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THE ROUND IDEAL COMPLETION VIA SOBRIFICATION

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Abstract

In this paper we consider an important order completion, the rounded-ideal completion, that has arisen in the modern theory of continuous domains. We show that it can be alternately viewed as a special case of a more general topological method of completion, namely taking the sobrification of a topological space. A number of important special cases and examples are included.

1. Introduction

The two standard methods of obtaining the real numbers from the rationals are via completions with Cauchy sequences or with Dedekind cuts. The first approach is a topological construction while the second is an order-theoretic one. Of course the method of completions via Cauchy sequences extends to all metric spaces, not just ordered spaces.

In this short paper we consider another important order completion of more recent origin, the rounded-ideal completion, and show that it can be alternately viewed as a special case of a more general topological method of completion, namely taking the sobrification of a topological space.

We quickly recall basic notions of ordered sets and continuous domain theory (see, for example, [AJ95] or [COMP]). Let

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$(P, \leq)$ be a partially ordered set (or poset). A *principal ideal* is a set of the form $\downarrow x := \{y \colon y \leq x\}$. A set A is a *lower set* if $A = \downarrow A := \{y : y \leq a \text{ for some } a \in A\}$. A non-empty subset D of P is *directed* if $x, y \in D$ implies there exists $z \in D$ with $x \leq z$ and $y \leq z$. An *ideal* of P is a directed lower set. The poset P is a *directed complete partially ordered set* (DCPO) if every directed subset of P has a supremum.

If x, y belong to a partially ordered set P, we write $x \ll y$ and say that x *approximates* y if for every directed set D with $y \leq \sup D$, we have $x \leq d$ for some $d \in D$. A partially ordered set is called a *continuous poset* if every element is the directed supremum of all elements which approximate it, $y = \bigvee^{\uparrow}\{x : x \ll y\}$ for all $y \in P$. The partially ordered set is a *continuous domain* if it is both a DCPO and a continuous poset.

Let P be a poset. A subset B of P is called a *basis* if for each $x \in P$ there exists a directed subset D of B with supremum x such that each member of D approximates x. It is a standard result that P is a continuous poset if and only if P possesses a basis. The set P itself is a basis in any continuous poset.

A continuous poset admits a useful T_0 topology called the *Scott topology*; a set is Scott closed by definition if it is a lower set which is closed with respect to taking suprema of directed subsets.

The open sets are the upper sets U with the property that if the supremum of a directed set lies in U, then some member of the directed set lies in U. It is well-known that the Scott topology has as a basis of open sets all sets of the form

$$\Uparrow x := \{y : x \ll y\}.$$

2. Rounded Ideal Completions

The material in this section is more-or-less standard with the seminal ideas dating back to the early work of D. Scott. We refer the reader to [Sm77], Section 2.2 of [AJ95], and Chapter

I.1 of [COMP].

Let $(P, \prec)$ denote a set equipped with a binary relation. For $F \subseteq P$ and $y \in P$, we write $F \prec y$ if $x \prec y$ for all $x \in F$. The binary relation $\prec$ is called *fully transitive* if it is transitive ($x \prec y, y \prec z \Rightarrow x \prec z$) and satisfies the *strong interpolation property*:

$$\forall |F| < \infty,\ F \prec z \Rightarrow \exists y \prec z \text{ such that } F \prec y.$$

For the case that $F = \emptyset$, we interpret the condition to mean that for all $z \in P$, there exists $y \in P$ such that $y \prec z$. A(n) (*abstract*) *basis* is a pair $(B, \prec)$ where B is a set equipped with a fully transitive relation $\prec$. We observe that if B is a basis for a continuous poset, then $(B, \ll)$, the restriction of the approximation relation to B, is an abstract basis.

Definition 2.1. Suppose that $(B, \prec)$ is an abstract basis. A non-empty subset I of B is a *round ideal* if (i) $y \in I$ and $x \prec y$ imply that $x \in I$, and (ii) $x, y \in I$ implies there exists $z \in I$ with $x \prec z$ and $y \prec z$. The *round ideal completion* is the partially ordered set $(\mathbf{RI(B)}, \sqsubseteq)$ consisting of all rounded ideals ordered by set inclusion.

Theorem 2.2. *Let $(B, \prec)$ be an abstract basis. Then the following hold:*

(i) *the round ideal completion $(\mathbf{RI(B)}, \sqsubseteq)$ is a continuous domain;*

(ii) *$I \ll J$ in $\mathbf{RI(B)} \Leftrightarrow \exists y \in \mathbf{J}$ such that $I \prec y$;*

(iii) *the mapping $j \colon B \to \mathbf{RI(B)}$ defined by $j(y) = \Downarrow y := \{x : x \prec y\}$ has the property that $x \prec y$ implies $j(x) \ll j(y)$;*

(iv) *the set $j(B)$ is a basis for $\mathbf{RI(B)}$.*

A proof may be found in Section 2.2 of [AJ95].

The following example shows that the round ideal completion may be viewed as a generalization of the ideal completion of a poset.

Example 2.3. Let $(P, \leq)$ be any partially ordered set, and let $\prec := \leq$. Then the round ideals of P are just the ideals, so the round ideal completion agrees with the ideal completion in this case, and the standard mapping $x \mapsto \downarrow x$ into the ideal completion is just the mapping j of Theorem 2.2.

There is a variant context for constructing the round ideal completion that one often encounters in practice. Let $(P, \leq)$ be a partially ordered set. A binary relation $\prec$ on P is called an *auxiliary relation* if (i) $\prec$ is fully transitive, (ii) $\prec \subseteq \leq$, and (iii) $w \leq x \prec y \leq z$ implies $w \prec z$. An auxiliary relation is said to be *approximating* if given any $y \not\leq x$, there exists $z \prec y$ such that $z \not\leq x$. We note that the order relation itself in any partially ordered set and the approximation relation in any continuous poset are always approximating auxiliary relations.

There is a slightly modified and sharpened version of Theorem 2.2 for posets equipped with auxiliary relations.

Theorem 2.4. *Let $(P, \leq)$ be a partially ordered set endowed with an auxiliary relation $\prec$. Then the following hold for the round ideal completion with respect to $\prec$:*

(i) *each round ideal is an ideal;*

(ii) *the round ideal completion $(\mathbf{RI(P)}, \sqsubseteq)$ is a continuous domain;*

(iii) $I \ll J$ *in* $\mathbf{RI(P)} \Leftrightarrow \exists y \in \mathbf{J}$ *such that* $I \subseteq \downarrow y$.

(iv) *the mapping* $j\colon P \to \mathbf{RI(P)}$ *defined by* $j(y) = \Downarrow y := \{x : x \prec y\}$ *is order-preserving and has the property that* $x \prec y$ *implies* $j(x) \ll j(y)$;

(v) *the set* $j(P)$ *is a basis for* $\mathbf{RI(P)}$;

(vi) *if $\prec$ is approximating, then j is injective, strictly order preserving, and satisfies $x \prec y$ if and only if $j(x) \ll j(y)$.*

Proof. We verify those assertions that do not appear already in Theorem 2.2.

Let I be a round ideal, let $y \in I$, and let $x \leq y$. There exists $w \in I$ such that $y \prec w$, and hence by property (iii) of an auxiliary relation $x \prec w$. Thus $x \in I$ and hence I is a lower set. Now assume that $x, y \in I$. By definition there exists $z \in I$ such that $x \prec z$ and $y \prec z$. Then by property (ii) of an auxiliary relation $x \leq z$ and $y \leq z$. Hence I is directed and is thus an ideal.

Suppose that $I \ll J$ in $\mathbf{RI(P)}$. By Theorem 2.2 there exists $y \in J$ such that $I \prec y$. But since $\prec \subseteq \leq$, it follows that $I \subseteq \downarrow y$. Conversely suppose that $I \subseteq \downarrow y$ for some $y \in J$. Since J is round, there exists $z \in J$ with $y \prec z$. Then $I \prec z$, and again from Theorem 2.2 $I \ll J$.

Suppose that $x \leq y$ in P. If $w \prec x$, then $w \prec y$ (by property (iii) of an auxiliary relation), and hence $j(x) \sqsubseteq j(y)$. Thus j is order-preserving.

We show j is strictly order-preserving under the hypothesis of (vi). Suppose that $j(x) \sqsubseteq j(y)$. If $x \not\leq y$, then there exists $w \prec x$ with $w \not\leq y$. But then $w \in j(x)$ and $w \notin j(y)$, which contradicts $j(x) \sqsubseteq j(y)$. Since strictly order preserving maps are injective, we conclude that j is injective.

Finally suppose that $\prec$ is approximating and $j(x) \ll j(y)$. By (iii) there exists $w \in j(y)$ such that $j(x) \subseteq \downarrow w$. If $x \not\leq w$, then $z \prec x$ but $z \not\leq w$ for some $z \in P$. But then $z \in j(x) \subseteq \downarrow w$, a contradiction. Thus $x \leq w \prec y$, and hence $x \prec y$. □

Example 2.5. Let $\mathbb{Q}$ denote the set of rational numbers with the usual order and with $\prec = <$, which is an approximating auxiliary relation. The round ideals are all sets of the form $(-\infty, r) \cap \mathbb{Q}$, $r \in \mathbb{R}$, and the set $\mathbb{Q}$ itself. Thus the round ideal completion may be naturally identified with $\mathbb{R} \cup \{\infty\}$, with ∞ added as a largest element.

The following proposition gives a universal property for the round ideal completion (see Proposition 2.2.24 of [AJ95] for a weaker universal property under weaker hypotheses).

Proposition 2.6. *Let $(B, \prec)$ be an abstract basis and let $f\colon B \to P$ be a function into a DCPO which satisfies:*

$$\forall b \in B,\ f(b) = \bigvee{}^{\uparrow} f(\Downarrow b).$$

Then there exists a unique Scott continuous function $F\colon \mathbf{RI(B)} \to P$ such that $F \circ j = f$, where $j\colon B \to \mathbf{RI(B)}$.

Proof. For each round ideal $I \in \mathbf{RI(B)}$, set $F(I) = \bigvee{}^{\uparrow} f(I)$.

It is straightforward to verify that F preserves directed suprema and it follows from the hypothesis that $F(\Downarrow b) = f(b)$ for $b \in B$, i.e., $F \circ j = f$. $\square$

3. Completions via Sobrification

We recall some standard topological facts about sober spaces (see, for example, [COMP]).

A subset E of a topological space X is said to be *irreducible* if for any two closed subsets A and B such that $E \subseteq A \cup B$, either $E \subseteq A$ or $E \subseteq B$. The closure of any singleton subset is always a closed irreducible subspace. The space X is said to be *sober* if every closed irreducible set C is the closure of a unique point, $C = \overline{\{x\}}$ for an unique $x \in X$.

Given any space X, a pair (X^s, j) is called a *sobrification* of X if X^s is a sober space and $j\colon X \to X^s$ is a continuous mapping such that $U \mapsto j^{-1}(U)$ is a lattice isomorphism from the lattice $\mathcal{O}(X^s)$ of open sets of X^s onto the lattice $\mathcal{O}(X)$ of open sets of X. Every space X admits a sobrification. One standard construction to obtain the sobrification of X is to take for the elements of X^s all closed irreducible subsets of X and for the closed subsets of X^s all closed irreducible subsets contained in a given closed subset of X; the mapping j sends a point to its closure (see Exercise V.4.9 of [COMP]). Furthermore, the sobrification of X is unique in the sense that if (X_i^s, j_i) are sobrifications for $i = 1,2$, then there exists a unique homeomorphism $h\colon X_1^s \to X_2^s$ such that $h \circ j_1 = j_2$.

Definition 3.1. Let $(B, \prec)$ be an abstract basis. For $b \in B$, set

$$\Uparrow b := \{x \in B : b \prec x\}.$$

It follows from the strong interpolation property that the sets $\Uparrow b$ form a basis for a topology on B, called the *pseudoScott topology*.

Lemma 3.2. *Let $(B, \prec)$ be an abstract basis equipped with the pseudoScott topology and let the mapping $j : B \to \mathbf{RI}(\mathbf{B})$ be given by $j(b) = \Downarrow b$. Then for each open set U in B, $\uparrow j(U)$ is open in $\mathbf{RI}(\mathbf{B})$ and $j^{-1}(\uparrow j(U)) = U$.*

Proof. Let $y \sqsupseteq j(b)$ for some $b \in U$. There exists $d \in B$ such that $b \in \Uparrow d \subseteq U$. Pick c such that $d \prec c \prec b$. Then $c \in \Uparrow d$ implies $c \in U$. By Theorem 2.2 $j(b) \in \Uparrow j(c)$ and hence $y \in \Uparrow j(c) \subseteq \uparrow j(c) \subseteq \uparrow j(U)$. Since $\Uparrow j(c)$ is open in $\mathbf{RI}(\mathbf{B})$, we conclude that $\uparrow j(U)$ is open.

Set $V := \uparrow j(U)$; we have just seen that V is open. Clearly $U \subseteq j^{-1}(V)$. Conversely suppose $j(q) \in V$. Then $j(q) \sqsupseteq j(b)$ for some $b \in U$. As in the preceding paragraph, there exists $c \in U$ such that $c \prec b$ and $\Uparrow c \subseteq U$. Then $c \in \Downarrow b \sqsubseteq \Downarrow q$ implies $c \prec q$ and hence $q \in U$. $\square$

Theorem 3.3. *Let $(B, \prec)$ be an abstract basis equipped with the pseudoScott topology. Then the mapping $j : B \to \mathbf{RI}(\mathbf{B})$ of Theorem 2.2 is a sobrification of B, where $\mathbf{RI}(\mathbf{B})$ is equipped with the Scott topology.*

Proof. By Theorem 2.2 $\mathbf{RI}(\mathbf{B})$ is a continuous domain, and it is a standard result that a continuous domain is sober with respect to the Scott topology (see [La79]).

Let $b \in B$ and let U be a Scott open set containing $j(b) = \Downarrow b$. Since $j(b)$ is the directed union of all $j(x)$, $x \in \Downarrow b$, there exists $x \prec b$ such that $j(x) \in U$. Since U is an upper set in $\mathbf{RI(B)}$, by part (iii) of Theorem 2.2, $j(\Uparrow x) \subseteq U$. We conclude that j is continuous and hence that $j^{-1}(U)$ is open in B for each Scott open set U.

Suppose that U and V are open sets in $\mathbf{RI(B)}$, $U \neq V$. Then there exists $x \in U \setminus V$ (or vice-versa). Since $j(B)$ is a basis by Theorem 2.2, there exists $j(b) \ll x$ with $j(b) \in U$. But then $j(b) \notin V$, for otherwise $x \in \uparrow V = V$. It follows that $j^{-1}(U) \neq j^{-1}(V)$.

To complete the proof, we need to show that each open set W in B is of the form $j^{-1}(U)$ for some U open in $\mathbf{RI(B)}$. But this follows directly from the preceding lemma. □

Example 3.4. Let $(P, \leq)$ be a partially ordered set. If we consider P as an abstract basis for the fully transitive relation $\leq$, then the pseudoScott topology is the topology of *all* upper sets, sometimes called the *Alexandroff discrete* topology. As we have seen in the in Example 2.3, the round ideal completion is the ideal completion, and by the previous Theorem 3.3 the sobrification of the Alexandroff discrete topology yields the ideal completion equipped with the Scott topology.

The following important special case of the above results was worked out by R.-E. Hoffmann [Ho79] and further treated by M. Mislove in Section 4.2 of [Mi98].

Example 3.5. Let $(P, \leq)$ be a continuous poset. Then the approximation relation $\ll$ on P is an approximating auxiliary relation. We may view the round ideal completion as attaching to P suprema for all rounded ideals which are not principal, i.e., not of the form $\Downarrow(x) = \{y : y \ll x\}$, so that P is enriched and transformed to a DCPO $\mathbf{RI(P)}$. The pseudoScott topology on P is just the Scott topology, and by Theorem 3.3 its sobrification gives $\mathbf{RI(P)}$, again equipped with the Scott topology. If the sobrification mapping is injective (i.e., if the original space is T_0), as it is in this case, then it is easily seen to be a homeomorphic embedding, and thus the restriction of the Scott topology on **RI**(**P**) to P (which we identify with $j(P)$) is again the Scott topology. Observing that $y \ll x$ in P resp. **RI**(**P**) if and only if x is in the Scott-interior (in P resp. **RI**(**P**)) of $\uparrow y$, we conclude directly from the preceding remarks that the restriction of the approximation or "way-below" relation on **RI**(**P**) to P agrees with the original approximation relation on P. This fact also follows from part (vi) of 2.4.

We recall the the sobrification functor is the adjoint functor for the inclusion of the category of sober spaces into the category of topological spaces. It follows that the sobrification satifies a universal property, namely that given any continuous map from the original space into another sober space Y, there exists a unique continuous function from the sobrification into Y which extends the original function.

Proposition 3.6. *Let* $(B, \prec)$ *be an abstract basis, let* Y *be a sober space, and let* $f : B \to Y$ *be a function which is continuous with respect to the pseudoScott topology. Then there exists a continuous function* $F\colon \mathbf{RI(B)} \to \mathbf{Y}$ *such that* $F \circ j = f$*, where* $j\colon B \to \mathbf{RI(B)}$ *is the sobrification mapping.*

Proof. The proof follows from the preceding general theory of sobrifications. However one can alternately check directly in this case that a round ideal is an irreducible subset of B, and hence its image in Y also is. The mapping F is then given by sending the round ideal to the unique point which generates the closure of its image. □

4. Computational Models for Topological Spaces

In an early paper D. Scott [Sc70] suggested that partially ordered structures such as the set of all closed subintervals of a closed real interval (viewed as a partially ordered set ordered by reverse inclusion) should be useful for the study of continuous and computable functions on the closed interval. Ideas of computability on metric spaces related to those of Scott's had surfaced much earlier, for example, in [Lac59] (see also [ML70]). Scott's suggestion was developed along various lines in [WS81] and [KT84]. Two recent developments have given a new impetus to such considerations. One of these is the introduction and ripening theory of the probabilistic power domain of a continuous poset [Sa80], [Jo89], [JP89]. If a continuous poset can be appropriately associated with a topological space, then one has at hand via the probabilistic power domain what can be a very effective and powerful tool for the study of a variety of problems associated with integration, measure and probability (see, for example, [Ed95a]). In addition, A. Edalat has effectively used such structures in a wide variety of applications involving measures, dynamical systems, and iterated function systems and fractals [Ed95b, Ed96, EH97].

In Edalat's work a compact space is realized as the set of maximal points of the continuous domain consisting of its compact non-empty subsets ordered by reverse inclusion (sometimes called the "upper space"). In [La97] the first construction was given for realizing any complete separable metric space as the set of maximal points of a continuous domain.

In this section we briefly point out how the ideas of this paper and of [La97] lead naturally to the introduction of a computational structure on a topological space. These ideas from domain theory provide a unified framework for a variety of classical recursive settings and provide immediate generalization to a much broader class of spaces.

Definition 4.1. Let $(B, \prec)$ be a countable abstract basis with a fixed enumeration, and suppose that $\prec$ is a recursive subset of $B \times B$. We say that $(B, \prec, h)$ is a *computational environment* for a topological space X if $h : X \to \mathbf{RI}(\mathbf{B})$ is a homeomorphism onto the set of maximal points of $\mathbf{RI}(\mathbf{B})$ endowed with the relative Scott topology.

For technical reasons it is also generally useful to require that $\uparrow x \cap \mathrm{Max}(\mathbf{RI}(\mathbf{B}))$ is relatively Scott closed in $\mathrm{Max}(\mathbf{RI}(\mathbf{B}))$ for all $x \in \mathbf{RI}(\mathbf{B})$.

The last requirement for a computational environment insures that the space of maximal points (with the relative Scott topology) is a Polish space.

The definition given above reduces the recursive theory that arises to recursion theory on the positive integers. Another convenient, often more direct, equivalent approach is via Post systems (see [ML70] for a particularly nice presentation).

Example 4.2. Let $(P, \leq)$ be the partially ordered set of finite strings of $\{0, 1\}$, where the partial order is the *prefix order*, $w_1 \leq w_1 w_2$, where w_1 and w_2 are finite strings of 0's and 1's. Taking $\leq$ as the fully transitive order, one sees that (via Post systems or a standard enumeration of P) it is a recursive relation. In this case, as we have seen in Example 2.3, the round ideal completion is just the ideal completion. The non-principal ideals are the maximal points of $\mathbf{RI}(\mathbf{P})$ and may be identified with the infinite strings of $\{0, 1\}$; the relative Scott topology of the set of maximal points is just the product topology, and hence the maximal points form a Cantor set. We have thus constructed the classical computational environment for the Cantor set.

Example 4.3. The classical computational environment for Baire space is constructed in a quite analogous way. In this case start with the set $\mathbb{N}$, and let P consist of finite strings of non-negative integers ordered by the prefix order. The infinite strings are again the maximal points, the relative Scott topology is the product topology, and hence we have the Baire space $\mathbb{N}^{\mathbb{N}}$ as the space of maximal points. This is the classical computational environment for Baire space.

Example 4.4. Consider all closed intervals of rational numbers $[p/q, r/s]$. We define a partial order by reverse inclusion and an auxiliary relation by strict reverse inclusion at both endpoints. The maximal points in the round ideal completion arise from strictly nested intervals of rational numbers with length approaching 0; these may be identified with their insersection on the real line, and hence correspond to $\mathbb{R}$. We thus have a classical computational model for the real numbers.

These examples extend to much broader classes of topological spaces using the techniques for creating domain environments for topological spaces (continuous domains in which the space appears as the set of maximal points) that have recently appeared. Once one has constructed a computational environment for a space, then one has at hand all necessary tools for a unified approach to the study of notions of a computable point in the space (the round ideal defining it is a recursively enumerable set), computable (i.e., recursive) functions, and other standard related notions. We close with one final example based on the approach of [EH97].

Example 4.5. We construct a computational environment for $\mathbb{R}^n$ by taking B to be all $n+1$-tuples of rational numbers, where the last rational entry is always positive. We think of the tuple as a closed ball in $\mathbb{R}^n$ with the first n-coordinates giving the center and the last coordinate giving the radius. The relation $b_1 \prec b_2$ holds if and only if the closed ball b_2 is contained in the interior of the ball b_1. This yields a recursive abstract basis, and the space of maximal points with the relative Scott topology is easily seen to be $\mathbb{R}^n$.

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