$. There are two cases.

Case 1. A is not stationary in μ.
Take a cub set C in $\mathrm{cf}\mu$ which is disjoint from $M^{-1}(A)$. For each $\gamma \in C$, since $V_{M(\gamma)}(X) \cap \nu$ is not stationary in ν, we find a

cub set D_γ in cfν which is disjoint from $N^{-1}(V_{M(\gamma)}(X))$. Then it follows from cf$\mu <$ cfν that $D = \bigcap_{\gamma\in C} D_\gamma$ is a cub set in cfν. It is straightforward to show $X \cap M(C) \times N(D) = \emptyset$. Put $\mathcal{X}_C = \{X_{(M(p_C(\gamma)),M(\gamma)]} : \gamma \in \mathrm{Succ}(C)\}$ and $\mathcal{X}^D = \{X^{(N(p_D(\delta)),N(\delta)]} : \delta \in \mathrm{Succ}(D)\}$. Then they are pairwise disjoint collections of orthocompact clopen subspaces of X. So by Lemma 1, $\bigcup \mathcal{X}_C \cup \bigcup \mathcal{X}^D$ is an orthocompact open subspace. It suffices to show $X^\nu_\mu \subset \bigcup \mathcal{X}_C \cup \bigcup \mathcal{X}^D$. To show this, let $\langle \alpha, \beta\rangle \in X^\nu_\mu$. Since $X \cap M(C) \times N(D) = \emptyset$, first assume $\alpha \notin M(C)$. Take the minimal $\gamma \in C$ such that $\alpha \leq M(\gamma)$. Assume $\gamma \in \mathrm{Lim}(C)$. Then $\gamma \in \mathrm{Lim}(\mathrm{cf}\mu)$. It follows from the normality of M and the minimality of γ that $\alpha = M(\gamma) \in M(C)$, a contradiction. So we have $\gamma \in \mathrm{Succ}(C)$. Therefore we have $M(p_C(\gamma)) < \alpha \leq M(\gamma)$. This implies $\langle \alpha, \beta\rangle \in \bigcup \mathcal{X}_C$.

Next assume $\beta \notin N(D)$. Then similarly we have $\langle \alpha, \beta\rangle \in \bigcup \mathcal{X}^D$. This shows $X^\nu_\mu \subset \bigcup \mathcal{X}_C \cup \bigcup \mathcal{X}^D$.

Case 2. A is stationary in μ.

First fix $\gamma \in A \cap \mathrm{Lim}(\mathrm{cf}\mu)$. For each $\delta \in N^{-1}(V_{M(\gamma)}(X)) \cap \mathrm{Lim}(\mathrm{cf}\nu)$, as $\langle M(\gamma), N(\delta)\rangle \in X$, fix $U(\gamma,\delta) \in \mathcal{U}$, $f(\gamma,\delta) < \gamma$ and $g(\gamma,\delta) < \delta$ such that

$$X^{(N(g(\gamma,\delta)),N(\delta)]}_{(M(f(\gamma,\delta)),M(\gamma)]} \subset U(\gamma,\delta).$$

By the PDL, there are a stationary set $T(\gamma) \subset N^{-1}(V_{M(\gamma)}(X)) \cap \mathrm{Lim}(\mathrm{cf}\nu)$ in cfν, $f(\gamma) < \gamma$ and $g(\gamma) <$ cfν such that $f(\gamma,\delta) = f(\gamma)$ and $g(\gamma,\delta) = g(\gamma)$ for each $\delta \in T(\gamma)$.

Next again applying the PDL to $A \cap \mathrm{Lim}(\mathrm{cf}\mu)$, we find a stationary set $S \subset A \cap \mathrm{Lim}(\mathrm{cf}\mu)$ in cfμ and $\gamma_0 <$ cfμ such that $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Put $\delta_0 = \sup\{g(\gamma) : \gamma \in S\}$. Then we have $X^{(N(\delta_0),N(\delta)]}_{(M(\gamma_0),M(\gamma)]} \subset U(\gamma,\delta)$ for each $\delta \in T(\gamma)$ with $\gamma \in S$. So $\mathcal{V} = \{X^{(N(\delta_0),N(\delta)]}_{(M(\gamma_0),M(\gamma)]} : \delta \in T(\gamma)$ with $\gamma \in S\}$ is an open refinement of $\mathcal{U}$ whose union is $X^{(N(\delta_0),\nu)}_{(M(\gamma_0),\mu)}$. Moreover, as $X^{(N(\delta_0),\beta]}_{(M(\gamma_0),\alpha]} \subset \bigcap(\mathcal{V})_{\langle\alpha,\beta\rangle}$ for each $\langle\alpha,\beta\rangle \in X^{(N(\delta_0),\nu)}_{(M(\gamma_0),\mu)}$, $\mathcal{V}$ is interior preserving. Finally, since $X_{M(\gamma_0)+1} \cup X^{N(\delta_0)+1}$ is an orthocompact clopen subspace, we can find an interior preserving open refinement which covers X_μ^ν. □

3 Proof of the Theorem.

In this section, we prove the Theorem.

The implication (1) → (2) of the Theorem is evident.

(2) → (3) : Assume X is a weakly suborthocompact subspace of $(\lambda+1)^2$ and $\langle \mu, \nu \rangle \in (\lambda+1)^2 \setminus X$ with $\omega_1 \leq \mathrm{cf}\mu = \mathrm{cf}\nu = \kappa$. Since $X_{\mu+1}^{\nu+1}$ is clopen in X, we may assume $X \subset (\mu+1) \times (\nu+1)$.

To show (a), assume $\Delta_{MN}(X)$ is not stationary in κ. As the closed subspace $X \cap \mathrm{ran}M \times \mathrm{ran}N$ is homeomorphic to $Y = \{\langle \gamma, \delta \rangle \in (\kappa+1)^2 : \langle M(\gamma), N(\delta) \rangle \in X\}$, Y is also weakly suborthocompact. By the assumption, $\Delta(Y) = \{\gamma < \kappa : \langle \gamma, \gamma \rangle \in Y\} = \Delta_{MN}(X)$ is not stationary in κ. So, by Lemma 2, there is a cub set C in κ such that $Y \cap C^2 = \emptyset$. Then it is easy to show $X \cap M(C) \times N(C) = \emptyset$.

(b) follows from Lemma 3. (c) also follows from a Lemma similar to Lemma 3.

(3) → (1): Assume (3) holds, but X is not orthocompact. Put

$$\mu = \min\{\zeta \leq \lambda : X_{\zeta+1} \text{ is not orthocompact }\},$$

$$\nu = \min\{\eta \leq \lambda : X_{\mu+1}^{\eta+1} \text{ is not orthocompact }\}.$$

Note that $X_{\mu+1}^{\nu+1}$ is not orthocompact, but $X_{\mu'+1}^{\nu+1}$ and $X_{\mu+1}^{\nu'+1}$ are orthocompact for each $\mu' < \mu$ and $\nu' < \nu$. Since $X_{\mu+1}^{\nu+1}$ is a clopen subspace of X, we may assume $X = X_{\mu+1}^{\nu+1}$. Then again note that X is not orthocompact, but $X_{\mu'+1}$ and $X^{\nu'+1}$ are orthocompact for each $\mu' < \mu$ and $\nu' < \nu$. So there is an open cover $\mathcal{U}$ of X which does not have an interior preserving open refinement which covers X. It follows from Lemma 4 that

$\langle\mu,\nu\rangle \notin X$ and $\omega \leq \mathrm{cf}\mu = \mathrm{cf}\nu$. If $\omega = \mathrm{cf}\mu = \mathrm{cf}\nu$, then X is covered by the two pairwise disjoint collections $\{X_{(M(n-1),M(n)]} : n \in \omega\}$ and $\{X^{(N(n-1),N(n)]} : n \in \omega\}$ of orthocompact clopen subspaces. Therefore X is orthocompact, a contradiction. So we have $\omega_1 \leq \mathrm{cf}\mu = \mathrm{cf}\nu$. Put $\kappa = \mathrm{cf}\mu = \mathrm{cf}\nu$.

Claim 1 *There is an interior preserving open refinement of $\mathcal{U}$ which covers X^ν_μ.*

Proof: There are two cases.

Case 1. $\triangle_{MN}(X)$ is not stationary in κ.

In this case, by the condition (a), there is a cub set C in κ such that $X \cap M(C) \times N(C) = \emptyset$. Then, as in Lemma 4-Claim 3-Case 1, we can find an interior preserving open refinement of $\mathcal{U}$ which covers X^ν_μ.

Case 2. $\triangle_{MN}(X)$ is stationary in κ.

For each $\gamma \in \triangle_{MN}(X) \cap \mathrm{Lim}(\kappa)$, fix $U(\gamma) \in \mathcal{U}$ and $f(\gamma) < \gamma$ such that $X^{(N(f(\gamma)),N(\gamma)]}_{(M(f(\gamma)),M(\gamma)]} \subset U(\gamma)$. By the PDL, there is a stationary set $S \subset \triangle_{MN}(X) \cap \mathrm{Lim}(\kappa)$ in κ and $\gamma_0 < \kappa$ such that $f(\gamma) = \gamma_0$ for each $\gamma \in S$. Then $\{X^{(N(\gamma_0),N(\gamma)]}_{(M(\gamma_0),M(\gamma)]} : \gamma \in S\}$ is an interior preserving open refinement of $\mathcal{U}$ whose union is $X^{(N(\gamma_0),\nu)}_{(M(\gamma_0),\mu)}$. Since, moreover, $X_{M(\gamma_0)+1} \cup X^{N(\gamma_0)+1}$ is an orthocompact clopen subspace, we can find an interior preserving open refinement of $\mathcal{U}$ which covers X^ν_μ. $\square$

Claim 2 *There is an interior preserving open refinement of $\mathcal{U}$ which covers $X^{\{\nu\}}$.*

Proof: There are two cases.

Case 1. $H_\nu(X) \cap \mu$ is not stationary in μ.

In this case, as in Lemma 4-Claim 2-Case 1, we can find such an interior preserving open refinement.

Case 2. $H_\nu(X) \cap \mu$ is stationary in μ.

In this case, by the condition (b), there is $\mu' < \mu$ such that $\mathrm{Cl}_X X^\nu \cap X^{\{\nu\}}_{(\mu',\mu]} = \emptyset$. This means $X^{\{\nu\}}_{(\mu',\mu]}$ is a clopen subspace of X. Since $X^{\{\nu\}}_{(\mu',\mu]}$ is homeomorphic to some subspace of an ordinal, it is orthocompact. So there is an interior preserving open (of course, open in X) refinement of $\mathcal{U}$ whose union is $X^{\{\nu\}}_{(\mu',\mu]}$. Since $X_{\mu'+1}$ is an orthocompact clopen subspace of X, we can easily find an interior preserving open refinement of $\mathcal{U}$ which covers $X^{\{\nu\}}$. □

Similarly we have:

Claim 3 *There is an interior preserving open refinement of $\mathcal{U}$ which covers $X_{\{\mu\}}$.*

Then Claims 1, 2 and 3 yield a contradiction. This completes the proof of the Theorem.

4 Corollaries and examples

Considering the normal functions M and N as the identity map on ω_1 in the Theorem, we have:

Corollary 1 *Let $X \subset (\omega_1+1)^2$. Then X is orthocompact iff, if $\langle\omega_1,\omega_1\rangle \notin X$, then the following* (a)-(c) *hold:*

(a) *If $\Delta(X) = \{\gamma < \omega_1 : \langle\gamma,\gamma\rangle \in X\}$ is not stationary in ω_1, then there is a cub set C in ω_1 such that $X \cap C^2 = \emptyset$.*

(b) *If $H_{\omega_1}(X)$ is stationary in ω_1, then there is $\mu' < \omega_1$ such that $\mathrm{Cl}_X X^{\omega_1} \cap X^{\{\omega_1\}}_{(\mu',\omega_1]} = \emptyset$.*

(c) *If $V_{\omega_1}(X)$ is stationary in ω_1, then there is $\nu' < \omega_1$ such that $\mathrm{Cl}_X X_{\omega_1} \cap X^{(\nu',\omega_1]}_{\{\omega_1\}} = \emptyset$.*

In particular, note that all subspaces of $(\omega_1+1)^2$ which contain the point $\langle\omega_1,\omega_1\rangle$ are orthocompact. Moreover:

Corollary 2 *Let $X \subset \omega_1^2$. Then X is orthocompact iff, if $\Delta(X)$ is not stationary in ω_1, then there is a cub set C in ω_1 such that $X \cap C^2 = \emptyset$.*

According to (2-2) of the Corollary in [KNSY, p 295], if $X \subset \omega_1^2$ is normal and $\Delta(X)$ is not stationary in ω_1, then there is a cub set C in ω_1 such that X_C and X^C are separated. As separated sets are disjoint, Corollary 2 yields:

Corollary 3 *For every subspace X of ω_1^2, normality of X implies its orthocompactness.*

Example 4 Put $X = \omega \times \omega_1 \cup \{\omega\} \times \mathrm{Succ}(\omega_1)$. Then X is not normal, in fact, $X_{\{\omega\}}$ and $X^{\mathrm{Lim}(\omega_1)}$ cannot be separated, see [KNSY, Example 1].

On the other hand, by putting $C = (\omega, \omega_1)$ in Corollary 2, we see X is orthocompact. So the reverse implication of Corollary 3 is not true.

Example 5 Put $X = \mathrm{Succ}(\omega_1)^2 \cup \omega_1 \times \{\omega_1\}$. Then X is a subspace of $(\omega_1 + 1)^2$. First we give a direct proof that X is normal. Let F and G be disjoint closed sets in X. Since $H_{\omega_1}(F)$ and $H_{\omega_1}(G)$ are disjoint closed sets of ω_1, they are separated by disjoint open sets U and V, respectively, in ω_1. As each point of $\mathrm{Succ}(\omega_1)^2$ is isolated in X, $[X_U \cup (F \cap X^{\omega_1})] \backslash G$ and $[X_V \cup (G \cap X^{\omega_1})] \backslash F$ are open and separate F and G. So X is normal.

Next, since $\mathrm{Succ}(\omega_1) \subset \{\alpha < \omega_1 : \langle \alpha, \omega_1 \rangle \in \mathrm{Cl}_X X^{\omega_1}\}$, the condition (b) of Corollary 1 is not satisfied. So X is not orthocompact. Therefore Corollary 3 cannot be extended for subspaces of $(\omega_1 + 1)^2$.

It is not known the existence of a non-normal subspace X of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$, see the Question 1 in [KNSY, p296]. But Lemma 4 yields:

Corollary 6 *Let $X \subset \omega_1 \times \omega_2$. If $X_{\alpha+1}$ and $X^{\beta+1}$ are orthocompact for each $\alpha < \omega_1$ and $\beta < \omega_2$, then X is orthocompact.*

Finally as σ-orthocompactness and suborthocmpactness (see

[KY] for definitions),are weaker than orthocompactnesss but stronger than weak suborthocompactness, we have:

Corollary 7 *For every $X \subset (\lambda+1)^2$, orthocompactness, σ-orthocompactness, suborthocompactness and weak suborthocompactness of X are all equivalent.*

We would like to ask:

Problem Characterize orthocompactness of subspaces of the finite power ω_1^n as in Corollary 2.

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