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# NORMALITY AND COLLECTIONWISE NORMALITY OF PRODUCT SPACES

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**Abstract**

We prove that for a collectionwise normal $P$-space $X$ and a paracompact $\Sigma$-space $Y$, the product $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal; this extends K. Nagami's theorem [8] with $Y$ being a paracompact $\sigma$-space as well as affirmatively answers to L. Yang's problem in [14].

## 1. Introduction

Throughout this paper we assume all spaces to be Hausdorff. For two collectionwise normal spaces $X$ and $Y$, results asserting that normality of $X \times Y$ implies its collectionwise normality have been proven by several authors. The cases $Y$ being metrizable, Lasňev, a paracompact $M$-space and $\sigma$-locally compact paracompact were proved by Okuyama [10], Hoshina [3], Rudin-Starbird [13] and Chiba [2], respectively. In [8], Nagami proved another case: for a collectionwise normal $P$-space $X$ and a paracompact $\sigma$-space $Y$, $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal. As is known, Nagami [9] proved that for a paracompact $P$-space $X$ and a paracompact $\Sigma$-space $Y$, $X \times Y$ is paracompact; this extends his result in [8] where $Y$ is assumed to be paracompact $\sigma$. Thus, it is natural to ask if it is true that for a collectionwise normal $P$-space $X$ and a paracompact $\Sigma$-space $Y$, normality of $X \times Y$ implies its collectionwise normality. Indeed, this was posed as a problem by Yang [14]. He showed there that the answer is affirmative if $X$ is normal countably compact, and recently Hoshina and the author [5] generalized it to the case where $X$ is collectionwise normal $\Sigma$. It should be noted that the answer seems to be unkown even in the case of $X$ being perfectly normal. In this paper, we solve this problem affirmatively without any such condition on $X$. That is, we show that for a collectionwise normal $P$-space $X$ and a paracompact $\Sigma$-space $Y$, $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.

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## 2. Preliminaries and the proof of the theorem

First let us recall $P$-spaces and $\Sigma$-spaces. Let $N$ denote the set of all positive integers.

A space $X$ is a *P-space* [7] if for any index set $\Omega$ and for any collection $\{G(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega; n \in N\}$ of open subsets of $X$ such that

$$G(\alpha_1,\ldots,\alpha_n)\subset G(\alpha_1,\ldots,\alpha_n,\alpha_{n+1}) \text{ for } \alpha_1,\ldots,\alpha_n,\alpha_{n+1}\in\Omega,$$

there exists a collection $\{F(\alpha_1,\ldots,\alpha_n) \mid \alpha_1,\ldots,\alpha_n\in\Omega;\ n\in N\}$ of closed subsets of $X$ such that the conditions (a), (b) below are satisfied :

(a) $F(\alpha_1,\ldots,\alpha_n)\subset G(\alpha_1,\ldots,\alpha_n)$ for $\alpha_1,\ldots,\alpha_n\in\Omega$,

(b) $X=\cup\{G(\alpha_1,\ldots,\alpha_n)\mid n\in N\}$
$\Longrightarrow X=\cup\{F(\alpha_1,\ldots,\alpha_n)\mid n\in N\}$.

A $\Sigma$-space [9] is a space $Y$ having a sequence, called a $\Sigma$-*net*, $\{\mathcal{E}_n|n\in N\}$ of locally finite closed covers of $Y$ which satisfies the following conditions:

(c) $\mathcal{E}_n$ is written as $\{E(\alpha_1,\ldots,\alpha_n)\mid\alpha_1,\ldots,\alpha_n\in\Omega\}$ with an index set $\Omega$,

(d) $E(\alpha_1,\ldots,\alpha_n)=\cup\{E(\alpha_1,\ldots,\alpha_n,\alpha_{n+1})\mid\alpha_{n+1}\in\Omega\}$ for $\alpha_1,\ldots,\alpha_n\in\Omega$,

(e) For every $y\in Y$, $C(y)$ is countably compact, and there exists a sequence $\alpha_1,\alpha_2,\ldots\in\Omega$ such that $C(y)\subset V$ with $V$ open implies $C(y)\subset E(\alpha_1,\ldots,\alpha_n)\subset V$ for some $n$, where $C(y)=\cap\{E|y\in E\in\mathcal{E}_n,\ n\in N\}$. We call $\{E(\alpha_1,\ldots,\alpha_n)\mid n\in N\}$ a *local net* of $C(y)$.

It is well-known that every perfect space (= a space each of whose open subset is $F_\sigma$) and every $\Sigma$-space are $P$-spaces, and for other related facts the reader is referred to [12] or [4].

To prove our theorem, we need the following lemma due to Katětov [6].

**Lemma 2.1** *Let $X$ be a countably paracompact normal space. Then $X$ is collectionwise normal if and only if for any locally finite closed collection $\mathcal{F}$ of $X$ there exists a locally finite open cover $\mathcal{U}$ of $X$ such that each $U\in\mathcal{U}$ intersects at most finitely many members of $\mathcal{F}$.*

Let us now prove our main theorem.

**Theorem 2.2** *Let $X$ be a collectionwise normal $P$-space and $Y$ a paracompact $\Sigma$-space. Then $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

**Proof:** It suffices to show the "if" part. Assume that $X \times Y$ is normal. From [9, Theorem 4.10], we have that $X \times Y$ is countably paracompact. Let $\{D_\lambda | \lambda \in \Lambda\}$ be a locally finite closed collection of $X \times Y$. Let $\{\mathcal{E}_n | n \in N\}$ be a $\Sigma$-net of Y, where we express $\mathcal{E}_n = \{E(\alpha_1, \ldots, \alpha_n) | \alpha_1, \ldots, \alpha_n \in \Omega\}$. Since $Y$ is paracompact, for each $n \in N$ there exists a locally finite open cover $\{L(\alpha_1, \ldots, \alpha_n) | \alpha_1, \ldots, \alpha_n \in \Omega\}$ of $Y$ such that

$$E(\alpha_1, \ldots, \alpha_n) \subset L(\alpha_1, \ldots, \alpha_n)$$

for each $\alpha_1, \ldots, \alpha_n \in \Omega$.

Define $\Delta = \{\Gamma \subset \Lambda | \text{ Card } \Gamma \text{ is finite}\}$. For each $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in N$ and $\Gamma \in \Delta$, let us put

$$G_\Gamma(\alpha_1, \ldots, \alpha_n) = \cup\{O \mid O \text{ is open in } X \text{ and } (O \times E(\alpha_1, \ldots, \alpha_n)) \cap (\cup\{D_\lambda | \lambda \notin \Gamma\}) = \emptyset\}.$$

Then $G_\Gamma(\alpha_1, \ldots, \alpha_n)$ is open in $X$ and we have

$$G_\Gamma(\alpha_1, \ldots, \alpha_n) \subset G_\Gamma(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

for each $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$ since $E(\alpha_1, \ldots, \alpha_n) \supset E(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$. Hence if we put

$$G(\alpha_1, \ldots, \alpha_n) = \cup\{G_\Gamma(\alpha_1, \ldots, \alpha_n) | \Gamma \in \Delta\},$$

then $G(\alpha_1, \ldots, \alpha_n)$ is an open subset of $X$ and we have

$$G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

for each $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$. Since $X$ is a $P$-space, there exists a collection

$$\{F(\alpha_1, \ldots, \alpha_n) | \alpha_1, \ldots, \alpha_n \in \Omega; n \in N\}$$

of closed subsets of $X$ such that

$$F(\alpha_1,\ldots,\alpha_n) \subset G(\alpha_1,\ldots,\alpha_n) \tag{1}$$

for each $\alpha_1,\ldots,\alpha_n \in \Omega,\ n \in N$, and

$$X = \cup\{G(\alpha_1,\ldots,\alpha_n)|n \in N\} \Longrightarrow X = \cup\{F(\alpha_1,\ldots,\alpha_n)|n \in N\}. \tag{2}$$

Here we may assume

$$F(\alpha_1,\ldots,\alpha_n) \subset F(\alpha_1,\ldots,\alpha_n,\alpha_{n+1}) \tag{3}$$

for each $\alpha_1,\ldots,\alpha_n,\alpha_{n+1} \in \Omega$.

Fix $\alpha_1,\ldots,\alpha_n \in \Omega$ and $n \in N$ arbitrarily. Define a subset $P_\Gamma(\alpha_1,\ldots,\alpha_n)$ of $X$ for $\Gamma \in \Delta$ by

$$P_\Gamma(\alpha_1,\ldots,\alpha_n) = \{x \in X|(\{x\} \times E(\alpha_1,\ldots,\alpha_n)) \cap D_\lambda \neq \emptyset \text{ if and only if } \lambda \in \Gamma\}.$$

We show that the collection

$$\{F(\alpha_1,\ldots,\alpha_n) \cap P_\Gamma(\alpha_1,\ldots,\alpha_n)|\Gamma \in \Delta\} \tag{4}$$

is locally finite in $X$. To prove this, let $x \in X$ and we show that this collection is locally finite at $x$. Since $F(\alpha_1,\ldots,\alpha_n)$ is closed in $X$, we may assume that $x \in F(\alpha_1,\ldots,\alpha_n)$. Then by (1) we have $x \in G(\alpha_1,\ldots,\alpha_n)$, and hence there exists a $\Gamma_x \in \Delta$ such that $x \in G_{\Gamma_x}(\alpha_1,\ldots,\alpha_n)$. Suppose

$$G_{\Gamma_x}(\alpha_1,\ldots,\alpha_n) \cap P_\Gamma(\alpha_1,\ldots,\alpha_n) \neq \emptyset.$$

Select a point $z \in G_{\Gamma_x}(\alpha_1,\ldots,\alpha_n) \cap P_\Gamma(\alpha_1,\ldots,\alpha_n)$. To prove $\Gamma \subset \Gamma_x$, let $\lambda \in \Gamma$. Since $z \in P_\Gamma(\alpha_1,\ldots,\alpha_n)$, we have $(\{z\} \times E(\alpha_1,\ldots,\alpha_n)) \cap D_\lambda \neq \emptyset$. Consequently $(G_{\Gamma_x}(\alpha_1,\ldots,\alpha_n) \times E(\alpha_1,\ldots,\alpha_n)) \cap D_\lambda \neq \emptyset$, which implies that $\lambda \in \Gamma_x$ by the definition of $G_{\Gamma_x}(\alpha_1,\ldots,\alpha_n)$. Thus, $\Gamma \subset \Gamma_x$ and since $\mathrm{Card}\Gamma_x$ is finite, we have shown that the collection (4) above is localy finite at $x$.

Since $X$ is collectionwise normal and countably paracompact, applying Lemma 2.1 to the above collection (4), there exist a locally finite open cover $\{H_\mu | \mu \in M(\alpha_1, \dots, \alpha_n)\}$ of $X$ such that each $H_\mu$ intersects at most finitely many members of $\{F(\alpha_1, \dots, \alpha_n) \cap P_\Gamma(\alpha_1, \dots, \alpha_n) | \Gamma \in \Delta\}$. For each $\mu \in M(\alpha_1, \dots, \alpha_n)$, let us put

$$\Delta_\mu = \{\Gamma \in \Delta | (F(\alpha_1, \dots, \alpha_n) \cap P_\Gamma(\alpha_1, \dots, \alpha_n)) \cap H_\mu \neq \emptyset\}.$$

Then Card $\Delta_\mu$ is finite. Define

$$C_\mu = \cup\{D_\lambda | D_\lambda \cap ((F(\alpha_1, \dots, \alpha_n) \cap P_\Gamma(\alpha_1, \dots, \alpha_n)) \times E(\alpha_1, \dots, \alpha_n)) = \emptyset \text{ for each } \Gamma \in \Delta_\mu\}$$

for each $\mu \in M(\alpha_1, \dots, \alpha_n)$.

Next we define

$$\mathcal{U} = \{H_\mu \times L(\alpha_1, \dots, \alpha_n) - C_\mu | \mu \in M(\alpha_1, \dots, \alpha_n); \alpha_1, \dots, \alpha_n \in \Omega, n \in N\}.$$

Then it is easy to see that $\mathcal{U}$ is a $\sigma$-locally finite open collection of $X \times Y$.

We shall show that each $U \in \mathcal{U}$ intersects at most finitely many members of $\{D_\lambda | \lambda \in \Lambda\}$. Let $U \in \mathcal{U}$. So we can express that

$$U = H_\mu \times L(\alpha_1, \dots, \alpha_n) - C_\mu$$

for some $n \in N$, $\alpha_1, \dots, \alpha_n \in \Omega$ and $\mu \in M(\alpha_1, \dots, \alpha_n)$. Suppose $U \cap D_\lambda \neq \emptyset$. Then, by the definition of $C_\mu$, we have

$$D_\lambda \cap ((F(\alpha_1, \dots, \alpha_n) \cap P_\Gamma(\alpha_1, \dots, \alpha_n)) \times E(\alpha_1, \dots, \alpha_n)) \neq \emptyset$$

for some $\Gamma \in \Delta_\mu$. Consequently $D_\lambda \cap (P_\Gamma(\alpha_1, \dots, \alpha_n) \times E(\alpha_1, \dots, \alpha_n)) \neq \emptyset$. Pick

$$(x, y) \in D_\lambda \cap (P_\Gamma(\alpha_1, \dots, \alpha_n) \times E(\alpha_1, \dots, \alpha_n)).$$

Since $x \in P_\Gamma(\alpha_1, \ldots, \alpha_n)$, we have $\lambda \in \Gamma$. Since $\Delta_\mu$ is finite, it follows that $\lambda$ is contained in the finite subset $\cup\{\Gamma | \Gamma \in \Delta_\mu\}$ of $\Lambda$. Hence we have shpwn that each $U \in \mathcal{U}$ intersects at most finitely members of $\{D_\lambda | \lambda \in \Lambda\}$.

Finally we shall show that $X \times Y = \cup \mathcal{U}$. To see this, let $(x, y) \in X \times Y$ be any point of $X \times Y$. Take $\alpha_1, \alpha_2, \ldots \in \Omega$ so that

$$\{E(\alpha_1, \ldots, \alpha_n) | n \in N\} \text{ is a local net of } C(y).$$

Before everything, we show that $X = \cup\{G(\alpha_1, \ldots, \alpha_n) | n \in N\}$. Let $a \in X$. We put

$$\Gamma_{ay} = \{\lambda \in \Lambda | (\{a\} \times C(y)) \cap D_\lambda \neq \emptyset\}.$$

Since $\{a\} \times C(y)$ is compact and $\{D_\lambda | \lambda \in \Lambda\}$ is locally finite, $\Gamma_{ay}$ is finite and there exist open sets $O$ and $O'$ of $X$ and $Y$, respectively, such that

$$\{a\} \times C(y) \subset O \times O' \subset X \times Y - \cup\{D_\lambda | \lambda \notin \Gamma_{ay}\}.$$

From the property of the local net, there exists an $n \in N$ such that

$$C(y) \subset E(\alpha_1, \ldots, \alpha_n) \subset O'.$$

Therefore we have

$$(O \times E(\alpha_1, \ldots, \alpha_n)) \cap (\cup\{D_\lambda | \lambda \notin \Gamma_{ay}\}) = \emptyset.$$

Thus we can verify $a \in O \subset G_{\Gamma_{ay}}(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$. Hence we have $X = \cup\{G(\alpha_1, \ldots, \alpha_n) | n \in N\}$. It follows from (2) that

$$X = \cup\{F(\alpha_1, \ldots, \alpha_n) | n \in N\}. \qquad (5)$$

Let

$$\Gamma_{xy} = \{\lambda \in \Lambda | (\{x\} \times C(y)) \cap D_\lambda \neq \emptyset\}.$$

Likewise the matter shown above, $\Gamma_{xy}$ is a finite set, that is, $\Gamma_{xy} \in \Delta$, and there exist open sets $O_x$ and $O_y$ of $X$ and $Y$, respectively, such that

$$\{x\} \times C(y) \subset O_x \times O_y \subset X \times Y - \cup\{D_\lambda | \lambda \notin \Gamma_{xy}\}.$$

From the property of the local net, there exists an $m \in N$ such that

$$\{x\} \times C(y) \subset O_x \times E(\alpha_1, \ldots, \alpha_m) \subset O_x \times O_y.$$

We note that

$$x \in P_{\Gamma_{xy}}(\alpha_1, \ldots, \alpha_i) \text{ for every } i \geq m \tag{6}$$

since $C(y) \subset E(\alpha_1, \ldots, \alpha_i) \subset E(\alpha_1, \ldots, \alpha_m)$. By (5), there exists an $\ell \in N$ such that

$$x \in F(\alpha_1, \ldots, \alpha_\ell).$$

Put $k = \max\{m, \ell\}$. Then it follows from (1) and (3) that

$$x \in F(\alpha_1, \ldots, \alpha_\ell) \subset F(\alpha_1, \ldots, \alpha_k) \subset G(\alpha_1, \ldots, \alpha_k). \tag{7}$$

Because $\{H_\mu | \mu \in M(\alpha_1, \ldots, \alpha_k)\}$ covers $X$, there exists a

$$\mu \in M(\alpha_1, \ldots, \alpha_k) \text{ such that } x \in H_\mu. \tag{8}$$

Note that $(x, y) \in H_\mu \times L(\alpha_1, \ldots, \alpha_k)$. To show $(x, y) \notin C_\mu$, we assume the contrary, that is, $(x, y) \in C_\mu$. Hence there exists a $\lambda \in \Lambda$ such that

$$(x, y) \in D_\lambda, \tag{9}$$

and

$$D_\lambda \cap (\cup\{F(\alpha_1, \ldots, \alpha_k) \cap P_\Gamma(\alpha_1, \ldots, \alpha_k) | \Gamma \in \Delta_\mu\} \times E(\alpha_1, \ldots, \alpha_k)) = \emptyset. \tag{10}$$

Because of (6) and (7), we have

$$x \in F(\alpha_1, \ldots, \alpha_k) \cap P_{\Gamma_{xy}}(\alpha_1, \ldots, \alpha_k). \tag{11}$$

Since $y \in E(\alpha_1, \ldots, \alpha_k)$, (9), (10) and (11) imply

$$\Gamma_{xy} \notin \Delta_\mu.$$

But we have

$$x \in F(\alpha_1, \ldots, \alpha_k) \cap P_{\Gamma_{xy}}(\alpha_1, \ldots, \alpha_k) \cap H_\mu$$

by (11) and (8), which implies

$$\Gamma_{xy} \in \Delta_\mu,$$

a contradiction. Thus

$$(x, y) \in H_\mu \times L(\alpha_1, \ldots, \alpha_k) - C_\mu \in \mathcal{U},$$

showing that $X \times Y = \cup\mathcal{U}$.

The above shows that $\mathcal{U}$ is a $\sigma$-locally finite open cover of $X \times Y$ such that each member of $\mathcal{U}$ intersects at most finitely many members of $\{D_\lambda | \lambda \in \Lambda\}$. Since $X \times Y$ is countably paracompact, there exists a locally finite open cover $\mathcal{U}'$ of $X \times Y$ which refines $\mathcal{U}$. Clearly $\mathcal{U}'$ has the same property. Hence it follows from Lemma 2.1 that $X \times Y$ is collectionwise normal. □

**Corollary 2.3.** *Let $X$ be a collectionwise normal perfectly normal space and $Y$ a paracompact $\Sigma$-space. Then $X \times Y$ is collectionwise normal if and only if $X \times Y$ is normal.*

**Remark** The product $X \times Y$ of a collectionwise normal perfect space $X$ and compact space $Y$ need not to be normal because of [11] (see [2]).

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