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PRODUCTS OF k-SPACES HAVING POINT-COUNTABLE k-NETWORKS

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Introduction

As is well-known, the concept of k-networks has played an important role in the theory of generalized metric spaces, characterizations of nice images of metric spaces, and products of k-spaces, and so on. The study on products of k-spaces having point-countable k-networks has been done in terms of the *Hypothesis* below, and many results have been obtained; see [LLi], [LT_2], [S_2], [T_2], etc. In this paper, we generalize and unify these results, and we give some affirmations and negations to the *Hypothesis* (Theorems 1, 2, and 3).

Hypothesis: Let X and Y be k-spaces with point-countable k-networks. Then, $X \times Y$ is k-space if and only if one of the following holds.

(K_1) *X and Y have point-countable bases.*

(K_2) *X or Y is locally compact.*

(K_3) *X and Y are locally k_ω-spaces.*

We note that, in the *Hypothesis*, as is well-known, the "if" part holds for all k-spaces X and Y. Also, it is possible to replace "k-space(s)" by "sequential space(s)".

Every Lašnev space, or every space which is a quotient s-image of a metric space; or more generally, every space dominated by these spaces is a k-space with a point-countable k-network (Remark 2).

The author posed the following question in [T_3] implicitly: For quotient s-images X and Y of metric spaces, what is a necessary and sufficient condition for $X \times Y$ to be a k-space? Then, he posed in 1994 the question whether the *Hypothesis* holds for quotient s-images X and Y of metric spaces. Also, in [LT_2], the question whether the *Hypothesis* is equivalent to "$BF(\omega_2)$ is false" was posed for k-spaces X and Y with point-countable (compact-countable, or σ-compact-finite) k-networks. For these two questions, the first one is affirmative if X and Y are Fréchet, and so is the second one if X and Y have compact-countable k-networks, or more generally, any σ-compact subset of X and Y is an $\aleph_0$-space (Theorem 1, etc). However, the questions are negative for k-spaces with point point-countable compact k-networks (Theorem 3).

We recall a basic axiom $BF(\omega_2)$. Let ${}^\omega\omega$ be the set of all functions from ω to ω. For $f,\ g \in{}^\omega \omega$, we define $f \geqq g$ if $\{n \in \omega : f(n) < g(n)\}$ is finite. Let $\mathbf{b} = \min\{\gamma$: there is an unbounded family $A \subset{}^\omega \omega$ with $|A| = \gamma\}$, here A is "unbounded" iff no $f \in{}^\omega \omega$ is $\geqq$ any $g \in A$. By $BF(\omega_2)$, we mean "$\mathbf{b} \geqq \omega_2$". In this paper, we use $\rceil BF(\omega_2)$ instead of "$BF(\omega_2)$ is false". Obviously $\rceil BF(\omega_2) \Leftrightarrow$ "$\mathbf{b} = \omega_1$", (CH)$\Rightarrow$ $\rceil BF(\omega_2)$. Also, $(MA{+}\rceil CH) \Rightarrow BF(\omega_2)$, for $(MA) \Rightarrow$ "$\mathbf{b} = c\ (= 2^\omega)$ ".

We assume that spaces are regular, T_1, and maps are continuous and onto.

Peliminaries

We recall some definitions around the *Hypothesis.* A cover $\mathcal{C}$ of a space is *point-countable (resp. compact-countable; star-countable)* if every point $x \in X$ (resp. compact $K \subset X$; element $C \in \mathcal{C}$) meets at most countable many $D \in \mathcal{C}$. *Compact-finite* covers are similarly defined.

Let X be a space, and let $\mathcal{C}$ be a cover of X. Then X *is determined by* $\mathcal{C}$ if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathcal{C}$. A space is a *k-space* (resp, *sequential space*) if it is determined by a cover of compact (resp. compact metric) subsets. For an infinite cardinal α, a space is a *k_α-space* if it is determined by a cover $\mathcal{C}$ of compact subsets with $|\mathcal{C}| \leqq \alpha$, and a space X is *locally* $< k_\alpha$ if each $x \in X$ has a nbd whose closure is a $k_{\alpha(x)}$-space, where $\alpha(x) < \alpha$; see [TZ$_1$]. Then, every locally $< k_{\omega_1}$-space is precisely locally k_ω. Let $\mathcal{C}$ be a closed cover of a space X. Then, X *is dominated by* $\mathcal{C}$ if for any subcollection $\mathcal{C}'$ of $\mathcal{C}$, $S = \cup\{C : C \in \mathcal{C}'\}$ is closed in X, and S is determined by $\mathcal{C}'$. As is well-known, every CW-complex is dominated by a cover of compact metric subsets.

Let X be a space, and let $\mathcal{P}$ be a cover of X. Recall that $\mathcal{P}$ is a *k-network* if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{P}^* \subset U$ for some finite $\mathcal{P}^* \subset \mathcal{P}$. Spaces with a countable (resp. σ-locally finite) k-network are called *$\aleph_0$-spaces* (resp. *$\aleph$-spaces*). A cover $\mathcal{P}$ of a space X is called a *cs**-network (resp. *cs-network*) if, for any open nbd U of x, and for any sequence L converging to x, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$, and P contains L frequently (resp. eventually). A k-network $\mathcal{P}$ is *closed* (resp.*compact*) if any element of $\mathcal{P}$ is closed (resp. compact). Every *cs*-network, or every closed k-network is a cs^*-network.

Let X be a space. Let α be an infinite cardinal. For each $\gamma < \alpha$, let A_γ be a sequence in X converging to $x_\gamma \notin A_\gamma$ such that the A_γ are disjoint. Let $B_\gamma = A_\gamma \cup \{x_\gamma\}$. Let S_α be the space obtained from the topological sum of $\{B_\gamma : \gamma < \alpha\}$ by identifying all the limit points to a single point. In particular, S_ω is called the *sequential fan.* Let $S = \cup\{A_\gamma : \gamma < \alpha\}$. Let K be a compact metric subset of X such that $K \cap S = \emptyset$, but every $x_\gamma \in K$. Let $K_\alpha = S \cup K$ be a space with $S/K = S_\alpha$, where S/K is a quotient space obtained from S by identifying all points of K to a single point. If K is a singleton, then $K_\alpha = S_\alpha$. If the $\{x_\gamma : \gamma < \alpha\}$ is distinct, then such a space is denoted by K^*_α. When K is an infinite convergent sequence, K^*_ω is called the *Arens' space* S_2.

In this paper, let us use "$X \supset K_\alpha$ (or $X \supset S_\alpha$)" instead of "X contains a closed copy of K_α (or S_α)", and we use "$X \not\supset K_\alpha$" instead of "X contains no closed copy of K_α".

First, we need some facts. Fact A is well-known, or routinely shown. In Fact B, for (1), recall a well-known fact that each compact subset of a space determined by a countable increasing cover $\mathcal{C}$ is contained in an element of $\mathcal{C}$ (this is routinely shown). For (2); (3) & (4), see [GMT]; [T_4] respectively.

Fact A:

(1) Let X be determined by a cover $\mathcal{C}$. If each $C \in \mathcal{C}$ is determined by a cover $\mathcal{P}_c$. Then, X is determined by a cover $\cup\{\mathcal{P}_c : C \in \mathcal{C}\}$.

(2) If X is determined by a cover $\mathcal{C}$, then X is a quotient image of a topological sum of $\mathcal{C}$.

(3) Let $f : X \to Y$ be a perfect map. Then, X is a k-space iff so is Y.

Fact B:

(1) Let X be determined by a countable closed cover $\mathcal{C}$ of $\aleph_0$-subspaces, here we can assume $\mathcal{C}$ is increasing. Then, X is an $\aleph_0$-space.

(2) Let X be a k-space with a point-countable k-network. Then, every countably compact subset of X is metric, thus, X is sequential.

(3) Let X be a sequential space, or a σ-space. Let $\mathcal{P}$ be a point-countable cover of X such that, for any nbd U of $x \in X$, and any sequence C converging to x, some $P \in \mathcal{P}$ contains a subsequence of C, and $P \subset U$. Then $\mathcal{P}$ is a k-network for X.

(4) A sequential (resp. Fréchet) space with a point-countable cs^*-network is precisely a quotient (resp. psedo-open) s-image of a metric space.

Fact C:

(1) (a) S_α is a Lašnev space which is the perfect image of K_α. K_α^* is a quotient, finite-to-one image of a locally compact metric space.

(b) S_α and K_α are dominated by compact metric subspaces, and they have star-countable k-networks.

(c) K_{ω_1} is not locally ω_1-compact, and $\chi(X) > \omega_1$. More generally, K_{ω_1} doesn't satisfy (#): Each point has a collection $\{V_\alpha : \alpha < \omega_1\}$ of open nbds such that if $x_\alpha \in V_\alpha$ with the x_α distinct, then $\{x_\alpha : \alpha < \omega_1\}$ has an accumulation point.

(d) $K_\omega \supset S_\omega$ or S_2. $K_{\omega_1} \supset S_{\omega_1}$ or $K_{\omega_1}^*$.

(e) S_{ω_1} has no point-countable cs^*-networks.

(f) K_{ω_1} has no compact-countable cs^*-networks, no point-countable cs-networks.

(g) $K^*_{\omega_1}$ has no σ-hereditarily closure preserving k-networks.

(2) (a) $BF(\omega_2)$ iff $K_{\omega_1} \times K_\omega$ is a k-space

(b) $K_{\omega_1} \times K_{\omega_1}$ is not a k-space.

(c) $K_c \times K_\omega$ is not a k-space.

Proof: In (1), for (a), K_α is determined by a point-finite cover $\{B_\gamma : \gamma < \alpha\} \cup \{K\}$ of compact metric subsets. Thus, K_α is a quotient, finite-to-one image of a locally compact metric space by Fact A(2). For (b), K_α is dominated by a cover $\{A_\gamma \cup K : \gamma < \alpha\}$ of compact metric subsets. Each A_γ and K have countable bases, then K_α has the obvious star-countable k-network. For (c), suppose K_{ω_1} satisfies (#). Then, each point of K has a nbd V_α which meets only countable number of the sets A_γ. Since K is compact, there exists an open set U such that $U \supset K$ and U meets only countably many A_γ's, a contradiction. (d) is routinely shown. (e) holds by [T$_4$; Lemma 2.4]. For (f), suppose that K_{ω_1} has a compact-countable cs^*-networks. Then, since $K_{\omega_1}/K = S_{\omega_1}$ with K compact, S_{ω_1} has obviously a point-countable cs^*-network. This is a contradiction to (e). To show that K_{ω_1} has no point-countable cs-networks, in view of (d) & (e), it suffices to show that $K^*_{\omega_1}$ has no point-countable cs-networks. But, this is shown in view of the proof of [LiT; Remark 14], because $K^*_{\omega_1}$ is determined by a point-finite cover of compact metric subsets, and K has a countable base. For (g), suppose that $K^*_{\omega_1}$ has a σ-hereditarly closure preserving closed k-network. But, $K^*_{\omega_1}$ is a quotient s-image of a metric space by (a) in (1). Then, $K^*_{\omega_1}$ is an $\aleph$-space by [JZ; Corollary 2.8]. Thus, $K^*_{\omega_1}$ has a compact-countable cs^*-network. This is a contradiction to (f). (2) holds by [G$_1$; Lemmas 1 & 5], (a) in (1), and Fact A(3) (it is well-known that $S_c \times S_\omega$ is not a k-space).

Results

First, for spaces with certain k-networks, we consider the following property (C); and let us consider (A_i) an (B_i) below as properties satisfying (C). Here, a space is *cosmic* if it has a countable network. Obviously, every σ-compact, cosmic space is precisely a countable union of compact metric subsets. Note that not every first countable, σ-compact, cosmic space is an $\aleph_0$-space as is seen by the "butterfly space".

(C) Space in which any closed σ-compact, cosmic subspace is an $\aleph_0$-space.

(A_1) Space with a σ-locally countable cs^*-network.

(A_2) Fréchet space with a point-countable cs^*-network.

(A_3) Spaces with a star-countable cs^*-network.

(A_4) Space with a point-countable cs-network.

(A_5) Space with a compact-countable cs^*-network.

(B_1) Fréchet space with a point-countable k-network.

(B_2) Space with a σ-hereditarily closure-preserving k-network.

(B_3) Space with a star-countable k-network.

(B_4) Space with a σ-compact-finite k-network.

(B_5) Space with a compact-countable k-network.

Remark 1.

(1) $(A_1) \Rightarrow (A_5)$; and $(B_4) \Rightarrow (B_5)$. $(A_2) \Rightarrow (B_1)$ by Fact B(3). (B_2) or $(B_3) \Rightarrow (B_4)$ by [LT_2] or [LT_4]. Among sequential spaces, $(A_3) \Rightarrow (B_3)$; and $(A_5) \Rightarrow (B_5)$ by Fact B(3), moreover, $(A_3) \Rightarrow (A_5)$ as in [IT; Theorem 1.4]. Among Fréchet spaces, $(B_4) \Rightarrow (B_2)$, in particular, first countable spaces satisfying (B_4) are metric by [L_1]. While, (A_i) or $(B_i) \Rightarrow$ (C), but the converse doesn't hold. Indeed, (A_3) or (B_3); (A_4); (B_1); or

(B_2) implies that every cosmic subspace is an $\aleph_0$-space in view of [LT$_5$]; [LT$_4$]; [GMT]; or [Li$_1$] respectively. The latter part is shown by the k-spaces $K^*_{\omega_1}$ and X_A with point-countable compact k-networks in view of Facts B(3) & C(1), and Remark 2 below.

(2) In the *Hypothesis*, if the X and Y satisfy one of (A_i) with i $\neq$ 4, or (B_i), then we can replace (K_3) by "X and Y are topological sums of k_ω-spaces" by (3) in the following proposition. But, for sequential spaces satisfying (A_4) or (C), we can't do such a replacement by the space X_A in Remark 2.

Proposition: (1) *X is a sequential space satisfying* (A_3) *iff X is a topological sum of k-and $\aleph_0$-spaces. If X is Fréchet, we can replace "sequential space satisfying* (A_3)*" by "locally separable space satisfying* (B_1)*".*

(2) *X is a k-space satisfying* (B_3) *iff X is dominated by k-and-$\aleph_0$-subspaces. Thus, every k-space satisfying* (B_3) *is paracompact.*

(3) *X is a locally k_ω-space satisfying* (B_5) *iff X is a topological sum of k_ω-and-$\aleph_0$-spaces. If X is Fréchet, we can replace* (B_5) *by* (B_1)*.*

Proof: First, we recall a fact that, for a space Y determined by a star-countable cover $\mathcal{C}$, Y is a topological sum of spaces $Y_\alpha(\alpha \in A)$, where each Y_α is a countable union of elements of $\mathcal{C}$ ([IT]). For (1), the "if" part is obvious. For the "only if" part, let $\mathcal{P}$ be a star-countable cs^*-network for X. Since X is sequential, X is determined by $\mathcal{P}$. Thus, by the above fact and Fact B(3), X is a topological sum of $\aleph_0$-subspaces. The latter part of (1) holds by means of [GMT; Proposition 8.8]. (2) is due to [Sa$_1$]. For (3), the " if" part is obvious. For the "only if" part, let $\mathcal{P}$ be a compact-countable k-network for X. Since X is locally σ-compact, and $\mathcal{P}$ is compact-countable, $\{P \in \mathcal{P} : \mathrm{cl}P \text{ is } \sigma\text{-compact}\}$ is a star-countable k-network for X. Then, X is paracompact by (2). But, X is locally K_ω. Thus, X has a locally finite closed cover of k_ω-spaces. Then, by Fact A(1), X is determined by a star-countable cover of compact subsets. Thus, by the above fact, X is a topological sum of k_ω-spaces. Then, X is a topological sum of k_ω-and $\aleph_0$-spaces by Fact B(1) & (2). The latter part of (3) holds by means of the latter part of (1).

Remark 2. For a property (P) on *k-spaces*, let us say that (P) is preserved under closed maps (resp. dominations) if every closed image of a k-space with (P) (resp. every space dominated by k-spaces with (P)) satisfies (P). Then the following proposition holds. Here, (B_6) is a property "space with a point-countable k-network".

Proposition: (1) (B_i) ($i \leqq 6$) *are preserved under closed maps, but for* (B_5), *the domain satisfies one of the following;* (i) *space with a countable closed cover of normal subspaces;* (ii) *space determined by a point-countable closed cover of normal subspaces;* (iii) *every boundary of the fibers is Lindelöf.*

(2) (B_3), (B_4), (B_5), (B_6), *and* (C) *are preserved under dominations.*

(3) (A_i) *are not necessarily preserved under closed maps nor dominations.*

(4) (a) (CH). (C) *is not necessarily preserved under closed maps* ([S_3]).

(b) (B_1) *and* (B_2) *are not necessarily preserved under dominations.*

Proof: For the preservations under closed maps for (B_1) & (B_6); (B_2) (without the k-ness); (B_3); and (B_4), see [LiT]; [T_6]; [IT]; amd [LT_4] respectively. The preservations under dominations for (B_3), (B_4), (B_5), (B_6), and (C) are shown as in the proof of Theorem 2.4(1) in [IT] (without the k-ness), but for (C) recall Fact B(1). Next, we show that (1) for (B_5), and (3) & (4) hold. To (1) for (B_5), let $f : X \to Y$ be a closed map such that X is a k-space with a compact-countable k-network $\mathcal{P}_{\mathrm{x}}$. For each $y \in Y$, pick $x_y \in X$ such that $x_y \in Bf^{-1}(y)$ if $Bf^{-1}(y) \neq \emptyset$, otherwise $x_y \in f^{-1}(y)$. Let

$A = \{x_y : y \in Y\}$. Since f is closed, and X is sequential by Fact B(2), $\mathcal{P}_Y = \{f(A \cap P) : P \in \mathcal{P}_X\}$ is a point-countable k-network for Y in view of Fact B(3). To show $\mathcal{P}_Y$ is compact-countable, suppose not. Then, some compact subset K of Y meets uncountable many elements of $\mathcal{P}_Y$. Since $\mathcal{P}_Y$ is point-countable, some uncountable subset B of A with $f(B) \subset K$ meets uncountable many elements of $\mathcal{P}_X$. Here, note that any infinite subset of B has an accumulation point in X, for f is closed with $f(B) \subset K$. For (i), some normal closed subset F_1 of X contains an uncountable subset C_1 of B. For (ii), let X be determined by a point-countable closed cover $\mathcal{C}$ of normal subspaces. For $x \in X$, let $\{C \in \mathcal{C} : x \in C\} = \{C_n(x) : n \in N\}$. Suppose that B is not contained in any finite union of elements of $\mathcal{C}$. Then, there exists an infinite subset $D = \{x_n : n \in N\}$ of B such that $x_n \in B - \cup\{C_i(x_j) : \ i,\ j \leqq n\}$. But, $D \cap C$ is finite for every $C \in \mathcal{C}$. Thus, D is closed discrete in X, a contradiction. Then, B is contained in a finite union of elements of $\mathcal{C}$. Then, some $F_2 \in \mathcal{C}$ contains an uncountable subset C_2 of B. But, any infinite subset of C_i $(i = 1, 2)$ is not closed discrete in F_i. Since F_i $(i = 1, 2)$ are normal, clC_i are countably compact, thus compact by Fact B(2). But, a compact subset clC_i of X meets uncountably many elements of $\mathcal{P}_{\mathrm{x}}$, a contradiction. For (iii), let $F = \cup\{C_y : y \in Y\}$, where $C_y = Bf^{-1}(y)$ if $Bf^{-1}(y) \neq \emptyset$, otherwise $C_y = \{x_y\}$. Since F is closed in X, $g = f|F$ is a closed map with every $g^{-1}(y)$ Lindelöf. Since K is compact, $g^{-1}(K)$ is Lindelóf. But, $B \subset g^{-1}(K)$ and $g^{-1}(K)$ is a normal closed subset of X. Thus, cl$B$ is a compact subset of X which meets uncountable many elements of $\mathcal{P}_{\mathrm{x}}$, a contradiction. Hence, $\mathcal{P}_Y$ is a compact-countable k-network for Y. Thus, (1) for (B_5) holds. (3); and (4)(b) are respectively shown by the spaces S_{ω_1}; and $K^*_{\omega_1}$ in view of Fact C(1).

Finally, we show (a) of (4) holds, and give related matters. Let $I = [0, 1]$, and $C = \{1/n : n \in N\} \cup \{0\}$. For $A \subset I$ with $|A| \geqq \omega_1$, let $\mathcal{C}_A = \{I \times \{1/n\} : n \in N\} \cup \{\{t\} \times C : t \in$

$A\}$, $\mathcal{C}_A^* = \mathcal{C}_A \cup \{A \times \{0\}\}$, and let $\mathcal{C}_A^\# = \{I \times (0,1]\} \cup \{\{t\} \times I : t \in A\}$. Let X_A; X_A^*; and $X_A^\#$ be a space determined by a cover $\mathcal{C}_A$; $\mathcal{C}_A^*$; and $\mathcal{C}_A^\#$ respectively. Let $Y_A = X_A/A$, $Y_A^* = X_A^*/A$, and $Y_A^\# = X_A^\#/A$, here $A = A \times \{0\}$. Then, X_A is closed in $X_A^\#$, and so is $Y_A = Y_A^*$ in $Y_A^\#$. X_A and $X_A^\#$ are regular, and X_A^*, Y_A, $Y_A^\#$ are *Hausdorff.* According to [S_3], under (CH), for some uncountable subset B of I, $Y_B^\#$ is regular (thus, Y_B^* and X_B^* are regular). The following observation holds, here spaces are *Hausdorff.* Thus, (a) is shown by the regular space Y_B (or $Y_B^\#$). (Note that, (a) is shown by a Hausdorff space Y_A (or $Y_A^\#$) without (CH)).

Observation: (a) Locally k_ω-spaces X_A and $X_A^\#$ have point-countable compact k-and cs-networks. Besides, X_A and $X_A^\#$ satisfy (C).

(b) A cosmic space X_A^* (resp. σ-compact space X_1^*) has a point-countable closed (resp.compact) k-network. But, X_A^* has no point-countable cs-networks.

(c) σ-compact spaces Y_A and $Y_A^\#$ have point-countable k-networks. But, neither Y_A nor $Y_A^\#$ has a point-countable cs^*-network. Thus, neither X_A, X_A^*, $X_A^\#$, Y_A, nor $Y_A^\#$ has a compact-countable k-network.

Proof: For (a), to show $X_A^\#$ is locally k_ω, for $p \in A$, let $V(p)$ be a basic nbd of p such that $V(p) \subset \{(x,y) : y > |x-p|\} \cup \{p\}$. Then, $\mathrm{cl}V(p)$ is determined by $\{\mathrm{cl}V(p) \cap (\{p\} \times I)$, $\mathrm{cl}V(p) \cap (I \times (0,1])\}$. But, $\mathrm{cl}V(p) \cap (I \times (0,1])$ is a k_ω-space, because it is a closed subset of a space $I \times (0,1]$ which is locally compact and Lindelöf, hence k_ω. Thus, $\mathrm{cl}V(p)$ is a k_ω-space by Fact A(1). This implies that$X_A^\#$ is locally k_ω. By means of Fact B(3), obviously $X_A^\#$ has a point-countable compact k-and-cs-network, and $X_A^\#$ satisfies (C) since A is closed discrete in $X_A^\#$. For (b), obviously X_A^* has a point-countable closed k-network by means of Fact B(3). To show X_A^* has no point-countable cs-networks, suppose X_A^* has a point-countable cs-network $\mathcal{P}$. Obviously, X_A^* has a countable dense set D such that, for each

$x \in X_A^*$, there exists a sequence in D converging to x. Thus, for any sequence S converging to $x \in X_A^*$, there exists a sequence T in D such that the sequence $S \cup T$ converges to x. This shows that $\mathcal{P}_D = \{P \in \mathcal{P} : P \cap D \neq \emptyset\}$ is a countable *cs*-network for X_A^*, thus it is a k-network in view of Fact B(3). This is a contradiction to next (c). For (c), obviously Y_A has a point-countable k-network. To show Y_A has no point-countable cs^*-networks, suppose Y_A has a point-countable cs^*-network $\mathcal{P}'$. Let $\mathcal{P}^* = \{P \in \mathcal{P}' : P \ni \infty\}$, $\infty = [A] \in Y_A$. Let $I_n = I \times \{1/n\}$ for $n \in N$, and $L_t = (\{t\} \times \{1/n : n \in N\}) \cup \{\infty\}$ for $t \in A$. Let $\mathcal{C} = \{I_n, L_t : n \in N,\ t \in A\}$. Let $\mathcal{P}_1 = \{P \in \mathcal{P}^* : P$ is not contained in any finite union of elements of $\mathcal{C}\}$ and let $\mathcal{P}_2 = \mathcal{P}^* - \mathcal{P}_1$. For each $P \in \mathcal{P}_2$, let $P \subset \cup\{I_n : n \in N_p\} \cup \{L_t : t \in A_p\}$ for some finite $N_p \subset N$, and finite $A_p \subset A$. Pick $L^* = L_{t'} \in \mathcal{C}$ such that $L^* \cap \cup\{L_t : t \in \cup\{A_p : P \in \mathcal{P}_2\}\} = \{\infty\}$. Let $\mathcal{P}_1 = \{P_n : n \in N\}$. Pick $(t_1, 1/m_1) \in P_1 - (L^* \cup I_1)$, and pick $(t_2, 1/m_2) \in P_2 - (\cup\{L^*, L_{t_1}\} \cup \{I_n : n \leqq m_1\})$. In this way we can choose a subset $D = \{(t_n, 1/m_n) : n \in N\}$ of $Y_A - L^*$ such that $(t_n, 1/m_n) \in P_n$, the t_n are distinct, and $m_n < m_{n+1}$. Then, $L^* \subset Y_A - D$ with $Y_A - D$ open in Y_A. Thus there exists $P \in \mathcal{P}^*$ such that $P \cap L^*$ is infinite. But, this is a contradiction. Then, Y_A doesn't have any point-countable cs^*-network, thus, neither does Y_A^*. Then, $Y_A^\#$ or Y_A has no countable k-networks, thus, no compact countable k-networks. Then, neither X_A, X_A^*, nor $X_A^\#$ has a compact-countable k-network in view of the proof of the previous proposition (under spaces being Hausdorff).

Note: Quite recently, the author found a problem whether the Hausdorff space X_I^* is regular or not. He has known the space X_I^* is *not regular.* (Indeed, for a subset A of I, if X_A^* is regular, A must be first category in I [Sa$_2$]; and also a characterization on A for X_A^* (or Y_A^*) to be regular is obtained by [Sa$_3$]). The author had used the space X_I^* in his (joint) papers in which spaces are assumed to be regular. Thus, the space

$X(= X_I^*)$ in [TZ$_2$; Example 1.6(2)], [IT: Example 4.1(7)], and [LT$_2$; Example 4], etc. should be *Hausdorff*. (Except the regularity, the assertions on X_I^* there remain true in view of the *Observation* in Remark 2).

We will give some affirmations to the *Hypothesis*. First, we need some lemmas. For the cardinal c, let us say that a space X is *locally* $< c$*-compact* if each $x \in X$ has a nbd whose closure is a union of at most $\alpha(x)$ many compact subsets, where $\alpha(x) < c$. We assume that point-countable k-networks are closed under finite intersections in Lemma 1(2), *Key Lemma*, and so on.

Lemma 1 *Let X be a k-space with a point-countable k-network. Then, (1) and (2) below hold.*

(1) *$X \not\supset K_\omega$ iff X has a point-countable base.*

(2) *In the following,* (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) & (f) *holds*

(a) *$X \times K_\omega$ is a k-space.*

(b) *Every first countable closed subset of X is locally compact.*

(c) *For any (or some) point-countable k-network $\mathcal{P}$ for $X, \{P \in \mathcal{P} : clP$ is compact$\}$ is a k-network for X.*

(d) *Every closed, $\aleph_0$-subspace of X is a k_ω-space.*

(e) *X is locally σ-compact iff $X \not\supset K_\omega$.*

(f) *X is locally $< c$-compact iff $X \not\supset K_c$, here c is regular.*

Proof: (1) holds in view of [Li$_2$] since $K \not\supset S_\omega$ and $K \not\supset S_2$ by Fact C(1). For (2), (b) $\Rightarrow$ (c) holds in view of [LLi] or [S$_2$], also, for (b) $\Rightarrow$ (d), similarly every closed $\aleph_0$-subspace F of X has a countable compact k-network. But, F is a k-space. Then F is determined by this cover, thus, F is a k_ω-space. For

(c) or (d) $\Rightarrow$ (b), suppose (b) doesn't hold. Then, by Fact B(2), it is routine to show that there exists a closed subset $S = \cup\{D_n : n \in N\} \cup \{x_0\}$ of X, here the D_n are pairwise disjoint, infinite countable discrete closed subsets, and each basic nbd of x_0 in S has a form $\{x_0\} \cup \{D_m : m \geqq n\}$ $(n \in N)$. Then, S is a countable and closed metric subset (thus, S is an $\aleph_0$-space). But, S is not a k_ω-space. In fact, suppose that S is determined by a cover of compact subsets C_n $(n \in N)$, here we assume $C_n \subset C_{n+1}$. Then, there is a subset $A = \{x_m : m \in N\}$ such that $x_m \in D_{n(m)} \cap (S - C_m)$ with $n(m) \geqq m$. But, each $A \cap C_n$ is finite thus A is discrete in S, a contradiction. Thus, (d) doesn't hold. Also, (c) doesn't hold, because, if (c) holds, then the countable closed subset S has a countable compact k-network, thus, S is a k_ω-space, a contradiction. Thus, (c) or (d) $\Rightarrow$ (b) holds. For (c) $\Rightarrow$ (e) & (f), the " only if " parts of (e) & (f) hold (without (c)), for K is compact. The "if" part of (e) hold by means of [S_2; Lemma 2.4], and the part of (f) holds as in the proof of Lemma 2.4 in [S_2] under the c being regular.

The following lemma holds by means of Fact C. For Lemma 3 below, (1) holds by Remark 1, and (2) holds by means of [S_2; Lemma 2.9] and Fact B(2). For the sequential order $so(X)$ (or $\sigma(X)$) of a space X, see [AF] or [S_2], for example.

Lemma 2 *If one of* (a) $\sim$ (d) *below holds, then* $X \not\supset K_{\omega_1}$.

(a) X *satisfies one of* (A_i).

(b) X *satisfies* (#) (*in Fact* C).

(c) $X \times K_\omega$ *is a k-space under* $\rceil BF(\omega_2)$.

(d) $X \times X$ *is a k-space.*

Lemma 3 (1) *Let* X *satisfy one of* (A_i), *of* (B_i). *Then,* X *satisfies* (C).

(2) *Let X be a sequential space with a point-countable k-network of compact subsets. If the sequential order $so(X) \leqq 2$, then X satisfies* (C).

Lemma 4 (1) *Let X be a k-space having a compact-countable k-network of sets with compact closures. Then, X is locally $< k_c$ iff X is locally $<$ c-compact.*

(2) (MA). *Let X have a point-countable k-network. If X is locally k_ω, and Y is locally $< k_c$, then $X \times Y$ is a k-space.*

Proof: For (1), the "only if" part is clear. To show the "if" part, let X be locally $<$ c-compact, and let $x \in X$. Then, there is a nbd V_x of x such that $F = \mathrm{cl}V_x$ is a union of α many compact subsets, here $\omega \leqq \alpha < c$. Then F has a k-network $\mathcal{P}$ of sets with compact closures with $|\mathcal{P}| \leqq \alpha$. Since $\mathcal{P}$ is a k-network, each compact subset of F is contained in a finite union of elements of $\mathcal{P}$. Since F is a k-space, it is determined by the cover $\mathcal{P}^*$ of all finite unions of elements of $\mathcal{P}$. So, F is determined by $\{\mathrm{cl}P : P \in \mathcal{P}^*\}$ with $|\mathcal{P}^*| \leqq \alpha$. Then, F is a k_α-space. Thus, X is locally $< k_c$. For (2), each point of X has a nbd determined by a countable cover of compact metric subsets by Fact B(2). Thus, $X \times Y$ is a locally k-space by [TZ_1; Lemma 2.2], hence $X \times Y$ is a k-space.

The following is *Key Lemma* to the *Hypothesis*, which is the essence of the theory of products of k-spaces having point-countable k-networks.

Key Lemma: For k-spaces X and Y with point-countable k-networks, let $X \times Y$ be a k-space. Then (K_1) *or* (K_2) *in the Hypothesis holds, otherwise, properties* (P) *and* (K) *below hold*

(P) *For any point-countable k-network $\mathcal{P}$ for X and Y, $\{P \in \mathcal{P} : clP$ is compact$\}$ is a k-network.*

(K) (i) *$X \not\supset K_{\omega_1}$ and $Y \not\supset K_{\omega_1}$:* (ii) *$X \subset K_\omega$ but $X \not\supset K_{\omega_1}$ and $Y \supset K_{\omega_1}$ but $Y \not\supset K_c$; or* (iii) *X and Y are changed in* (ii).

Proof: We have cases: (c_1) $X \not\supset K_\omega$ and $Y \not\supset K_\omega$: (c_2) $X \not\supset K_\omega$ and $Y \supset K_\omega$ (or change X and Y); (c_3) $X \supset K_\omega$ and $Y \supset K_\omega$. But, (c_1); (c_2) implies (K_1); (K_2) in the *Hypothesis* by Lemma 1 respectively. (c_3) implies property (P) by Lemma 1(2). Moreover, for (c_3), we have cases: (c_{31}) $X \not\supset K_{\omega_1}$ and $Y \not\supset K_{\omega_1}$; ($c_{32}$) $X \not\supset K_{\omega_1}$ and $Y \supset K_{\omega_1}$ (or, X and Y are changed); (c_{33}) $X \supset K_{\omega_1}$ and $Y \supset K_{\omega_1}$. But, (c_{33}) doesn't hold by Fact C(2). While, (c_3) implies that $X \not\supset K_c$ and $Y \not\supset K_c$ by Fact C(2). Thus, (c_{31}) holds, or the following holds: $X \supset K_\omega$ but $\not\supset K_{\omega_1}$ and $Y \supset K_{\omega_1}$ but $\not\supset K_c$ (or, X and Y are changed). Thus, (c_3) also implies (K).

The following theorem holds by *Key Lemma* together with Lemmas 1 & 2. In the theorem below, the X (or Y) means the space X (or Y) in the *Hypothesis*; that is, X and Y are k-spaces with point-countable k-networks.

Theorem 1 *Suppose that the X and Y satisfy* (C). *Then* (1) *holds, then so do* (2) *and* (3).

(1) *The Hypothesis is valid if $X \not\supset K_{\omega_1}$ and $Y \not\supset K_{\omega_1}$.*

(2) *The Hypothesis is valid iff $\rceil BF(\omega_2)$.*

(3) *If $X = Y$, the Hypothesis is valid.*

Remark 3 (1) The previous theorem is not valid if we replace the property (C) (or (B_5)) by " space with a point-countable compact k-network" in view of Theorem 3(3) below. Thus, (C) is essential in the previous theorem.

(2)Let us review the previous theorem in terms of a property (C^+): Same as (C), but containing no closed copy of K_{ω_1}. The following shows implicitly that (C^+) is a fairly suitable condition for the "only if" part of the *Hypothesis.*

Proposition: Suppose that neither the X nor Y is locally compact. Then the following are equivalent. Under $\rceil BF(\omega_2)$ or $X = Y$, we can replace (C^+) *by* (C).

(a) *$X \times Y$ is a k-space with X and Y satisfying* (C^+).

(b) *$X \times Y$ is a k-space with X and Y satisfying: Same as* (C^+)*, but replace "closed σ-compact, cosmic" by "cosmic"* (*or "separable Lindelöf"*).

(c) *X and Y have point-countable bases, or they are locally k_ω-spaces.*

Proof: (b) $\Rightarrow$ (a) is clear. (a) $\Rightarrow$ (c) holds by Theorem 1. For (c) $\Rightarrow$ (b), let (c) hold. Since X and Y are first countable or locally Lindelöf, neither X nor Y contains a closed copy of K_{ω_1}. We show that any separable Lindelöf subset F of X (and Y) is an $\aleph_0$-space. If X and Y have point-countable bases, then F is separable metric, for F is separable. Let X and Y be locally k_ω. Then, X and Y are locally $\aleph_0$-spaces by Fact B(1) & (2). But, F is Lindelöf in X (and Y). Then, F has a countable, locally finite closed cover of $\aleph_0$-subspaces. Thus, F is an $\aleph_0$-space. Thus (b) holds.

As applications of Theorem 1, we have Theorems 1.1, 1.2, and 1.3 below. Here, ($C^\#$) is a property (C) satisfying (#) (in Fact C), and ($C^\#$) is a property "Space in which any cosmic subspace is an $\aleph_0$-space". Theorems 1.1 and 1.2 hold by Theorem 1, Lemmas 2 & 3(1), and Fact B(3). Theorem 1.3 holds by Theorem 1, *Key Lemma*, and Lemma 3(2). Theorems 1.1 and 1.2 improve [LT_2; Theorem 9].

In terms of the products of k-spaces having point-countable k-networks Theorems 1.1, 1.2, and 1.3 contain all results as far as the author knows. For Theorem 1.1, the result for (A_1); (A_2); and (A_4) is due to [S_2]; [S_2;LLi]; and [LiL] respectively, and, the result for (A_5) is due to [LLi], where spaces have compact-countable closed k-networks. For Theorem 1.2, the result for (B_2); (B_3); (B_5); and (C^*) is due to [L_2]; [LT_1]; [LT_3]; and [L_3] respectively. Theorem 1.3 is shown in [S_2] under (CH).

Theorem 1.1 *Suppose that the X satisfies one of* (A_i), *or* ($C^{\#}$), *and so does the Y. Then, the Hypothesis is valid.*

Theorem 1.2 *Suppose that the X satisfies one of* (B_i), *or* (C^*), *and so does the Y. Then, the Hypothesis is valid iff* $\rceil BF(\omega_2)$. *If $X = Y$, the Hypothesis is valid.*

Theorem 1.3 *Suppose that X and Y are k-spaces with point-countable closed k-networks such that $so(X) \leqq 2$, and $so(Y) \leqq 2$. Then, the Hypothesis is valid iff $\rceil BF(\omega_2)$. If $X = Y$, the Hypothesis is valid.*

As for the k-ness of $X^2; X^\omega$, the following corollary 1; 2 holds respectively. Corollary 1 follows from Theorem 1.2 and Remark 1(2). Corollary 2 is due to [LT_3], but let us give a simple proof here.

Corollary 1 *Let X be a k-space with a compact-countable k-network. Then the following are equivalent.*

(a) *X^2 is a k-space .*

(b) *X^2 is a paracompact k-and-$\aleph$-space, or X^2 has a point-countable base.*

(c) *X is a topological sum of k_ω-and-$\aleph_0$-spaces, or X has a point-countable base.*

Corollary 2 *Let X be a k-space with a point-countable k-network. Then X^ω is a k-space iff X has a point-countable base.*

Proof: For the "only if" part, by Lemma 1(1), it suffices to show that $X \not\supset K_\omega$, so suppose not. Then, $X^\omega \supset (K_\omega)^\omega$, so $(K_\omega)^\omega$ is a k-space . But, $(K_\omega)^\omega$ has a countable cs^*-network, and $(K_\omega)^\omega \times (K_\omega)^\omega = (K_\omega)^\omega$ is a k-space . Thus, $(K_\omega)^\omega$ is locally σ-compact by Theorem 1.1, thus, it is σ-compact. But, since K_ω is not compact, obviously $(K_\omega)^\omega$ is not σ-compact. This is a contradiction.

We consider the *Hypothesis* under (MA) as a model of $BF(\omega_2)$. The following theorem holds. In this result, it is essential to replace (K_3) by (K_3^*) in view of Theorem 3 below (under (MA $+\rceil CH$)).

Theorem 2 **(MA)** *Suppose that the X satisfies one of* (B_i), *and the Y satisfies one of* (A_i), (Bi), *or* ($C^\#$). *Then, the* Hypothesis *holds, but replace* (K_3) *by* (K_3^*)*: One of X and Y is locally K_ω, and another is locally $< k_c$.*

Proof: Let $X \times Y$ be a k-space , but neither (K_1) nor (K_2) holds. For cases (A_i), (B_i) with $i \neq 1$, or ($C^\#$), (K_3^*) holds by means of *Key Lemma*, Lemmas 1(2), 2, and 4(1), and Remark 1(1). Here, the cardinal c is regular under (MA). For X (or Y) satisfying (B_1), X has a point-countable k-network of separable subsets by means of *Key Lemma*, thus, X satisfies (B_3) by [Sa$_1$; Theorem 3.5]. Hence, (K_3^*) also holds. Conversely, (K_1), (K_2), or (K_3^*) implies that $X \times Y$ is a k-space by means of Lemma 4(2).

Remark 4 For a cardinal α with $\alpha \geqq \omega_1$, let us consider the following (C_α) which is a modification of (C).

(C_α): Space in which any closed subset having a cover $\mathcal{C}$ of compact metric subsets with $|\mathcal{C}| < \alpha$ has a point-countable k-network $\mathcal{P}$ with $|\mathcal{P}| < \alpha$.

Clearly, (C_{ω_1}) = (C). (B_5) implies (C_α). More generally, for $\beta < \alpha$, every space with a point-countable k-network such that each compact set meets at most β many elements of the k-network satisfies (C_α).

In Theorem 2, we can replace "(B_5)" by "(C_{ω_1}) and (C_c)" in view of the theorem. However, the author doesn't know whether (C_{ω_1}) or (C_c) is omitted.

Remark 5 Let us consider sufficient conditions for $X \times Y$ to be a k-space . For an infinite cardinal α, let us say that a space is an *lk_α-space* if it is determined by a cover $\mathcal{C}$ of closed locally compact subsets with $|\mathcal{C}| \leqq \alpha$, also that a space X is *locally*

lk_α, if each $x \in X$ has a nbd which is an $lk_{\alpha(x)}$-space, where $\alpha(x) < \alpha$. Then, a locally $< lk_{\omega_1}$-space is precisely a locally lk_ω-space. For (countably) bi-k-spaces, see [M]. As sufficient conditions for $X \times Y$ to be a k-space, besides the conditions (K_1), (K_2), (K_3), and (K_3^*), we have the following conditions in view of $[T_2]$: (i) X is a bi-k-space, and Y is a countably bi-k-space; or (ii) X and Y are locally lk_ω-spaces. But, among spaces with point-countable k-networks, (i) $\Leftrightarrow$ (K_1) by [GMT; Corollary 3.6]. The following proposition shows that (ii) $\Leftrightarrow$ (K_3) holds, and that, in the *Hypothesis*; or Theorem 2, etc., it is possible to replace "locally k_ω-space" by "locally lk_ω-space"; or "locally $< k_c$-space" by "locally $< lk_c$-space".

Proposition: (1) *A space X is a locally lk_ω-space iff X is a locally k_ω-space.*

(2) *Let X be a space with a point-countable k-network* (*resp. X be a paracompact space*). *Then, X is a locally $< lk_\alpha$-space iff X is a locally $< k_\alpha$-space* (*resp. X is a topological sum of k_β-spaces, here $\beta < \alpha$*).

Proof: We show that the "only if" parts of (1) and (2) hold. First, we need the following facts. (b) is well-known; see [GMT].

(a): Every space S determined by a countable closed cover $\{L_n : n \in N\}$ of locally compact subspaces is locally σ-compact (hence, locally paracompact).

(b): Every locally compact space with a point-countable k-network is metric (hence, paracompact).

(c): Every paracompact locally compact space is determined by a compact-finite closed cover of compact subsets.

Indeed, for (a), we can assume that $L_n \subset L_{n+1}$ for each $n \in N$. For $p \in S$, let $p \in L_m$. Since each L_n $(n \geqq m)$ is locally compact, there exists a sequence $\{V_n : n \geqq m\}$ of subsets such that $p \in V_m \subset V_n \subset V_{n+1}$, and each V_n is open in L_n with $\mathrm{cl}V_n$ compact. Let $V = \cup\{V_n : n \geqq m\}$. Since $V \cap L_n$, is open in L_n for each $n \in N$, V is open in X with

$V \ni p$. Besides, V is contained in a σ-compact subset $\cup\{\mathrm{cl}V_n : n \in N\}$. This implies that S is locally σ-compact. For (c), every paracompact locally compact space has a locally finite closed cover of compact subsets, thus, it is determined by this compact-finite cover.

Now, we show that the "ony if" parts of (1) and (2) hold. By means of the above facts and Fact A(1), it follows that each $x \in X$ has a nbd $V(x)$ which is determined by a cover $\mathcal{C}$ of compact subsets such that each elements meets at most $\beta < \alpha$ many other elements. Here, for (1), $\alpha = \omega_1$. Let $\mathcal{C}_1 = \{C \in \mathcal{C} : x \in \mathcal{C}\}$, and let $\mathcal{C}_n = \{C \in \mathcal{C} : C$ meets some element of $\mathcal{C}_{n-1}\}$ for $n \geqq 2$. Then, $|\mathcal{C}_n| \leqq \beta$ for each $n \in N$. Let $\mathcal{C}^* = \cup\{\mathcal{C}_n : n \in N\}$, and $W = \cup\{C : C \in \mathcal{C}^*\}$. Then, $|\mathcal{C}^*| < \alpha$. Since $W \cap C = C$ or ϕ for each $C \in \mathcal{C}$, W is open in $V(x)$ with $W \ni x$, and W is determined by the cover $\mathcal{C}^*$ of compact subsets (note that $V(x)$ is a topological sum of these clopen subsets W). This shows that X is locally $< k_\alpha$. For the parenthetic part, since X is paracompact, X is locally $< k_\alpha$ in view of the above. Then, X has a locally finite closed cover of k_β-spaces, here $\beta < \alpha$. Thus, X is determined by a cover of compact subsets such that each element meets at most $\beta < \alpha$ many other elements. Thus, in view of the above, X is a topological sum of k_β-spaces, here $\beta < \alpha$.

Let us say that an operation is CD if it is the decomposition of operators "closed maps" and "dominations", and let CD (metric) be the class of all spaces obtained from metric spaces under the operation CD. For example, every space dominated by Lašnev spaces belongs to CD(metric). The following holds. In (k_3) or (k_3^*), we can replace "dominated by a countable cover of locally compact metric spaces" by "a topological sum of k_ω-and-$\aleph_0$-spaces".

Corollary 3 *Let X and Y be spaces in the class CD(metric). Then, the following* (1) $\sim$ (3) *hold.*

(1) *Under* $\rceil BF(\omega_2), X \times Y$ *is a* k*-space iff* (k_1) X *and* Y *are metric;* (k_2) X *or* Y *is locally compact metric; otherwise* (k_3) X *is dominated by a countable cover of locally compact metric spaces, and so is* Y. *If* $X = Y$, *then the result holds without* $\rceil BF(\omega_2)$.

(2) *Under* (MA), *the result in* (1) *holds, but replaces* (k_3) *by* (k_3^*)*: One of* X *and* Y *is dominated by a countable cover of locally compact metric spaces, and another is a topological sum of* k_α*-spaces, here* $\alpha < c$.

(3) X^ω *is a* k*-space iff* X *is metric.*

Proof: As is well-known, paracompactness is preserved under closed maps and dominations. Thus, by Remark 2, every space in CD(metric) is a paracompact space staisfying (B_4). While, every first countable (or locally compact) space satisfying (B_4) is metric; see Remark 1(1). Thus, in view of Remark 5, (1) $\sim$ (3) holds by Theorems 1.2 & 2, Corollary 2.

Finally, we give some negations to the *Hypothesis* under $BF(\omega_2)$, (CH). In the following theorem, (1) is routine by Fact C(2). (2); (3) is due to [LiL]; [S_2] respectively.

Theorem 3 (1) *Under* $BF(\omega_2)$, *the Hypothesis is not valid for Lašnev spaces* $X = S_{\omega_1}$ *and* $Y = S_\omega$.

(2) *Under* $BF(\omega_2)$, *the Hypothesis is not valid for spaces* $X = K_{\omega_1}^*$ *and* $Y = S_2$ *with point-countable compact* k*-networks.* (X *and* Y *are quotient, compact images of locally compact metric spaces*).

(3) *Under* (*CH*), *the Hypothesis is not valid for a space* $X = S_\omega$, *and a* σ*-compact,* k*-space* Y *with a point-countable compact* k*-network such that* $so(Y) = 3$ *and* $X \times Y$ *is a* k*-space, but* Y *is not a locally* k_ω*-space.* (X *and* Y *are quotient* s*-images of locally compact metric spaces*).

Note: The following statement was suggested in [H]: Under (CH) and (MC) (=Existance of an uncountable measurable cardinal), the *Hypothesis* holds for k-spaces X and Y with point-countable closed k-networks. But, there is a gap (wrong application of (MC), p 72; +11) in the proof as it was pointed out in [G_2]. Under (CH) alone, the statement is false by Theorem 3(3).

In view of Theorem 1 and 2, and Remark 5, the following question is posed.

Question: let X and Y be k-spaces with point-countable k-networks. Then,

(1) If X^2 is a k-space , then does X satisfy (K_1), or (K_3)?

(2) As a characterization for $X \times Y$ to be a k-space, are there different types of properties on X, Y from the properties (K_1), (K_2), (K_3), and (K_3^*)?

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