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Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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# COMPACTNESS AND SOBRIETY IN BITOPOLOGICAL SPACES

Bruce S. Burdick

## Abstract

We define asymmetric notions of compactness and sobriety for bitopological spaces and show how these properties interact. Some applications are given to the hyperspace operator introduced in an earlier paper. This hyperspace operator makes possible a bitopological sobrification contruction. We explore certain problems with this bitopological sobrification which make it different from the traditional sobrification.

## Introduction

When one begins to look at the literature of bitopological spaces one becomes aware of a wealth of competing definitions for the terms that one knows well from traditional topology. In this paper we suggest some interesting definitions for well known properties and explore the usefulness of those definitions. We pay particular attention to results that would be of interest to researchers working with hyperspaces, and because certain hyperspaces with the order of reverse inclusion are important examples of partially ordered sets we endeavor to find connections to the theory of continuous partial orders.

A *bitopological space* is a triple $(X, \mathcal{T}, \mathcal{T}^*)$ where $\mathcal{T}$ and $\mathcal{T}^*$ are topologies on $X$. Bitopological spaces were introduced by Kelly [Ke] as a tool for analyzing quasimetric spaces. Our

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terminology is primarily influenced by Kopperman [K1] and secondarily by Deák [De].

The *dual* of a bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is the bitopological space $(X, \mathcal{T}^*, \mathcal{T})$. When convenient we may designate a bitopological space by $X$ and its dual by $X^*$.

The $\mathcal{T}$-closure operator in $X$ is designated by $c$ and the $\mathcal{T}^*$-closure operator is $c^*$. Similarly, the $\mathcal{T}$-saturation operator is designated by sat and the $\mathcal{T}^*$-saturation operator is designated by sat$^*$, where the *saturation* of a set $A$ with respect to a topology $\mathcal{T}$ is $\cap\{O \in \mathcal{T} | \ A \subseteq O\}$.

A map $f : X \to Y$ is a *continuous* map $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ of bitopological spaces if it is continuous with respect to first topologies $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ and with respect to second topologies $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U}^*)$. $(X, \mathcal{T}, \mathcal{T}^*)$ is a *subspace* of $(Y, \mathcal{U}, \mathcal{U}^*)$ if $X \subseteq Y$ and $\mathcal{T} = \mathcal{U}|_X$, $\mathcal{T}^* = \mathcal{U}^*|_X$. A *product* $(X, \mathcal{T}, \mathcal{T}^*)$ of bispaces $(X_\alpha, \mathcal{T}_\alpha, \mathcal{T}^*_\alpha)$ is formed by letting $X = \times X_\alpha$, $\mathcal{T} = \times \mathcal{T}_\alpha$, and $\mathcal{T}^* = \times \mathcal{T}^*_\alpha$.

The word "pairwise" has been used extensively in the literature and with several different meanings (compare for example [De, 0.9] with [K1, Def. 2.1]). We will use this term only in the following case: $A$ is *pairwise dense* in $X$ if it is dense with respect to both topologies.

When we use a result about spaces with only one topology we will refer to this as a result from *traditional* topology.

## 1. Asymmetric Definitions for Compactness and Sobriety

A number of different versions of compactness have been defined for bitopological spaces. Among these is *sup-compactness*, defined by Deák [De, 0.8] to be the property that the supremum of the two topologies is compact. We wish to introduce a different definition of compactness and to show that it has certain desirable properties.

**Definition 1.1.** A bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is *compact* if whenever $\mathcal{C}$ is a cover of $X$ consisting of one member of $\mathcal{T}^*$ and arbitrarily many members of $\mathcal{T}$ then $\mathcal{C}$ has a finite subcover.

Equivalently, we could say that $(X, \mathcal{T}, \mathcal{T}^*)$ is compact if whenever $\mathcal{C}$ is a cover of $X$ consisting of finitely many members of $\mathcal{T}^*$ and arbitrarily many members of $\mathcal{T}$ then $\mathcal{C}$ has a finite subcover, or we could say that $(X, \mathcal{T}, \mathcal{T}^*)$ is compact if every $\mathcal{T}^*$-closed subset of $X$ is $\mathcal{T}$-compact.

A bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is defined to be *stable* if every $\mathcal{T}^*$-closed proper subset of $X$ is $\mathcal{T}$-compact. So we are saying that $(X, \mathcal{T}, \mathcal{T}^*)$ is compact iff it is stable and $(X, \mathcal{T})$ is compact. (See [K1] for some other properties of stable bispaces.)

**Definition 1.2.** A space $(X, \mathcal{T}, \mathcal{T}^*)$ is an $R_1$ space if for any $x, y \in X$, if $x \notin cy$ then there exist $O_1 \in \mathcal{T}$ and $O_2 \in \mathcal{T}^*$ such that $O_1 \cap O_2 = \phi$, $x \in O_1$, and $y \in O_2$. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a *regular* space if for any $x \in X$ and any $O \in \mathcal{T}$ if $x \in O$ then there is an $O' \in \mathcal{T}$ such that $x \in O'$ and $c^*O' \subseteq O$. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a *normal* space if for any $\mathcal{T}^*$-closed $A \subseteq X$ and any $O \in \mathcal{T}$ if $A \subseteq O$ then there is an $O' \in \mathcal{T}$ such that $A \subseteq O'$ and $c^*O' \subseteq O$.

We note that by the Alexander subbase theorem a bispace $X$ is sup-compact if and only if both $X$ and $X^*$ are compact. We note also that compact regular bispaces are normal, and that if $X$ is compact and $X^*$ is $R_1$ then $X^*$ is regular. (See for example Theorem 3.6 in [K1].)

**Definition 1.3.** A space $(X, \mathcal{T}, \mathcal{T}^*)$ is an $R_0$ space if for any $x \in X$ and any $O \in \mathcal{T}$ if $x \in O$ then $c^*x \subseteq O$. We will designate by $R_0^*$ the property that the dual space is $R_0$.

**Definition 1.4.** A non-empty set $A \subseteq X$ is $\mathcal{T}$-*irreducible* if it is $\mathcal{T}$-closed and for any $\mathcal{T}$-closed sets $B$ and $C$, if $A = B \cup C$ then either $A = B$ or $A = C$. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is *quasisober* if for any $\mathcal{T}$-irreducible $A \subseteq X$, there is a point $x \in A$ such that for any $O \in \mathcal{T}^*$ if $x \in O$ then $A \subseteq O$, i.e., $A \subseteq \text{sat}^* x$.

This property "quasisober" for bispaces is a weakening of the property "sober" which we introduced in [B1]. We will explore its consequences throughout this paper, and a further application of this property can be found in [B2].

Recall that a traditional space $(X, \mathcal{T})$ is quasisober if each irreducible set is the closure of a point, and it is sober if it is quasisober and $T_0$. Note that if $(X, \mathcal{T})$ is quasisober and $(X, \mathcal{T}, \mathcal{T}^*)$ is $R_0^*$ then $(X, \mathcal{T}, \mathcal{T}^*)$ is quasisober. On the other hand, if $(X, \mathcal{T}, \mathcal{T}^*)$ is quasisober and $R_0$ then $(X, \mathcal{T})$ is quasisober.

**Definition 1.5.** A bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is *cover regular* if whenever $\mathcal{C} \subseteq \mathcal{T}$ is a cover of a $\mathcal{T}^*$-closed set $A$ then there is a $\mathcal{T}$-open cover $\mathcal{C}'$ of $A$ such that for each $O' \in \mathcal{C}'$ there is an $O \in \mathcal{C}$ with $c^*O' \subseteq O$.

**Proposition 1.1.** *A bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is regular if and only if it is $R_0$ and cover regular.*

*Proof.* That regular implies $R_0$ and cover regular is clear.

On the other hand, suppose $(X, \mathcal{T}, \mathcal{T}^*)$ is $R_0$ and cover regular. Given $x \in O \in \mathcal{T}$ we see that $\{O\}$ is a $\mathcal{T}$-open cover of $c^*x$. So there is a $\mathcal{T}$-open cover $\mathcal{C}$ of $c^*x$ such that each $O' \in \mathcal{C}$ satisfies $c^*O' \subseteq O$. One of these $O' \in \mathcal{C}$ contains $x$. □

**Proposition 1.2.** *If a bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is compact and cover regular then $X^*$ is quasisober.*

*Proof.* Suppose $A \subseteq X$ is a $\mathcal{T}^*$-irreducible set and that for each $x \in A$ there is an $O_x \in \mathcal{T}$ with $x \in O_x$ and $A \not\subseteq O_x$. By cover regularity there is a $\mathcal{T}$-open cover $\mathcal{C}$ of $A$ such that for each $O \in \mathcal{C}$ there is an $x \in A$ with $c^*O \subseteq O_x$. By compactness there is a finite subcover $\mathcal{C}'$ of $\mathcal{C}$. But now we have $A \subseteq \cup_{O \in \mathcal{C}'} c^*O$ with no $c^*O$ containing $A$. This contradicts the irreducibility of $A$. □

**Proposition 1.3.** *A $\mathcal{T}$-closed or $\mathcal{T}^*$-closed subspace of a compact space $(X, \mathcal{T}, \mathcal{T}^*)$ is compact.*

**Proposition 1.4.** *A $\mathcal{T} \vee \mathcal{T}^*$-closed subspace of a quasisober space $(X, \mathcal{T}, \mathcal{T}^*)$ is quasisober.*

*Proof.* Given $A \subseteq X$, closed in $\mathcal{T} \vee \mathcal{T}^*$, suppose $B \subseteq A$ is $\mathcal{T}|_A$-irreducible. Then $cB$ is $\mathcal{T}$-irreducible. So there is an $x \in cB$ such that $B \subseteq \text{sat}^* x$. Now if $O \in \mathcal{T}$ and $O^* \in \mathcal{T}^*$ are neighborhoods of $x$ then we have $O \cap B \neq \phi$, and since $B \subseteq O^*$ we have $O \cap O^* \cap B \neq \phi$. So we have $x \in B$. □

**Lemma 1.1.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a quasisober space. Let $\mathcal{F}$ be a collection of non-empty, $\mathcal{T}$-compact, $\mathcal{T}^*$-closed subsets of $X$ which is closed under finite intersections. Then $\cap\mathcal{F} \neq \phi$.*

*Proof.* Extend $\mathcal{F}$ to $\mathcal{M}$, maximal with respect to the properties of containing non-empty, $\mathcal{T}$-compact sets and being closed under finite intersections. Let $C \subseteq X$ be the set of $\mathcal{T}$-cluster points of the filterbase $\mathcal{M}$. We must have $C \cap D \neq \phi$ for each $D \in \mathcal{M}$ since each such $D$ is $\mathcal{T}$-compact. $C$ is $\mathcal{T}$-closed, so each $C \cap D$ for $D \in \mathcal{M}$ is $\mathcal{T}$-compact. By the maximality of $\mathcal{M}$ we have $C \cap D \in \mathcal{M}$ for each $D \in \mathcal{M}$.

Let $C_1$ and $C_2$ be $\mathcal{T}$-closed sets such that $C_1 \cup C_2 = C$. If there were a $D_1 \in \mathcal{M}$ disjoint from $C_1$ and a $D_2 \in \mathcal{M}$ disjoint from $C_2$ then $D_1 \cap D_2 \cap C$ would be empty. So without loss of generality we assume that $C_1 \cap D \neq \phi$ for each $D \in \mathcal{M}$. Using again the maximality of $\mathcal{M}$ we have $C_1 \cap D \in \mathcal{M}$ for each $D \in \mathcal{M}$. But, since for each $x \in C$ and each $D \in \mathcal{M}$ we have $x \in cD$, we must have $C_1 = C$. This shows that $C$ is $\mathcal{T}$-irreducible.

Let $x \in C$ be a point with $C \subseteq \text{sat}^*\{x\}$. For every $F \in \mathcal{F}$ we have $C \cap F \neq \phi$ and therefore we have $x \in c^*F = F$. □

**Proposition 1.5: Hofmann-Mislove Theorem for Bispaces.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a quasisober space. Let $\mathcal{F}$ be a col-*

*lection of $\mathcal{T}$-compact, $\mathcal{T}^*$-closed subsets of $X$ which is closed under finite intersections. Then:*

*(1) If $\cap\mathcal{F} \subseteq O \in \mathcal{T}$ then there is some $F \in \mathcal{F}$ with $F \subseteq O$.*

*(2) $\cap\mathcal{F}$ is $\mathcal{T}$-compact.*

*Proof.* (1) Suppose $F - O \neq \phi$ for each $F \in \mathcal{F}$. Then the set $\mathcal{F}|_{X-O} = \{F - O|\ F \in \mathcal{F}\}$ would be a collection, which is closed under finite intersections, of non-empty, $\mathcal{T}|_{X-O}$-compact, $\mathcal{T}^*|_{X-O}$-closed subsets of the quasisober space $X - O$. Then by Lemma 1.1 and Proposition 1.4 we have $\cap\mathcal{F}|_{X-O} \neq \phi$, a contradiction.

(2) If $\mathcal{C}$ is a $\mathcal{T}$-covering of $\cap\mathcal{F}$ then by (1) we have $F \subseteq \cup\mathcal{C}$ for some $F \in \mathcal{F}$. $F$ is $\mathcal{T}$-compact so $\mathcal{C}$ has a finite subcover of $F$, and therefore of $\cap\mathcal{F}$. □

Proposition 1.5 is the bispace version of a theorem of Hofmann and Mislove [HM, Proposition 2.19] for traditional sober spaces. If $(X, \mathcal{T})$ is quasisober, and $\mathcal{T}^*$ is the topology whose closure operator is the $\mathcal{T}$-saturation operator, then $(X, \mathcal{T}, \mathcal{T}^*)$ is quasisober, and applying our result to $(X, \mathcal{T}, \mathcal{T}^*)$ gives the Hofmann and Mislove theorem for the traditional space $(X, \mathcal{T})$.

**Corollary 1.1.** *If a bispace $X$ is compact and quasisober then $X^*$ is compact.*

*Proof.* Let $(X, \mathcal{T}, \mathcal{T}^*)$ be compact and quasisober. Given a $\mathcal{T}$-closed set $C$ we show that $C$ is $\mathcal{T}^*$-compact. Suppose $\mathcal{F}$ is a filterbase of $\mathcal{T}^*$-closed sets each of which intersects $C$. Then each $F \in \mathcal{F}$ is $\mathcal{T}$-compact and so by Proposition 5 we have that if $\cap\mathcal{F} \subseteq X - C$ then there would be some $F \in \mathcal{F}$ with $F \subseteq X - C$, a contradiction. So $\cap\mathcal{F} \cap C \neq \phi$. □

**Proposition 1.6: Tikhonov Product Theorem for Bispaces.** *If $(X, \mathcal{T}, \mathcal{T}^*)$ is the product of the family of bispaces $\{(X_\alpha, \mathcal{T}_\alpha, \mathcal{T}_\alpha^*)|\alpha \in L\}$ then we have the following:*

*(1) $X$ is quasisober if and only if each of the bispaces $X_\alpha$ is quasisober.*

*(2) $X$ is sup-compact if and only if each of the bispaces $X_\alpha$ is sup-compact.*

*(3) $X$ is quasisober and compact if and only if each of the bispaces $X_\alpha$ is quasisober and compact.*

*Proof.* (1) Suppose $X$ is quasisober and suppose that $A_{\alpha'} \subseteq X_{\alpha'}$ is $\mathcal{T}_{\alpha'}$-irreducible. For each $\alpha \neq \alpha'$ choose a $\mathcal{T}_\alpha$-irreducible set $A_\alpha$ and let $A$ be the product of the $A_\alpha$'s. Then $A$ is $\mathcal{T}$-irreducible. Choose $x \in X$ with $A \subseteq \text{sat}^* x$ and let the $\alpha'$th coordinate of $x$ be $x_{\alpha'}$. Then $A_{\alpha'} \subseteq \text{sat}^* x_{\alpha'}$.

Conversely, suppose each $X_\alpha$ is quasisober. Suppose $A \subseteq X$ is $\mathcal{T}$-irreducible. For each $\alpha$ let $A_\alpha$ be the $\alpha$th projection of $A$. We claim that $A$ is the product of the $A_\alpha$'s. For if $x \in (\times A_\alpha) - A$ then there are $O_{\alpha_1} \in \mathcal{T}_{\alpha_\infty}, \ldots, \mathcal{O}_{\alpha_\backslash} \in \mathcal{T}_{\alpha_\backslash}$ with $x \in \pi_{\alpha_1}^{-1}[O_{\alpha_1}] \cap \cdots \cap \pi_{\alpha_n}^{-1}[O_{\alpha_n}] \subseteq X - A$. But each $\pi_{\alpha_i}^{-1}[O_{\alpha_i}]$ must intersect $A$, contradicting the irreducibility of $A$. So $A = \times A_\alpha$. Hence each $A_\alpha$ is $\mathcal{T}_\alpha$-closed and so it is also $\mathcal{T}_\alpha$-irreducible. For each $\alpha$ choose $x_\alpha \in A_\alpha$ with $A_\alpha \subseteq \text{sat}^* x_\alpha$ and let $x \in X$ be the point whose $\alpha$th coordinate is $x_\alpha$. Then $x \in A$ and $A \subseteq \text{sat}^* x$.

(2) This follows from the traditional Tikhonov Theorem by noting that $(X, \mathcal{T} \vee \mathcal{T}^*)$ is the product of the family of traditional spaces $\{(X_\alpha, \mathcal{T}_\alpha \vee \mathcal{T}_\alpha^*) | \alpha \in L\}$.

(3) This follows from (1) and (2) since by Corollary 1.1, compact and quasisober is equivalent to sup-compact and quasisober. □

**Example 1.1.** The product of two compact bispaces need not be compact. Let $X$ be the interval $[0, 1)$. Let $\mathcal{T}$ be generated by sets of the form $(a, 1)$ and let $\mathcal{T}^*$ be the usual order topology. Then $(X, \mathcal{T}, \mathcal{T}^*)$ is compact, since the $\mathcal{T}^*$-closed sets each contain a least element. However the product $(X, \mathcal{T}, \mathcal{T}^*) \times (X, \mathcal{T}, \mathcal{T}^*)$ is not compact. We see this by noting

$D = \{(x, y) \in X \times X |\ x + y = 1\}$ is $\mathcal{T}^* \times \mathcal{T}^*$-closed but not $\mathcal{T} \times \mathcal{T}$-compact. In fact $D$ with the topology it inherits from $\mathcal{T} \times \mathcal{T}$ is homeomorphic to an open interval in the reals with the usual topology.

## 2. Compactness and Other Properties in Bitopological Hyperspaces

We constructed a hyperspace operator for bitopological spaces in [B1] (See also [VG], [GS] for other such operators).

**Definition 2.1.** Given a space $(X, \mathcal{T}, \mathcal{T}^*)$ let $2^X$ be the set of non-empty subsets of $X$ which are closed relative to $\mathcal{T}$. If $\{A_1, A_2, \ldots, A_n\}$ is a family of subsets of $X$, let $< A_1, A_2, \ldots, A_n > = \{B \in 2^X |\ B \subseteq \cup_{i=1}^n A_i$ and for each $i = 1, \ldots, n,\ A_i \cap B \neq \phi\}$. Let $L(\mathcal{T})$ be the topology on $2^X$ generated by the subbasis consisting of sets of the form $< O, X >$ where $O \in \mathcal{T}$ and let $U(\mathcal{T}^*)$ be the topology on $2^X$ generated by the basis consisting of sets of the form $< O >$ where $O \in \mathcal{T}^*$. Then $(2^X, L(\mathcal{T}), \mathcal{U}(\mathcal{T}^*))$ is the *hyperspace* of $(X, \mathcal{T}, \mathcal{T}^*)$. Sometimes it will be convenient to denote this hyperspace by $(2^X, \mathcal{L}, \mathcal{U})$.

For a traditional space $(X, \mathcal{T})$ the topologies $L(\mathcal{T})$ and $U(\mathcal{T})$ are respectively the *lower Vietoris* and *upper Vietoris* topologies on $2^X$, and their supremum is the *Vietoris* topology [Vi].

**Proposition 2.1.** *For a bispace $(X, \mathcal{T}, \mathcal{T}^*)$ the following are equivalent:*

*(1) $(X, \mathcal{T}, \mathcal{T}^*)$ is compact.*

*(2) $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is compact.*

*(3) $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is sup-compact.*

*Proof.* (1) implies (3): Suppose $(X, \mathcal{T}, \mathcal{T}^*)$ is compact. Note that $\mathcal{S} = \{< X, O > \mid O \in \mathcal{T}\}$ and $\mathcal{S}^* = \{< O > \mid O \in \mathcal{T}^*\}$

are subbases for $L(\mathcal{T})$ and $U(\mathcal{T}^*)$, respectively, and so $\mathcal{S} \cup \mathcal{S}^*$ is a subbase for $\sup\{L(\mathcal{T}), U(\mathcal{T}^*)\}$. Given $\mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{C}^* \subseteq \mathcal{T}^*$ such that

$$\mathbf{C} = \{< X, O > \mid O \in \mathcal{C}\} \cup \{< O > \mid O \in \mathcal{C}^*\}$$

is a cover of $2^X$, let $A = X - \cup\mathcal{C}$.

If $A \neq \phi$ then $A \in 2^X$ and so $A \subseteq O$ for some $O \in \mathcal{C}^*$. Then $\mathcal{C}$ is a cover of $X - O$ and so it has a finite subcover $\{O_1, \ldots, O_n\}$. Then $\{< X, O_1 >, \ldots, < X, O_n >, < O >\}$ will be a finite subcover of $\mathbf{C}$.

If $A = \phi$ then $\mathcal{C}$ covers $X$, and we complete the proof in a similar fashion.

(3) implies (2): Trivial.

(2) implies (1): Suppose $A$ is $\mathcal{T}^*$-closed and $\mathcal{C}$ is a $\mathcal{T}$-open cover of $A$. Then $< X, A >$ is $U(\mathcal{T}^*)$-closed and $\{< X, O > \mid O \in \mathcal{C}\}$ is an $L(\mathcal{T})$-open cover of $< X, A >$. Choose a finite $\mathcal{C}' \subseteq \mathcal{C}$ such that $\{< X, O > \mid O \in \mathcal{C}'\}$ covers $< X, A >$. Then $\mathcal{C}'$ covers $A$. □

In the next section we will observe that hyperspaces are always quasisober. This explains (in view of Corollary 1.1) why compactness and sup-compactness are equivalent for hyperspaces.

**Example 2.1.** The dual of a hyperspace may be compact even when the conditions in Proposition 2.1 do not hold. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a bispace where $\mathcal{T}$ is any non-compact topology for $X$, and $\mathcal{T}^*$ is indiscrete. Then $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is not compact but the dual $(2^X, U(\mathcal{T}^*), L(\mathcal{T}))$ is compact.

In [B1] we showed that if the base space $(X, \mathcal{T}, \mathcal{T}^*)$ is regular then the hyperspace $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is *symmetrically* $T_2$, i.e., $T_0$, $R_1$, and $R_1^*$. In [K1], Kopperman calls the combination of sup-compact and symmetrically $T_2$ by the name *joincompact*, and therefore we can say that if a bispace is compact and regular then its hyperspace is joincompact. By Kopperman's results in [K2] we can make a number of inferences about a hyperspace of a compact regular bispace $(X, \mathcal{T}, \mathcal{T}^*)$. Among the most interesting of these is that each of the topologies $L(\mathcal{T})$ and $U(\mathcal{T}^*)$ is the *de Groot dual* of the other.

**Definition 2.2.** A topology $\mathcal{T}$ for $X$ is the *de Groot dual* of a topology $\mathcal{T}^*$ if $\mathcal{T}$ is generated by the complements of $\mathcal{T}^*$-saturated $\mathcal{T}^*$-compact subsets of $X$.

**Lemma 2.1.** *Suppose $(X, \mathcal{T}, \mathcal{T}^*)$ is an $R_0$ space; then the de Groot dual of $U(\mathcal{T}^*)$ is finer than $L(\mathcal{T})$. Suppose $(X, \mathcal{T}, \mathcal{T}^*)$ is a normal space; then the de Groot dual of $U(\mathcal{T}^*)$ is coarser than $L(\mathcal{T})$.*

In a related vein, it is important to point out the overlap between this investigation and the study of continuous partial orders. For this we will need some terminology, for which our principal reference will be the survey article of Lawson and Mislove [LM].

**Definition 2.3.** Let $(P, \leq)$ be a partial order. We create a convergence structure such that if $D \subseteq P$ is an upward directed set with supremum $x$ then $D$ converges to any $y \leq x$. The topology generated by this convergence structure is called the *Scott topology* for $(P, \leq)$. From this we get the *Lawson topology* for $(P, \leq)$ by taking the supremum of the Scott topology with the weakest topology for which the sets $\uparrow x = \{y | \ y \geq x\}$ are closed.

**Definition 2.4.** A bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ has two *specialization orders*: $x \leq y$ if and only if $x \in c\{y\}$ and $x \leq^* y$ if and only if $x \in c^*\{y\}$.

For a hyperspace $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ the statement $A \leq B$ is the same as $A \subseteq B$. If $(X, \mathcal{T}, \mathcal{T}^*)$ is $R_0$ then so is $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ (see [B1]) and so the statement $A \leq^* B$

is equivalent to $A \supseteq B$.

**Lemma 2.2.** *If $(X, \mathcal{T}, \mathcal{T}^*)$ is stable then $U(\mathcal{T}^*)$ is coarser than the Scott topology relative to $\supseteq$. (In fact, $(X, \mathcal{T}, \mathcal{T}^*)$ is compact if and only if all upward directed sets in $(2^X, \supseteq)$ have suprema and $U(\mathcal{T}^*)$ is coarser than the Scott topology relative to $\supseteq$.)*

**Lemma 2.3.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a regular space. Then $U(\mathcal{T}^*)$ is finer than the Scott topology relative to $\supseteq$.*

**Proposition 2.2.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a compact regular space. Then $U(\mathcal{T}^*)$ is the Scott topology on $2^X$ relative to $\supseteq$. The de Groot dual of $U(\mathcal{T}^*)$ is $L(\mathcal{T})$ and vice versa. $L(\mathcal{T}) \vee U(\mathcal{T}^*)$ is the Lawson topology relative to $\supseteq$.*

The case where $\mathcal{T} = \mathcal{T}^*$ in Proposition 2.2 is Example 7.4 in [LM].

**Proposition 2.3: Alexander Subbase Theorem for Quasisober Bispaces.** *Suppose $(X, \mathcal{T}, \mathcal{T}^*)$ is a quasisober bispace with $\mathcal{S}$ a subbase for $\mathcal{T}$ and $\mathcal{S}^*$ a subbase for $\mathcal{T}^*$. Then $(X, \mathcal{T}, \mathcal{T}^*)$ is sup-compact if and only if any cover of $X$ containing finitely many members of $\mathcal{S}^*$ and arbitrarily many members of $\mathcal{S}$ has a finite subcover.*

*Proof.* If $X$ is sup-compact then the latter condition follows trivially.

Conversely, suppose $X$ satisfies the subbase covering condition in the proposition. Then by the traditional Alexander Theorem it follows that any finite union of members of $\mathcal{S}^*$ has a $\mathcal{T}$-compact complement. Let $\mathcal{B}^*$ be the set of finite intersections of finite unions of members of $\mathcal{S}$. Then the complement of any member of $\mathcal{B}^*$ is a finite union of $\mathcal{T}$-compact sets and hence is $\mathcal{T}$-compact. $\mathcal{B}^*$ is clearly a basis for $\mathcal{T}^*$. Moreover,

$\mathcal{B}^*$ is closed under finite unions, since any finite union of finite intersections may be rewritten as a finite intersection of finite unions. It now follows from Proposition 1.5 that any $\mathcal{T}^*$-open set has a $\mathcal{T}$-compact complement and so $X$ is compact. Then by Corollary 1.1 $X$ is sup-compact. □

**Proposition 2.4.** *For a bispace $(X, \mathcal{T}, \mathcal{T}^*)$, if $(2^{X^*}, U(\mathcal{T}), L(\mathcal{T}^*))$, the dual of the hyperspace of the dual, is quasisober, then it is compact (and hence sup-compact).*

*Proof.* It suffices by Proposition 2.3 to show that if we are given $\mathcal{C} \subseteq \mathcal{T}$ and a finite $\mathcal{C}^* \subseteq \mathcal{T}^*$ such that

$$\mathbf{C} = \{< O > \mid O \in \mathcal{C}\} \cup \{< X, O > \mid O \in \mathcal{C}^*\}$$

is a cover of $2^{X^*}$, then $\mathbf{C}$ has a finite subcover. If $\mathcal{C}^*$ covers $X$ then we are done. Otherwise let $A = X - \cup\mathcal{C}^* \in 2^{X^*}$. Then $A \subseteq O'$ for some $O' \in \mathcal{C}$. So

$$\{< O' >\} \cup \{< X, O > \mid O \in \mathcal{C}^*\}$$

is a finite subcover for $\mathbf{C}$. □

**Corollary 2.1.** *For a regular bispace $(X, \mathcal{T}, \mathcal{T}^*)$ the following are equivalent:*

*(1) $(X, \mathcal{T}, \mathcal{T}^*)$ is compact.*

*(2) $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is compact.*

*(3) $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is sup-compact.*

*– $(2^X, U(\mathcal{T}^*), L(\mathcal{T}))$, the dual of the hyperspace, is quasisober.*

**Example 2.2.** A hyperspace may be sup-compact, but its dual may fail to be quasisober. Let $X = \{0, 1\}$ and let $\mathcal{T}$ be the discrete topology on $X$ and $\mathcal{T}^*$ be the Sierpinski topology $\{\phi, \{1\}, X\}$ on $X$. Then $2^X$ is sup-compact since it is finite, but $(2^X)^*$ is not quasisober. We observe that $2^X$ is a

$U(\mathcal{T}^*)$-irreducible set which can be covered by $L(\mathcal{T})$ open sets $< X, \{0\} >= \{\{0\}, X\}$ and $< X, \{1\} >= \{\{1\}, X\}$, neither of which contains $2^X$.

In this last example $X^*$ itself is not quasisober. We show in Proposition 2.5 that this is significant, but in the meantime we need a lemma generalizing the Hofmann-Mislove Theorem for bispaces. The idea of this lemma and the method of proof come from the recent paper of Keimel and Paseka [KP].

**Lemma 2.4.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a quasisober space and $\mathcal{F}$ be a collection of $\mathcal{T}$-compact, $\mathcal{T}^*$-closed subsets of $X$. Suppose that for any two $U, V \in \mathcal{T}$, if $U$ and $V$ each contain a member of $\mathcal{F}$ then so does $U \cap V$. Then the two conclusions of Proposition 1.5 hold.*

*Proof.* Suppose there is an $O \in \mathcal{T}$ with $\cap\mathcal{F} \subseteq O$ but no member of $\mathcal{F}$ is contained within $O$. Then among the $\mathcal{T}$-open sets which do not contain any member of $\mathcal{F}$ there is a maximal set $O'$ with $O \subseteq O'$. The complement $X - O'$ is $\mathcal{T}$-irreducible so there is a point $x \in X - O'$ with $X - O' \subseteq \operatorname{sat}^* x$. Every $\mathcal{T}^*$-closed set which doesn't contain $x$ must be contained in $O'$ and therefore $x \in \cap\mathcal{F}$, which is a contradiction. So any $O \in \mathcal{T}$ with $\cap\mathcal{F} \subseteq O$ must contain some member of $\mathcal{F}$. The other conclusion, that $\cap F$ is compact, follows easily. □

**Proposition 2.5.** *For a compact, quasisober bispace $(X, \mathcal{T}, \mathcal{T}^*)$ we have that $(2^{X^*}, U(\mathcal{T}), L(\mathcal{T}^*))$, the dual of the hyperspace of the dual, is quasisober (hence sup-compact).*

*Proof.* Suppose $\mathcal{F} \subseteq 2^{X^*}$ is $U(\mathcal{T})$-irreducible. Then $\mathcal{F}$ consists of non-empty, $\mathcal{T}$-compact, $\mathcal{T}^*$-closed sets, and for any two $U, V \in \mathcal{T}$ if $< U >$ and $< V >$ both intersect $\mathcal{F}$ then $< U > \cap < V > = < U \cap V >$ must intersect $\mathcal{F}$. Thus by the lemma we conclude that for any $O \in \mathcal{T}$ with $\cap\mathcal{F} \subseteq O$ we must have some $C \in \mathcal{F}$ with $C \subseteq O$. $\cap\mathcal{F} \neq \phi$ since otherwise some $C \in \mathcal{F}$ satisfies $C \subseteq \phi$. So $\cap\mathcal{F} \in 2^{X^*}$. Furthermore we see that every $U(\mathcal{T})$-neighborhood of $\cap\mathcal{F}$ intersects $\mathcal{F}$. So $\cap\mathcal{F} \in \mathcal{F}$. Finally, note that every $L(\mathcal{T}^*)$ neighborhood of $\cap\mathcal{F}$ contains all of $\mathcal{F}$. □

## 3. Sobriety and Sobrification

**Definition 3.1.** A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a $T_0$ space if for any two distinct points in $X$ there is an element of $\mathcal{T} \cup \mathcal{T}^*$ which contains one point but not the other. A bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is *sober* if it is $T_0$, $R_0^*$, and for any $\mathcal{T}$-irreducible $A \subseteq X$, there is a point $x \in A$ such that $A = c\{x\}$.

Note that $(X, \mathcal{T}, \mathcal{T}^*)$ sober is equivalent to saying that $(X, \mathcal{T})$ is sober and $(X, \mathcal{T}, \mathcal{T}^*)$ is $R_0^*$. Sober implies quasisober, $T_0$, and $R_0^*$ but is stronger than the conjunction of these three properties, as the following example shows.

**Example 3.1.** Let $X$ be an infinite set, let $\mathcal{T}$ be the cofinite topology on $X$, and let $\mathcal{T}^*$ be the indiscrete topology on $X$. Then $(X, \mathcal{T}, \mathcal{T}^*)$ is $T_0$, $R_0^*$, and quasisober but not sober.

Propositions 3.1–3.3 were proved in [B1] (as Propositions 4–6) and are repeated here.

**Proposition 3.1.** *Any hyperspace* $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ *is sober.*

This explains why in Proposition 2.1 we saw that compact and sup-compact are equivalent for hyperspaces—it's because they are all quasisober.

**Proposition 3.2.** *Any* $T_0$*,* $R_0^*$ *bispace* $(X, \mathcal{T}, \mathcal{T}^*)$ *may be embedded as a pairwise dense subspace of a sober bispace* $(Y, \mathcal{S}, \mathcal{S}^*)$*.* $Y \subseteq 2^X$ *is the set of* $\mathcal{T}$*-irreducible subsets of* $X$ *and* $\mathcal{S}$ *and* $\mathcal{S}^*$ *are* $L(\mathcal{T})|_Y$ *and* $U(\mathcal{T}^*)|_Y$*, respectively. The embedding* $e_X : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{S}, \mathcal{S}^*)$ *is defined by* $e_X(x) = c\{x\}$*.*

**Definition 3.2.** The sober bispace $Y$ guaranteed by Proposition 3.2 is the *sobrification* of $X$. (See [AGV] for an early reference to sobrification in traditional topology.) The construction may be generalized to spaces which are not $T_0$, abandoning the claim that $e_X$ is an embedding. So for any $R_0^*$ bispace $(X, \mathcal{T}, \mathcal{T}^*)$ we define a sobrification $(Y, \mathcal{S}, \mathcal{S}^*)$ and a mapping $e_X : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{S}, \mathcal{S}^*)$ as in Proposition 3.2.

**Proposition 3.3.** *A sober bispace $(X, \mathcal{T}, \mathcal{T}^*)$ is homeomorphic to its sobrification.*

One suspects that there is a categorical discussion that is possible at this point, and that the sobrification can be cast as a reflection into a category of sober spaces. This approach is initially tripped up by the observation that this bitopological sobrification (like the hyperspace) is not a functor on the category of bispaces and continuous maps.

**Definition 3.3.** Let $(X, \mathcal{T}, \mathcal{T}^*)$ and $(Y, \mathcal{S}, \mathcal{S}^*)$ be bispaces and let $(\hat{X}, \hat{\mathcal{T}}, \hat{\mathcal{T}}^*)$ and $(\hat{Y}, \hat{\mathcal{S}}, \hat{\mathcal{S}}^*)$ be their respective sobrifications. For a function $f : X \to Y$, define by $\hat{f}(A) = c(f[A])$ a function $\hat{f} : \hat{X} \to \hat{Y}$. (For this to be well defined it suffices for $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ to be continuous, since then the closure of the image of a $\mathcal{T}$-irreducible set is $\mathcal{S}$irreducible.) A function $f : X \to Y$ is an *S-map* if both $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{S}, \mathcal{S}^*)$ and $\hat{f} : (\hat{X}, \hat{\mathcal{T}}, \hat{\mathcal{T}}^*) \to (\hat{Y}, \hat{\mathcal{S}}, \hat{\mathcal{S}}^*)$ are continuous mappings of bispaces.

**Example 3.2.** A continuous map of bispaces may not be an S-map. Let $X$ be any infinite set. Let $\mathcal{T}$ and $\mathcal{T}^*$ be the cofinite and indiscrete topologies on $X$, respectively. Let $Y$ be $X \cup \{\infty\}$ and let $\mathcal{S} = \{\phi\} \cup \{O \subseteq Y |\ \infty \in O \ \&\ Y - O$ is finite$\}$ and $\mathcal{S}^* = \{\{\phi\}, X, Y\}$. The inclusion map $f : X \to Y$ is continuous. Consider the induced map $\hat{f} : (\hat{X}, \hat{\mathcal{T}}, \hat{\mathcal{T}}^*) \to (\hat{Y}, \hat{\mathcal{S}}, \hat{\mathcal{S}}^*)$. The points of $\hat{X}$ are each $\{x\}$ for $x \in X$, together with $X$. $\hat{f}(\{x\}) = \{x\}$ for each $x \in X$ but $\hat{f}(X) = Y$. So $\hat{f}^{-1}[< X >$

$] = \{\{x\} | \ x \in X\}$, which is not open in $\hat{\mathcal{T}}^*$. So $f$ is continuous but not an S-map. Note that $X$ and $Y$ are both $R_0^*$ spaces in this example.

**Definition 3.4.** Let $\mathbf{R}$ be the category of $R_0^*$ bitopological spaces and S-maps and let $\mathbf{S}$ be the category of sober bitopological spaces and S-maps. Define a functor $\sigma : \mathbf{R} \to \mathbf{S}$ by letting $\sigma$ of an object be its sobrification and $\sigma$ of a morphism $f$ be the induced map $\hat{f}$. Define a functor $\iota : \mathbf{S} \to \mathbf{R}$ to be the inclusion functor.

There are two technicalities which are entailed by the claim that $\sigma$ is a functor. That $\sigma$ respects the composition of morphisms belongs to traditional topology. However, unique to our bitopological treatment of sobrification is the implicit claim that the induced map of an S-map is an S-map. We will show this in Proposition 3.4.

**Definition 3.5.** Let $(X, \mathcal{T}, \mathcal{T}^*)$ and $(Y, \mathcal{S}, \mathcal{S}^*)$ be bispaces. A function $f : X \to Y$ is an *N-map* if for any $\mathcal{T}$-irreducible $A \subseteq X$ and any $O \in \mathcal{S}^*$, if $f[A] \subseteq O$ then $c(f[A]) \subseteq O$.

**Lemma 3.1.** *Any continuous N-map is an S-map.*

*Proof.* Given a continuous N-map $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{S}, \mathcal{S}^*)$ it suffices to show that $\hat{f} : (\hat{X}, \hat{\mathcal{T}}^*) \to (\hat{Y}, \hat{\mathcal{S}}^*)$ is continuous. Given $O \in \mathcal{S}^*$ we note that $\hat{f}^{-1}[< O >] = \{A \in \hat{X} | \ f[A] \subseteq O\}$ by the definition of an N-map. So $\hat{f}^{-1}[< O >] = \ < f^{-1}[O] > \ \in \mathcal{T}^*$. □

**Lemma 3.2.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ and $(Y, \mathcal{S}, \mathcal{S}^*)$ be bispaces. For any function $f : X \to Y$ (even a discontinuous one) the function $\tilde{f} : 2^X \to 2^Y$ defined by $\tilde{f}(A) = c(f[A])$ is an N-map with respect to $L(\mathcal{T})$ and $U(\mathcal{S}^*)$.*

*Proof.* The proof of Proposition 3.1 included showing that any $L(\mathcal{T})$-irreducible set in $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is of the form $2^A$ for some $A \in 2^X$. Now $\tilde{f}[2^A] = \{c(f[B]) | \ B \in 2^A\} \subseteq 2^{c(f[A])}$, so $c(\tilde{f}[2^A]) = 2^{c(f[A])}$. Any $U(\mathcal{S}^*)$-open neighborhood of $\tilde{f}[2^A]$ must contain a basic open set $< O >$ with $c(f[A]) \subseteq O$. But then $2^{c(f[A])} \subseteq < O >$. □

**Lemma 3.3.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ and $(Y, \mathcal{S}, \mathcal{S}^*)$ be bispaces. The restriction $f|_A : A \to Y$ of an N-map $f : X \to Y$ to a $\mathcal{T}$-closed subspace $A$ of $X$ is an N-map.*

**Proposition 3.4.** *The induced map of an S-map is an S-map.*

*Proof.* Suppose $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{S}, \mathcal{S}^*)$ is an S-map. Then $\hat{f} : (\hat{X}, \hat{\mathcal{T}}, \hat{\mathcal{T}}^*) \to (\hat{Y}, \hat{\mathcal{S}}, \hat{\mathcal{S}}^*)$ is continuous. $\hat{f}$ is also the restriction of the function $\tilde{f}$ to the closed subspace $\hat{X}$ of $2^X$. $\tilde{f}$ is an N-map and so $\hat{f}$ is an N-map. Since $\hat{f}$ is a continuous N-map it is an S-map. □

**Proposition 3.5.** *For an $R_0^*$ space $(X, \mathcal{T}, \mathcal{T}^*)$ the mapping $e_X : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{S}, \mathcal{S}^*)$ is an S-map.*

*Proof.* We have already observed that $e_X$ is continuous. That $e_X$ is an S-map then follows if we show that $\widehat{e_X} = e_{\hat{X}}$. (Note that $\hat{X}$ is $R_0^*$.) For a $\mathcal{T}$-irreducible set $A$, $\widehat{e_X}(A) = c(\{c(x) | \ x \in A\}) \in \hat{\hat{X}}$. We claim that the $\hat{\mathcal{T}}$-closure of $\{c(x) | \ x \in A\}$ is $\hat{A}$, since $\hat{A}$ is the smallest $\hat{\mathcal{T}}$-irreducible set containing $\{c(x) | \ x \in A\}$. But $\hat{A}$ is the $\hat{\mathcal{T}}$-closure of $\{A\}$, which is $e_{\hat{X}}(A)$. So $\widehat{e_X}(A) = e_{\hat{X}}(A)$. □

**Proposition 3.6.** *$\sigma$ is left-adjoint to $\iota$.*

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Roger Williams University

*E-mail address*: bruce@alpha.rwu.edu