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SUBMETACOMPACTNESS OF β-SPACES

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Abstract

It is proved that a β-space X is submetacompact if and only if every well-monotone open cover of X has a σ-closure-preserving closed refinement. We also show that this is not true without the assumption of β-spaces.

1 Introduction

Worrell and Wicke [10] introduced the concept of submetacompactness, which is a generalization of metacompactness and

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subparacompactness. First, it seems that submetacompactness had been investigated as a sufficient condition in the study of generalized metric spaces. After that, Junnila [6, 7] gave several nice characterizations for submetacompactness (and metacompactness). In particular, our motivation of this paper comes from the following.

Theorem 1.1 [6, 7] *The following are equivalent for a space X.*

(a) *X is submetacompact (metacompact).*

(b) *Every well-monotone open cover of X has a θ-sequence of open refinements (a point-finite open refinement).*

(c) *Every interior-preserving directed open cover of X has a σ-closure-preserving (a closure-preserving) closed refinement.*

(d) *Every directed open cover of X has a σ-closure-preserving (a closure-preserving) closed refinement.*

Observe that every well-monotone open cover is (interior-preserving and) directed. However, as is shown in the last section, a space X is not necessarily submetacompact even if every well-monotone open cover of X has a σ-closure-preserving closed refinement. So it seems to be natural to ask the following question:

(†) If every well-monotone open cover of a space X has a σ-closure-preserving closed refinement, when is X submetacompact?

Considering that submetacompactness plays important roles in the study of generalized metric spaces, we consider the class of β-spaces which is a class of generalized metric spaces containing the classes of Σ-spaces and semi-stratifiable spaces (see [3], Theorem 7.8 (i)). Our main result is to give an answer to the question (†) under the assumption of X being a β-space. Moreover, as another remarkable result for submetacompactness, Jiang [5] proved that strict p-spaces are submetacompact. This is called the solution of the strict p-space problem, where one observes submetacompactness is a necessary condition different from others. Here we use the technique due to Jiang in the proof of our main result.

In the next section, we consider the following question:

(‡) Characterize spaces whose every well-monotone open cover has a (σ-)closure- preserving closed refinement.

We give an answer to the question (‡) by the (sub)orthocompactness of certain products, which is an analogue of a result for $\mathcal{B}$-property by Yasui [14].

Throughout this paper, no separation axiom is assumed without special mention. We use the following notations: Let A be a set. $|A|$ denotes the cardinality of A. $[A]^{<\omega}([A]^n)$ denotes the collection of finite subsets (of cardinality n) in A. Moreover, $A^{<\omega}$ denotes the collection of finite sequences of members of A. Let X be a space and $\mathcal{U}$ an open cover of X. Top(X) denotes the topology of X. For each $x \in X$, let $\mathcal{U}(x) = \{U \in \mathcal{U} : x \in U\}$. Moreover, let $\mathrm{ord}(x,\mathcal{U}) = |\mathcal{U}(x)|$ and let $\mathrm{St}(x,\mathcal{U}) = \bigcup \mathcal{U}(x)$. For each $Y \subset X$, let $\mathcal{U} \upharpoonright Y = \{U \cap Y : U \in \mathcal{U}\}$. The letter κ denotes an infinite cardinal.

2 Main result

Let X be a space and $\mathcal{U}$ a cover of X. A cover $\mathcal{V}$ of X is a *refinement (point-star refinement)* of $\mathcal{U}$ if each member of $\mathcal{V}$ (each $\mathrm{St}(x,\mathcal{V}), x \in X$,) is contained in some member of $\mathcal{U}$. A collection $\mathcal{W}$ of (open) subsets of X is a *partial (open) refinement* of $\mathcal{U}$ if each member of $\mathcal{W}$ is contained in some member of $\mathcal{U}$, where $\mathcal{W}$ is not necessarily a cover of X.

Recall that an open cover $\mathcal{V}$ of X is *interior-preserving* if $\bigcap \mathcal{V}'$ is open in X for each $\mathcal{V}' \subset \mathcal{V}$.

Lemma 2.1 [6] *An interior-preserving open cover* $\mathcal{U}$ *of a space* X *has a closure-preserving closed refinement if and only if* $\mathcal{U}$ *has an interior-preserving point-star open refinement.*

The proof was done in that of ([6], Lemma 2.3).

A space X is called a *β-space* if there is a function $g : X \times \omega \to \mathrm{Top}(X)$, satisfying

(i) $x \in \bigcap_{n\in\omega} g(x, n)$,

(ii) if $x \in g(x_n, n)$ for each $n \in \omega$, then $\{x_n\}$ has a cluster point in X.

Such a function g is called a *β-function* of X.

A cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X is *well-monotone* if the index set A is well-ordered by $<$ such that $U_\beta \subset U_\alpha$ if $\beta < \alpha$.

The following was essentially proved by Jiang. However, we state the proof here for reader's convenience.

Lemma 2.2 [5] *Let* X *be a* β*-space and* $\mathcal{U}$ *a well-monotone open cover of* X*. If* $\mathcal{H}$ *is an open refinement of* $\mathcal{U}$*, there is a sequence* $\{\mathcal{G}_{\mathcal{H},s} : s \in \omega^{<\omega}\}$ *of partial open refinements of* $\mathcal{U}$*, satisfying*

(1) $\mathcal{G}_{\mathcal{H},s} \subset \mathcal{G}_{\mathcal{H},s'}$ *for* $s \subset s'$,

(2) *if* $x \in X$ *with* $ord(x, \mathcal{H}) \leq n$*, then* $x \in \bigcup \mathcal{G}_{\mathcal{H},s}$ *for each* $s \in \omega^{n+1}$,

(3) *for each* $x \in X$*, there is some* $\sigma \in \omega^\omega$ *such that* $ord(x, \mathcal{G}_{\mathcal{H},(\sigma\restriction n)}) < \omega$ *for each* $n \in \omega$.

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ be such that $\alpha' < \alpha$ implies $U_{\alpha'} \subset U_\alpha$. Let g be a β-function of X. Let $\alpha(x) = \min\{\alpha \in \kappa : x \in U_\alpha\}$ for each $x \in X$. Let $U_{x,n} = U_{\alpha(x)} \cap g(x, n)$ for each $x \in X$ and $n \in \omega$. Then each $U_{x,n}$ is an open neighborhood of x in X.

Let $\mathcal{G}_{\mathcal{H},\emptyset} = \emptyset$. Take any $s \in \omega^{n+1}$. Assume that $\mathcal{G}_{\mathcal{H},(s\restriction n)}$ has been already constructed. For each $\eta \in [\mathcal{H}]^n$, let

$$G_{\eta,s} = \bigcup\{U_{x,s(n)} : \eta = \mathcal{H}(x) \text{ and } x \in X \setminus \bigcup \mathcal{G}_{\mathcal{H},(s\restriction n)}\}.$$

Here we put $\mathcal{G}_{\mathcal{H},s} = \mathcal{G}_{\mathcal{H},(s\restriction n)} \cup \{G_{\eta,s} : \eta \in [\mathcal{H}]^n\}$. Thus we have constructed a sequence $\{\mathcal{G}_{\mathcal{H},s} : s \in \omega^{<\omega}\}$ of collections of open sets in X. We show this is a desired one.

As in ([5], p.312), it is easy to see that each $\mathcal{G}_{\mathcal{H},s}$ is a partial open refinement of $\mathcal{U}$. Clearly, (1) is satisfied. From the choice of $\mathcal{G}_{\mathcal{H},s}$, it is also easily verified by induction that (2) is satisfied.

Pick $x \in X$. Assume $\sigma \restriction n$ has been defined. Let $s_i = (\sigma \restriction n)^\frown(i)$ for each $i \in \omega$. Assume that $\mathrm{ord}(x, \mathcal{G}_{\mathcal{H},s_i}) \geq \omega$ for each $i \in \omega$. There are distinct members $\eta_0, \eta_1, \cdots \in [\mathcal{H}]^n$ such that $x \in G_{\eta_i,s_i}$ for each $i \in \omega$. For each $i \in \omega$, we can choose $x_i \in X \setminus \bigcup \mathcal{G}_{\mathcal{H},(\sigma\restriction n)}$ such that $\eta_i = \mathcal{H}(x_i)$ and $x \in U_{x_i,i}$. There is a cluster point y of $\{x_i\}$ in X. Now, assume $\mathrm{ord}(y, \mathcal{H}) \geq n$. Take some $\eta^* \in [\mathcal{H}]^n$ with $y \in \bigcap \eta^*$. Find $k, j \in \omega$ with $k \neq j$ and $x_k, x_j \in \bigcap \eta^*$. Since $\eta^* \subset \mathcal{H}(x_k) \cap \mathcal{H}(x_j) = \eta_k \cap \eta_j$, we have $\eta^* = \eta_k = \eta_j$. This contradicts that η_k and η_j are distinct. Hence we obtain $\mathrm{ord}(y, \mathcal{H}) < n$. So it follows from (2) that $y \in \bigcup \mathcal{G}_{\mathcal{H},(\sigma\restriction n)}$. On other hand, by the choices of x_i's and y, we have $y \notin \bigcup \mathcal{G}_{\mathcal{H},(\sigma\restriction n)}$. This is a contradiction. □

Recall that a sequence $\{\mathcal{V}_n\}$ of open covers of a space X is a *θ-sequence* if for each $x \in X$ there is $n \in \omega$ such that $\mathcal{V}_n$ is point-finite at x.

A basic idea for the proof of the following is also due to Jiang [5].

Lemma 2.3 *Let X be a β-space and $\mathcal{U}$ a well-monotone open cover of X. If $\mathcal{U}$ has a closure-preserving closed refinement, then it has a θ-sequence of open refinements.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ be such that $\alpha < \alpha'$ implies $U_\alpha \subset U_{\alpha'}$. Let g be a β-function of X such that $g(x, n+1) \subset$

$g(x,n)$ for each $x \in X$ and $n \in \omega$. It follows from the assumption and Lemma 2.1 that there is an interior-preserving point-star open refinement $\mathcal{V}$ of $\mathcal{U}$. Let $\beta(x) = \min\{\alpha \in \kappa : \mathrm{St}(x,\mathcal{V}) \subset U_\alpha\}$ for each $x \in X$. Let $W_{x,n} = (\bigcap \mathcal{V}(x)) \cap g(x,n)$ for each $x \in X$ and $n \in \omega$. Then each $W_{x,n}$ is an open neighborhood of x in X.

Let $\Theta_0 = \{\mathcal{U}\}$. Assume that we have already constructed a sequence Θ_i of open refinements of $\mathcal{U}$ with $\Theta_{i-1} \subset \Theta_i$ for each $i \leq m$. Take an $\mathcal{H} \in \Theta_m$. It follows from Lemma 2.2 that there is a sequence $\{\mathcal{G}_{\mathcal{H},s} : s \in \omega^{<\omega}\}$ of partial open refinements of $\mathcal{U}$, satisfying (1), (2) and (3) of Lemma 2.2. Let $\Xi_{m+1} = [\Theta_m \times \omega^{<\omega}]^{<\omega}$. Let $\mathcal{G}_\xi = \bigcup\{\mathcal{G}_{\mathcal{H},s} : (\mathcal{H},s) \in \xi\}$ for each $\xi \in \Xi_{m+1}$. Let

$$H_{\xi,\alpha} = \bigcup\{W_{x,m+1} : x \in X \setminus \bigcup \mathcal{G}_\xi \text{and } \beta(x) = \alpha\}$$

for each $\xi \in \Xi_{m+1}$ and $\alpha \in \kappa$. Here we set $\mathcal{H}_\xi = \mathcal{G}_\xi \cup \{H_{\xi,\alpha} : \alpha \in \kappa\}$ for each $\xi \in \Xi_{m+1}$. Moreover, we set $\Theta_{m+1} = \Theta_m \cup \{\mathcal{H}_\xi : \xi \in \Xi_{m+1}\}$. It is easy to verify $H_{\xi,\alpha} \subset U_\alpha$ for each $\alpha \in \kappa$. So $\mathcal{H}_\xi$ is an open refinement of $\mathcal{U}$. Thus we have constructed $\{\Theta_m : m \in \omega\}$ by induction. Then $\Theta = \bigcup_{m\in\omega} \Theta_m$ is a sequence of open refinements of $\mathcal{U}$.

Now, to show that Θ is a θ-sequence, assume the contrary. Let

$$Y = \{x \in X : \mathrm{ord}(x,\mathcal{H}) < \omega \text{ for some } \mathcal{H} \in \Theta\}.$$

We pick some $p \in X \setminus Y$ with $\beta(p) = \min\{\beta(x) : x \in X \setminus Y\}$. Let $\Theta_i = \{\mathcal{H}_{i,j} : j \in \omega\}$ for each $i \in \omega$. Let $\mathcal{H}_{\xi_0} = \mathcal{U} \in \Theta_0$. By (3) in Lemma 2.2, for each $i,j \in \omega$, there is $\sigma_{ij} \in \omega^\omega$ such that $\mathrm{ord}(p, \mathcal{G}_{\mathcal{H}_{ij},(\sigma_{ij}\restriction n)}) < \omega$ for each $n \in \omega$. Take $m \in \mathbf{N}$, where $\mathbf{N} = \omega \setminus \{0\}$. Let $\xi_m = \{(\mathcal{H}_{ij}, \sigma_{ij} \restriction m) : i,j < m\}$. Then $\mathcal{H}_{\xi_m} = \mathcal{G}_{\xi_m} \cup \{H_{\xi_m,\alpha} : \alpha \in \kappa\} \in \Theta_m$. By the choice of p, we have $\mathrm{ord}(p, \{H_{\xi_m,\alpha} : \alpha \in \kappa\}) \geq \omega$. By $\mathcal{G}_{\xi_m} \subset \mathcal{G}_{\xi_{m+1}}$ and $W_{x,m+1} \subset W_{x,m}$, note that $H_{\xi_{m+1},\alpha} \subset H_{\xi_m,\alpha}$ for each $\alpha \in \kappa$. So we can choose $\beta_1 < \beta_2 < \cdots < \kappa$ such that $p \in H_{\xi_m,\beta_m}$ for each $m \in \mathbf{N}$. For each $m \in \mathbf{N}$, we can pick $x_m \in X \setminus \bigcup \mathcal{G}_{\xi_m}$ such that $\beta(x_m) = \beta_m$ and $p \in W_{x_m,m}$. Then there is a cluster point y of $\{x_m\}$ in X.

Claim. $\beta(y) < \beta(p)$.

Proof: First, assume that there is $k \in \mathbf{N}$ with $\beta_k > \beta(p)$. Since $\mathrm{St}(x_k, \mathcal{V}) \not\subset U_{\beta(p)}$, we can find $V_0 \in \mathcal{V}$ such that $x_k \in V_0$ and $V_0 \not\subset U_{\beta(p)}$. Then we have

$$p \in W_{x_k,k} \subset \bigcap \mathcal{V}(x_k) \subset V_0 \subset \mathrm{St}(p, \mathcal{V}) \subset U_{\beta(p)}.$$

This is a contradiction. Hence $\beta_m < \beta(p)$ for each $m \in \mathbf{N}$. Since $\bigcap \mathcal{V}(y)$ is an open neighborhood of y, it contains some x_ℓ. By $\mathcal{V}(y) \subset \mathcal{V}(x_\ell)$, we have $\mathrm{St}(y, \mathcal{V}) \subset \mathrm{St}(x_\ell, \mathcal{V}) \subset U_{\beta_\ell}$. Therefore, $\beta(y) \leq \beta_\ell < \beta(p)$.

By Claim, we have $y \in Y$. There is $\mathcal{H}^* \in \Theta$ with $\mathrm{ord}(y, \mathcal{H}^*) < \omega$. There is some $i_0, j_0, n_0 \in \omega$ such that $\mathcal{H}^* = \mathcal{H}_{i_0,j_0}$ and $\mathrm{ord}(y, \mathcal{H}^*) = n_0$. Let $m_0 = \max\{i_0, j_0, n_0\} + 1$. Let $s^* = \sigma_{i_0,j_0} \restriction m_0$. Since $(\mathcal{H}^*, s^*) \in \xi_{m_0}$, it follows from (1) and (2) in Lemma 2.2 that $y \in \bigcup \mathcal{G}_{\mathcal{H}^*,s^*} \subset \bigcup \mathcal{G}_{\xi_{m_0}}$. So we can find $k_0 \in \omega$ with $k_0 \geq m_0$ and $x_{k_0} \in \bigcup \mathcal{G}_{\xi_{m_0}}$. Then we have $x_{k_0} \in \bigcup \mathcal{G}_{\xi_{k_0}}$. This contradicts the choice of x_{k_0}. □

Recall that a space X is *submetacompact* if every open cover of X has a θ-sequence of open refinements.

Now, we obtain a main result.

Theorem 2.4 *A β-space X is submetacompact if and only if every well-monotone open cover of X has a σ-closure-preserving closed refinement.*

Proof: The "only if" part immediately follows from Theorem 1.1. We show the "if" part. Let $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ be a well-monotone open cover of X such that $\alpha' < \alpha$ implies $U_{\alpha'} \subset U_\alpha$. There is a σ-closure-preserving closed refinement $\bigcup_{n\in\omega} \mathcal{F}_n$ of $\mathcal{U}$. Let $X_n = \bigcup \mathcal{F}_n$ for each $n \in \omega$. Pick $n \in \omega$. Then X_n is a closed set in X and $\mathcal{F}_n$ is a closure-preserving closed refinement of $\mathcal{U} \upharpoonright X_n$. Since $\mathcal{U} \upharpoonright X_n$ is a well-monotone open cover of the β-space X_n, it follows from Lemma 2.3 that there is a θ-sequence $\{\mathcal{V}_{n,k}\}$ of open refinements of $\mathcal{U} \upharpoonright X_n$. For each $V \in \mathcal{V}_{n,k}$, choose $\alpha(V) \in \kappa$ with $V \subset U_{\alpha(V)} \cap X_n$, and let $V^* = (V \cup (X \setminus X_n)) \cap U_{\alpha(V)}$. Then V^* is an open set in X with $V^* \cap X_n = V$ and $V^* \subset U_{\alpha(V)}$. Here we set $\mathcal{V}^*_{n,k} = \{V^* : V \in \mathcal{V}_{n,k}\} \cup \mathcal{U} \upharpoonright (X \setminus X_n)$ for each $n, k \in \omega$. Then it is easy to see that $\mathcal{V}^*_{n,k}$ is a θ-sequence of open refinements of $\mathcal{U}$. Hence it follows from Theorem 1.1 that X is submetacompact. □

Remark. It follows from ([13], Theorem 3.3) that each regular space whose every well-monotone open cover has a σ-closure-preserving closed refinement is isocompact (that is, each countably compact closed subspace in it is compact). So, if every well-monotone open cover of a regular Σ-space X has such a refinement, then X is a strong Σ-space, hence X is subparacompact (see [3], Theorem 4.14). On the other hand, each semi-stratifiable space is always subparacompact (see [3], Theorem 5.11).

3 Another result

Here we discuss the property of such a space that every well-monotone open cover has a (σ-)closure-preserving closed refinement.

Lemma 3.1 [4, 13] *For a space X, the following are equivalent.*

(a) *For every well-monotone open cover $\{U_\alpha : \alpha \in \kappa\}$ of X, there is a well-monotone closed cover $\{F_\alpha : \alpha \in \kappa\}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \kappa$.*

(b) *Every well-monotone open cover of X has a cushioned refinement.*

(c) *Every infinite open cover* $\mathcal{U}$ *of* X *has an open refinement* $\mathcal{V}$ *with* $ord(x, \mathcal{V}) < |\mathcal{U}|$ *for each* $x \in X$.

This was first proved in ([4], Theorem 3.1), and after that, it was restated in ([13], Theorem 2.4).

Lemma 3.2 [9] *Assume that a space* Y *has a sequence* $\{y_\alpha : \alpha \in \kappa\}$ *of length* κ *and its cluster point* z *such that* $z \notin \mathrm{Cl}\{y_\beta : \beta < \alpha\}$ *for each* $\alpha \in \kappa$. *If* $X \times Y$ *is orthocompact, then every open cover* $\mathcal{U}$ *of* X *with* $|\mathcal{U}| = \kappa$ *has an open refinement* $\mathcal{V}$ *such that* $ord(x, \mathcal{V}) < \kappa$ *for each* $x \in X$.

The proof is similar to that of ([9], Lemma 1.3).

Recall that a space X is *orthocompact* if every open cover of X has an interior-preserving open refinement.

Lemma 3.3 [5, 6, 7] *Let* X *be an orthocompact space and let* $\mathcal{U}$ *be an open cover of* X. *Then* $\mathcal{U}$ *has a* (σ-)*closure-preserving closed refinement if and only if it has a* (σ-)*cushioned refinement.*

For a space X, we denote by $L(X)$ the Lindelöf degree of X. For an ordinal λ, we denote by $(\lambda + 1)'$ the space of all ordinals $\leq \lambda$ with the topology such that the point λ has a neighborhood base in the usual order topology and all other points are isolated.

Using the product $X \times (\lambda + 1)'$, we can obtain an analogue of ([14], Theorem) as follows.

Theorem 3.4 *For an orthocompact space* X, *the following are equivalent.*

(a) *Every well-monotone open cover of* X *has a closure-preserving closed refinement.*

(b) $X \times (\kappa + 1)'$ *is orthocompact for each cardinal* $\kappa \leq L(X)$.

(c) $X \times (\lambda+1)'$ *is orthocompact for each ordinal* λ.

Proof: (b)⇒(a): Let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| = \kappa$. We may assume $\omega \leq \kappa \leq L(X)$. Since the space $(\kappa+1)'$ satisfies the condition of Y in Lemma 3.2, it follows that $\mathcal{U}$ has an open refinement $\mathcal{V}$ such that $\mathrm{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$. So Lemmas 3.1 and 3.3 assure this implication.

(c)⇒(b): Obvious.

(a)⇒(c): Let $\mathcal{G} = \{G_\xi : \xi \in \Xi\}$ be an open cover of $X \times (\lambda+1)'$. We show that there is an interior-preserving (in $X \times (\lambda+1)'$) partial open refinement $\mathcal{H}$ of $\mathcal{G}$, covering $X \times \{\lambda\}$. For each $\mu < \lambda$ and $\xi \in \Xi$, let

$$U_{\mu,\xi} = \bigcup\{U : U \text{ is open in } X \text{ with } U \times (\mu, \lambda] \subset G_\xi\}.$$

Then $\{U_{\mu,\xi} : \langle\mu,\xi\rangle \in \lambda \times \Xi\}$ is an open cover of X. Since X is orthocompact, there is an interior-preserving open cover $\{V_{\mu,\xi} : \langle\mu,\xi\rangle \in \lambda \times \Xi\}$ of X such that $V_{\mu,\xi} \subset U_{\mu,\xi}$ for each $\langle\mu,\xi\rangle \in \lambda\times\Xi$. Well-order $\lambda\times\Xi$ by $\prec$ such that, for each $\langle\mu',\xi'\rangle, \langle\mu,\xi\rangle \in \lambda\times\Xi$, $\mu' < \mu$ implies $\langle\mu',\xi'\rangle \prec \langle\mu,\xi\rangle$. Since $\{\bigcup_{\langle\mu',\xi'\rangle\prec\langle\mu,\xi\rangle} V_{\mu',\xi'} : \langle\mu,\xi\rangle \in \lambda \times \Xi\}$ is a well-monotone open cover of X, there is a closure-preserving closed cover $\{F_{\mu,\xi} : \langle\mu,\xi\rangle \in \lambda\times\Xi\}$ of X such that $F_{\mu,\xi} \subset \bigcup_{\langle\mu',\xi'\rangle\prec\langle\mu,\xi\rangle} V_{\mu',\xi'}$ for each $\langle\mu,\xi\rangle \in \lambda\times\Xi$. We may assume that $\{F_{\mu,\xi} : \langle\mu,\xi\rangle \in \lambda\times\Xi\}$ is well-monotone by $\prec$ on $\lambda\times\Xi$. Let $W_{\mu,\xi} = V_{\mu,\xi}\setminus F_{\mu,\xi}$ for each $\langle\mu,\xi\rangle \in \lambda\times\Xi$, and let $\mathcal{W} = \{W_{\mu,\xi} : \langle\mu,\xi\rangle \in \lambda\times\Xi\}$. Then it is easy to see that $\mathcal{W}$ is an interior-preserving open cover of X. Let $\mathcal{H} = \{W_{\mu,\xi} \times (\mu,\lambda] : \langle\mu,\xi\rangle \in \lambda\times\Xi\}$. Then $\mathcal{H}$ is a partial open refinement of $\mathcal{G}$, covering $X\times\{\lambda\}$. For each $\langle x,\eta\rangle \in X\times(\lambda+1)'$ with $\eta < \lambda$, $\bigcap\mathcal{H}(x,\eta)$ contains $(\bigcap\mathcal{W}(x)) \times \{\eta\}$. Pick $x \in X$. Take a $\langle\mu_0,\xi_0\rangle \in \lambda\times\Xi$ with $x \in F_{\mu_0,\xi_0}$. By the choice of $W_{\mu,\xi}$'s, we have $x \notin \bigcup_{\langle\mu,\xi\rangle\succ\langle\mu_0,\xi_0\rangle} W_{\mu,\xi}$. Then we have

$$\begin{aligned}\bigcap\mathcal{H}(x,\lambda) &= \bigcap\{W_{\mu,\xi}\times(\mu,\lambda] : x \in W_{\mu,\xi} \text{ and } \langle\mu,\xi\rangle \preceq \langle\mu_0,\xi_0\rangle\},\\ &\supset \bigcap\{W_{\mu,\xi}\times(\mu_0,\lambda] : x \in W_{\mu,\xi}\},\end{aligned}$$

$$= (\bigcap \mathcal{W}(x)) \times (\mu_0, \lambda].$$

Since $\mathcal{W}$ is an interior-preserving open cover of X, we have shown $z \in \mathrm{Int}(\bigcap \mathcal{H}(z))$ for each $z \in X \times (\lambda + 1)'$. Hence $\mathcal{H}$ is interior-preserving in $X \times (\lambda + 1)'$.

Note that $X \times \{\mu\}$ is closed-open in $X \times (\lambda + 1)'$ for each $\mu \in \lambda$. Since X is orthocompact, there is an interior-preserving open refinement $\mathcal{H}_\mu$ of $\mathcal{G} \restriction (X \times \{\mu\})$. Let $\mathcal{O} = \mathcal{H} \cup (\bigcup_{\mu \in \lambda} \mathcal{H}_\lambda)$. It is easy to check that $\mathcal{O}$ is an interior-preserving open refinement of $\mathcal{G}$. Hence $X \times (\lambda + 1)'$ is orthocompact. □

A space X is *suborthocompact* [12] if every open cover $\mathcal{U}$ of X has a sequence $\{\mathcal{V}_n\}$ of open refinements such that, for each $x \in X$, there is $n \in \omega$ with $x \in \mathrm{Int} \bigcap \mathcal{V}_n(x)$.

Using this, we can obtain

Theorem 3.5 *Let X be a suborthocompact space.*

(1) *If every well-monotone open cover of X has a σ-closure-preserving closed refinement, then $X \times (\lambda + 1)'$ is suborthocompact for each ordinal λ.*

(2) *If $X \times (\lambda + 1)'$ is suborthocompact for each cardinal $\kappa \leq L(X)$, then every well-monotone open cover of X has a σ-cushioned refinement.*

The proof of (1) in the above is similar to that of (a)⇒(c) in Theorem 3.4. However, the former is more complicated than the latter. Modifying Lemmas 3.1 and 3.2, we can easily get (2) in the above as well as (b)⇒(a) in Theorem 3.4.

By Lemma 3.3 and Theorem 3.5, we immediately have

Corollary 3.6 *For an orthocompact space X, the following are equivalent.*

(a) *Every well-monotone open cover of X has a σ-closure-preserving closed refinement.*

(b) $X \times (\kappa+1)'$ *is suborthocompact for each cardinal* $\kappa \leq L(X)$.

(c) $X \times (\lambda+1)'$ *is suborthocompact for each ordinal* λ.

4 Example

A function V from a space X into $\mathrm{Top}(X)$ is a *neighbornet* if $x \in V(x)$ for each $x \in X$. A neighbornet V of X is *co-countable (co-finite)* if $\{y \in X : x \in V(y)\}$ is at most countable (finite) for each $x \in X$.

Lemma 4.1 [8, 11] *A space X has a closure-preserving closed cover by countable (finite) sets if and only if X has a co-countable (co-finite) neighbornet.*

The following example shows that our Theorem 2.4 is not true without the assumption of X being a β-space.

Example 4.2 There is a normal orthocompact space X which is not submetacompact, but which is such that every well-monotone open cover has a closure-preserving closed refinement.

The example is a modification of Bing's example in [1], which is seen in ([2], Example 4.9 (iii)). We restate it here for the reader's convenience.

Let $\mathcal{P}(\omega_1)$ be the family of all subsets of ω_1. Let ${}^{\mathcal{P}(\omega_1)}2$ be the set of all functions from $\mathcal{P}(\omega_1)$ into $\{0,1\}$. For each $\alpha \in \omega_1$, we define $\hat{\alpha} \in {}^{\mathcal{P}(\omega_1)}2$ by $\hat{\alpha}(Q) = 1$ for $\alpha \in Q$ and $\hat{\alpha}(Q) = 0$ for $\alpha \notin Q$. Let $E = \{\hat{\alpha} : \alpha \in \omega_1\}$. For each $\alpha \in \omega_1$ and each $r \in [\mathcal{P}(\omega_1)]^{<\omega}$, let $U(\hat{\alpha}, r) = \{f \in {}^{\mathcal{P}(\omega_1)}2 : f(Q) = \hat{\alpha}(Q)$ for each $Q \in r\}$. Let Y be the space ${}^{\mathcal{P}(\omega_1)}2$ with the topology defined such that each $\hat{\alpha} \in E$ has a neighborhood base

$\{U(\hat{\alpha}, r) : r \in [\mathcal{P}(\omega_1)]^{<\omega}\}$ and each $f \in^{\mathcal{P}(\omega_1)} 2 \backslash E$ is an isolated point. Now, we take the subspace X of Y defined by

$$X = E \cup \{x \in Y : \{Q \in \mathcal{P}(\omega_1) : x(Q) = 1\} \text{ is at most countable}\}.$$

Then it is known that X is normal, but not submetacompact (see [2], Example 4.9 (iii)). Since E is closed discrete in X and $X \backslash E$ is open discrete in X, it is easy to check that X is orthocompact. For each $x \in X$, we take the neighbornet V of X defined by $V(\hat{\alpha}) = U(\hat{\alpha}, \{\{\alpha\}\})$ for each $\alpha \in \omega_1$ and $V(x) = \{x\}$ for each $x \in X \backslash E$. Then $V(\hat{\alpha}) \cap E = \{\hat{\alpha}\}$ for each $\alpha \in \omega_1$. Now, we need to verify the following two facts.

Fact 1. V is a co-countable neighbornet of X.

If $\hat{\alpha} \in V(y)$, then y must be $\hat{\alpha}$. If $x \in V(y)$ and $y \in X \backslash E$, then $y = x$. Pick $x \in X \backslash E$. Let $A = \{\alpha \in \omega_1 : x \in V(\hat{\alpha})\}$. Then $x(\{\alpha\}) = \hat{\alpha}(\{\alpha\}) = 1$ for each $\alpha \in A$. Hence we have $|\{Q \in \mathcal{P}(\omega_1) : x(Q) = 1\}| \geq |A|$. However, by $x \in X \backslash E$, we also have $|\{Q \in \mathcal{P}(\omega_1) : x(Q) = 1\}| \leq \omega$. Hence $\{y \in X : x \in V(y)\}$ is at most countable.

Fact 2. X is countably metacompact.

Let $\{U_n : n \in \omega\}$ be a well-monotone countable open cover of X. Let $E_n = E \cap (U_n \backslash U_{n-1})$ for each $n \in \omega$, where $U_{-1} = \emptyset$. Then $\{E_n : n \in \omega\}$ is a discrete collection of closed sets in X. Since X is normal, there is a discrete collection $\{V_n : n \in \omega\}$ of open sets in X such that $E_n \subset V_n \subset U_n$ for each $n \in \omega$. Then $\{V_n : n \in \omega\} \cup \{\{x\} : x \in X \backslash \bigcup_{n \in \omega} V_n\}$ is a point-finite open refinement of $\{U_n : n \in \omega\}$.

It follows from Fact 1 and Lemma 4.1 that X has a closure-preserving closed cover by countable sets. So, when $\mathrm{cf}(\kappa) > \omega$, every well-monotone open cover of X with cardinality κ has a closure-preserving closed refinement (by countable sets). Moreover, by Fact 2, every well-monotone countable open cover of X has a well-monotone countable closed refinement. Therefore, every well-monotone open cover of X has a closure-preserving closed refinement. $\square$

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