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ON CELLULARITY IN HOMOMORPHIC IMAGES OF BOOLEAN ALGEBRAS

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Abstract

$c_{\mathrm{Hr}}A = \{(\mu,\nu) : |A/I| = \nu \geq \omega$ and $c(A/I) = \mu$ for some ideal I of A$\}$ for A an infinite Boolean algebra. Special cases of the main results are: (1) If $(\omega_1,\omega_2) \in c_{\mathrm{Hr}}A$ and $(\omega,\omega_2) \notin c_{\mathrm{Hr}}A$, then $(\omega_1,\omega_1) \in c_{\mathrm{Hr}}A$. (2) There is a model with a BA A such that $c_{\mathrm{Hr}}A = \{(\omega,\omega), (\omega_1,\omega_1), (\omega,\omega_2), (\omega_2,\omega_2)\}$. (3) Under GCH, there is a BA A such that $c_{\mathrm{Hr}}A = \{(\omega,\omega_1), (\omega_1,\omega_1), (\omega_1,\omega_2), (\omega_2,\omega_2)\}$. (4) If $cA \geq \omega_2$ and $(\omega,\omega_2) \in c_{\mathrm{Sr}}A$, then $(\omega_1,\omega_2) \in c_{\mathrm{Sr}}A$ for the notion c_{Sr} analogous to c_{Hr}.

For any infinite Boolean algebra A, let $c_{\mathrm{Hr}}A = \{(\mu,\nu) : |A/I| = \nu \geq \omega$ and $c(A/I) = \mu$ for some ideal I of $A\}$. Here for any Boolean algebra A, cA is the *cellularity* of A, which is defined to be the supremum of the cardinalities of families of pairwise disjoint elements of A. We call c_{Hr} the *homomorphic cellularity relation of A*. In topological terms, we are dealing with compact zero-dimensional Hausdorff spaces X, with

$$c_{\mathrm{Hr}}X = \{(\mu,\nu) : \text{there is an infinite closed subspace } Y \text{ of } X \text{ with weight } \nu \text{ and cellularity } \mu\}.$$

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It is natural to try to characterize these relations in cardinal number terms. This appears to be a difficult task, but one can give various properties of the relations. We mention some known facts; see Monk [6] for references and more details.

(1) (Shapirovskiĭ, Shelah) If $(\lambda, (2^\kappa)^+) \in \mathrm{c_{Hr}}A$ for some $\lambda \leq \kappa$, then $(\omega, (2^\kappa)^+) \in \mathrm{c_{Hr}}A$.

(2) (Koszmider) If $(\kappa', \lambda') \in \mathrm{c_{Hr}}A$, κ' is not inaccessible, and $\kappa' < \mathrm{cf}|A|$, then there is a $\kappa'' \geq \kappa'$ such that $(\kappa'', |A|) \in \mathrm{c_{Hr}}A$.

(3) (Todorčević) Assuming $V = L$, for each infinite κ there is a BA A such that $\mathrm{c_{Hr}}A = \{(\lambda, \lambda) : \omega \leq \lambda \leq \kappa\} \cup \{(\kappa, \kappa^+)\}$.

(4) (Malyhin, Shapirovskiĭ) Under MA, if $|A| < 2^\omega$, then A has a countable homomorphic image (implying obvious things about $\mathrm{c_{Hr}}A$).

(5) (Koszmider) There is a model with BA's A, B, C, D having respective homomorphic cellularity relations $\{(\omega, \omega_2)\}$, $\{(\omega, \omega_1)\}$, $\{(\omega, \omega_2), (\omega_1, \omega_2)\}$, $\{(\omega, \omega_1), (\omega_1, \omega_1)\}$.

In this paper we give some more properties of these relations.

(6) If $(\omega_1, \omega_2) \in \mathrm{c_{Hr}}A$ and $(\omega, \omega_2) \notin \mathrm{c_{Hr}}A$, then $(\omega_1, \omega_1) \in \mathrm{c_{Hr}}A$. This was mentioned without proof in Monk [6]. We prove a generalization of this to higher cardinalities.

(7) There is a model with a BA A such that $\mathrm{c_{Hr}}A = \{(\omega, \omega), (\omega_1, \omega_1), (\omega, \omega_2), (\omega_2, \omega_2)\}$. This was also mentioned without proof in Monk [6]. The model is a standard one used to adjoin a big maximal almost disjoint family of sets of integers, and we give the construction of that model, and a property it has that is crucial for this application, in a general form.

(8) Under CH, there is a BA A such that $\mathrm{c_{Hr}}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_2, \omega_2)\}$. This solves problem 8(i) of Monk [6] positively. This BA is the algebra of countable and cocountable subsets of ω_2, and we describe $\mathrm{c_{Hr}}$ for algebras $\langle [\kappa]^{\leq \rho} \rangle$ in general, in ZFC.

(9) Under GCH, there is a BA A such that $c_{\mathrm{Hr}}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}$. This solves problem 8(i) of Monk [6] positively. The BA is obtained from one of the previous algebras by adjoining a family of almost disjoint sets.

There is an analogous notion for subalgebras: $c_{\mathrm{Sr}}A = \{(\mu, \nu) :$ A has a subalgebra of size $\nu \geq \omega$ and cellularity $\mu\}$. Concerning this notion we give one result, a special case of which is

(10) If $cA \geq \omega_2$ and $(\omega, \omega_2) \in c_{\mathrm{Sr}}A$, then $(\omega_1, \omega_2) \in c_{\mathrm{Sr}}A$. This solves problem 4 of Monk [6] negatively.

Results about the relations $c_{\mathrm{Hr}}A$ and $c_{\mathrm{Sr}}A$ are described thoroughly in Monk [6]. In particular, the situation for algebras of size at most ω_2 is thoroughly discussed. After the results in the present paper, there remain six natural open problems, which can be concisely described as follows:

(1) Can one prove in ZFC that there is a BA A such that
$c_{\mathrm{Hr}}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$?
It is consistent that such a BA exists.

(2) Can one prove in ZFC that there is a BA A such that
$c_{\mathrm{Hr}}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}$?
Again it is consistent that such a BA exists.

(3) Is it consistent that there is a BA A such that
$c_{\mathrm{Hr}}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$?
It is consistent that no such BA exists.

(4) Is it consistent that there is a BA A such that
$c_{\mathrm{Hr}}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega, \omega_2), (\omega_2, \omega_2)\}$?
It is consistent that no such BA exists.

(5) Can one prove in ZFC that there is a BA A such that
$c_{\mathrm{Sr}}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2)\}$?
It is consistent that such a BA exists.

(6) Can one prove in ZFC that there is a BA A such that
$c_{\mathrm{Sr}}A = \{(\omega, \omega), (\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}$?

It is consistent that such a BA exists.

Notation. For set theory, we follow Kunen [5], with the following changes and additions. If $f : A \to B$ and $X \subseteq A$, then the f-image of X is denoted by $f[X]$. A family of sets $\mathscr{A}$ is *almost disjoint* if $|X \cap Y| < |X|, |Y|$ for any two distinct $X, Y \in \mathscr{A}$; it is *μ-almost disjoint* or *μ-ad* if the intersection of any two distinct members has size less than μ. A subset X of a set A is called *co-κ* if $|A \backslash X| < \kappa$.

For any topological space X, the collection of all closed and open subsets of X is denoted by clop$X$.

For Boolean algebras we follow Koppelberg [4]. If I is an ideal in a BA A and $x \in I$, then $[x]_I$ is the equivalence class of x under the equivalence relation determined by I. The subalgebra of A generated by X is denoted by $\langle X\rangle_A$, or simply $\langle X\rangle$ if A is clear. The free algebra on κ free generators is denoted by Frκ. The algebra of finite and cofinite subsets of a cardinal κ is denoted by Fincoκ. The completion of an algebra A is denoted by $\overline{A}$. We need a slight generalization of a result of Juhász and Shelah [2]; their result corresponds to successor λ in Theorem 2.

Let $\prec$ be a binary relation on a set X, and let τ and μ be infinite cardinal numbers. For any subset a of X and any $x \in X$, let $\text{Pred}_a x = \{y \in a : y \prec x\}$. We say that $\prec$ is *$(<\tau)$-full* if for every $a \in [X]^{<\tau}$ there is an $x \in X$ such that $a = \text{Pred}_a x$. And we say that $\prec$ is *μ-local* if for every $x \in X$ we have $|\text{Pred}_X x| \leq \mu$.

Lemma 1. *Let $\prec$ be a binary relation on an infinite cardinal ρ that is both $(<\tau)$-full and μ-local. Then for every $\sigma < \tau$ and every almost disjoint family $\mathscr{A} \subseteq [\rho]^\sigma$ we have $|\mathscr{A}| \leq \rho \cdot \mu^{<\tau}$.*

Proof. Since $\prec$ is $(<\tau)$-full, for every $a \in \mathscr{A}$ there is a $\xi_a < \rho$ such that $a = \text{Pred}_a \xi_a$. Thus $a \in [\text{Pred}_\rho \xi_a]^{<\tau}$. So $\mathscr{A} \subseteq \bigcup_{\xi<\rho} [\text{Pred}_\rho \xi]^{<\tau}$, and the latter has size at most $\rho \cdot \mu^{<\tau}$. □

Theorem 2. *Suppose that κ and λ are infinite cardinals, $\lambda \leq$*

κ^+, λ regular. Let f be a homomorphism from $\langle[\kappa]^{<\lambda}\rangle_{\mathscr{P}\kappa}$ onto an infinite BA B. Then $|B| < 2^{<\lambda}$ or $|B|^{<\lambda} = |B|$.

Proof. Let $\rho = |B|$ and $C = f[[\kappa]^{<\lambda}]$. Thus $|C| = \rho$ too. Suppose that $2^{<\lambda} \leq \rho$.

(1) $\leq_B$ restricted to C is $(<\lambda)$-full.

For, suppose that $a \subseteq C$ and $|a| < \lambda$. Then there is an $x \in [[\kappa]^{<\lambda}]^{<\lambda}$ such that $a = f[x]$. Since λ is regular, also $b \stackrel{\text{def}}{=} \bigcup x \in [\kappa]^{<\lambda}$, so $f(b) \in C$. Now $a \subseteq \text{Pred}_C f(b)$. For, if $u \in a$, say $u = f(c)$ with $c \in x$. Then $c \subseteq b$, so $f(c) \leq f(b)$. Hence $a = \{y \in a : y \leq f(b)\}$, and (1) follows.

(2) $\leq_B$ restricted to C is $2^{<\lambda}$-local.

In fact, suppose that $c \in C$; say $c = f(x)$ with $x \in [\kappa]^{<\lambda}$. If $b \in C$ and $b \leq c$, say $b = f(y)$ with $y \in [\kappa]^{<\lambda}$. Then $f(y \cap x) = f(y) \cap f(x) = b$. Thus $b \in f[\mathscr{P}x]$; and $|\mathscr{P}x| \leq 2^{<\lambda}$, as desired in (2).

Now by lemma 1 we have

(3) For every $\tau < \lambda$, and every almost disjoint $\mathscr{A} \subseteq [\rho]^\tau$ we have $|\mathscr{A}| \leq \rho \cdot (2^{<\lambda})^{<\lambda} = \rho$.

Now we are ready to show that $\rho^{<\lambda} = \rho$. For, suppose that $\rho^{<\lambda} > \rho$. Since $\lambda \leq \rho$, it follows that $\rho^\tau > \rho$ for some $\tau < \lambda$; let τ be minimum with this property. Then by a well-known argument, there is an almost disjoint $\mathscr{A} \subseteq [\rho]^\tau$ of size ρ^τ. This contradicts (3). □

Lemma 3. *Suppose that κ and λ are cardinals, $\omega \leq \lambda \leq \kappa^+$, λ regular. Let $A = \langle[\kappa]^{<\lambda}\rangle_{\mathscr{P}\kappa}$. Let I be an ideal on A, and assume that $|A/I| > 2^{<\lambda}$. Then*

(i) $\forall a \in I(|a| < \lambda)$.

(ii) Suppose that $\mathscr{A} \subseteq A$, $\forall a \in \mathscr{A}(|a| < \lambda)$, $\langle[a]_I : a \in \mathscr{A}\rangle$ is pairwise disjoint, and $\mathscr{A}$ is maximal with these properties. Then $\sum_{a\in\mathscr{A}}[a]_I = 1$.

(iii) Continuing (ii), $|A/I| \leq |\bigcup\mathscr{A}|^{<\lambda}$.

(iv) $|A/I| \leq \mathrm{c}(A/I)^{<\lambda}$.
(v) $2^{<\lambda} < \mathrm{c}(A/I)$.

Proof. For (i), suppose that $a \in I$ and $|-a| < \lambda$. Then the mapping $x \mapsto [x]_I$ for $x \subseteq -a$ is a homomorphism from $\mathscr{P}(-a)$ onto A/I. But $|\mathscr{P}(-a)| \leq 2^{<\lambda}$, contradicting $|A/I| > 2^{<\lambda}$.

For (ii), suppose not: say $[b]_I \neq 0$, while $[b]_I \cdot [a]_I = 0$ for all $a \in \mathscr{A}$. Then for all $c \in [b]^{<\lambda}$ we have $[c]_I = 0$. Hence $|b| \geq \lambda$, so $|-b| < \lambda$. So $[c]_I = [c \backslash b]_I$ for all $c \in [\kappa]^{<\lambda}$. Hence $\{[c]_I : c \in [\kappa]^{<\lambda}\} = \{[c]_I : c \in [-b]^{<\lambda}\}$ has size at most $\mu^{<\lambda}$, where $\mu = |-b|$. And $\mu < \lambda$, so $\mu^{<\lambda} \leq 2^{<\lambda}$. Hence $|A/I| \leq 2^{<\lambda}$, contradiction.

For (iii), note that if $b \in [\kappa \backslash \bigcup \mathscr{A}]^{<\lambda}$, then $b \in I$ by the maximality of $\mathscr{A}$. So

$$\{[b]_I : b \in A, |b| < \lambda\} = \{[b \cap \bigcup \mathscr{A}]_I : |b| < \lambda\},$$

so (iii) holds.

For (iv), note that if $\mathrm{c}(A/I) < \lambda$, then $|\bigcup \mathscr{A}| < \lambda$ by regularity of λ, and so $|\bigcup \mathscr{A}|^{<\lambda} \leq 2^{<\lambda}$, and (iii) gives a contradiction. So $\lambda \leq \mathrm{c}(A/I)$. Hence $|\bigcup \mathscr{A}| \leq \mathrm{c}(A/I)$. Then (iii) yields (iv).

Finally, (v) follows from (iv) and the hypothesis. □

Theorem 4. *Suppose that* $\omega \leq \rho \leq \kappa$. *Let* $A = \langle [\kappa]^{\leq \rho} \rangle_{\mathscr{P}\kappa}$. *Then* $\mathrm{c}_{\mathrm{Hr}}(A) = S \cup T \cup U$, *where*

$$S = \{(\mu, \nu) : \omega \leq \mu \leq \nu \leq 2^\rho, \nu^\omega = \nu\};$$
$$T = \{(\mu, \mu^\rho) : 2^\rho < \mu \leq \kappa\};$$
$$U = \{(\mu, \kappa^\rho) : 2^\rho < \mu, \mu^\rho = \kappa^\rho, \kappa < \mu\}.$$

Proof. First suppose that $(\mu, \nu) \in S$. The mapping $a \mapsto a \cap \rho$ gives a homomorphism of A onto $\mathscr{P}\rho$. Since $\mathscr{P}\rho$ has an independent subset of size 2^ρ, there is a homomorphism of $\mathscr{P}\rho$ onto an algebra B such that $\mathrm{Fr}\nu \leq B \leq \overline{\mathrm{Fr}\nu}$. Since $\nu^\omega = \nu$, we have $|B| = \nu$. Now there is a homomorphism of B onto an algebra C such that $\mathrm{Fr}\nu \times \mathrm{Finco}\mu \leq C \leq \overline{\mathrm{Fr}\nu \times \mathrm{Finco}\mu}$. Thus $|C| = \nu$ and $\mathrm{c}(C) = \mu$, so $(\mu, \nu) \in \mathrm{c_{Hr}}(A)$.

Second, suppose that $2^\rho < \mu \leq \kappa$. The mapping $a \mapsto a \cap \mu$ gives a homomorphism of A onto $\langle [\mu]^{\leq\rho} \rangle$, which has size μ^ρ and cellularity μ. So $(\mu, \mu^\rho) \in \mathrm{c_{Hr}}(A)$.

Third, suppose that $2^\rho < \mu$, $\mu^\rho = \kappa^\rho$, and $\kappa < \mu$. Note that $2^\rho < \kappa$, for if $\kappa \leq 2^\rho$ then $\kappa^\rho \leq 2^\rho \leq \kappa^\rho$, so $\kappa^\rho = 2^\rho < \mu \leq \mu^\rho = \kappa^\rho$, contradiction. Now let ν be minimum such that $\kappa \leq \nu^\rho$. Since $2^\rho < \kappa$ and $\kappa < \kappa^\rho$, it follows from Jech [1], Theorem 19, that $\mathrm{cf}\nu \leq \rho < \nu$ and $\kappa^\rho = \nu^{\mathrm{cf}\nu}$. Now if $\sigma < \mathrm{cf}\nu$, then $\nu^\sigma \leq \kappa$, for

$$\nu^\sigma = |{}^\sigma\nu| = \left|\bigcup_{\delta<\nu} {}^\sigma\delta\right| \leq \sum_{\delta<\nu} |{}^\sigma\delta| \leq \kappa.$$

Hence $\left|\bigcup_{\sigma<\mathrm{cf}\nu} {}^\sigma\nu\right| \leq \kappa$, so there is an $\mathscr{A} \subseteq [\kappa]^{\mathrm{cf}\nu}$ which is $\mathrm{cf}\nu$-ad and of size $\nu^{\mathrm{cf}\nu} = \kappa^\rho$. Let $I = [\kappa]^{<\mathrm{cf}\nu}$. Then $\langle [a]_I : a \in \mathscr{A} \rangle$ is isomorphic to $\mathrm{Finco}(\kappa^\rho)$. Hence there is a homomorphism of $\langle [a]_I : a \in \mathscr{A} \rangle$ onto $\mathrm{Finco}\mu$. By the Sikorski extension theorem we get a homomorphism h of A onto a BA B with $\mathrm{Finco}\mu \leq B \leq \overline{\mathrm{Finco}\mu}$. Thus $\mathrm{c}(B) = \mu$, and by Theorem 2, $|B|^\rho = |B|$. Since $\kappa < \mu \leq |B|$, it follows that $\kappa^\rho \leq |B|^\rho = |B| \leq \kappa^\rho$. So $|B| = \kappa^\rho$. Thus $(\mu, \kappa^\rho) \in \mathrm{c_{Hr}}(A)$.

Finally, suppose conversely that $(\mu, \nu) \in \mathrm{c_{Hr}}(A)$. Since A is σ-complete, it is well-known that $\nu^\omega = \nu$. So if $\nu \leq 2^\rho$, then $(\mu, \nu) \in S$. Suppose that $2^\rho < \nu$. By Theorem 2 $\nu^\rho = \nu$, and by Lemma 3, $2^\rho < \mu$ and $\nu \leq \mu^\rho$. Hence $\mu^\rho \leq \nu^\rho \leq \mu^\rho$, so $\nu = \nu^\rho = \mu^\rho$. If $\mu \leq \kappa$, then $(\mu, \nu) \in T$. Suppose that $\kappa < \mu$. Then $\kappa^\rho \leq \mu^\rho = \nu \leq \kappa^\rho$, so $\nu = \kappa^\rho$ and $(\mu, \nu) \in U$. □

Theorem 4 provides a positive solution of Problem 8(i) of Monk [6]. Namely, assume CH and let $\kappa = \omega_2$ and $\rho = \omega$ in the theorem. Thus with $A = \langle [\omega_2[^{\leq\omega} \rangle_{\mathscr{P}\omega_2}$, under CH we have

$$\mathrm{c_{Hr}}A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_2, \omega_2)\}.$$

Under GCH, there is a simpler description of $\langle [\kappa]^{\leq\rho} \rangle_{\mathscr{P}\kappa}$:

Corollary 5. (GCH) *Suppose that* $\omega \leq \rho \leq \kappa$. *Let* $A = \langle[\kappa]^{\leq\rho}\rangle_{\mathscr{P}\kappa}$. *Then*

$$\begin{aligned}\mathrm{c_{Hr}}A = &\{(\mu,\nu) : \omega \leq \mu \leq \nu \leq \rho^+, \mathrm{cf}\nu > \omega\}\\ &\cup \{(\mu,\mu) : \rho^+ < \mu, \rho < \mathrm{cf}\mu, \mu \leq \kappa\}\\ &\cup \{(\mu,\mu^+) : \rho^+ < \mu, \mathrm{cf}\mu \leq \rho, \mu \leq \kappa\}\\ &\cup \{(\kappa^+,\kappa^+) : \mathrm{cf}\kappa \leq \rho < \kappa\}.\end{aligned}$$ □

It is natural to also consider the algebra $A = \langle[\kappa]^{<\lambda}\rangle$ for λ limit. For λ singular the situation is unclear. Note that if $\mathrm{cf}\lambda = \omega$, it is possible that A has a countable homomorphic image. For example, let $\kappa = \lambda = \aleph_\omega$. For each $n \in \omega$ let F_n be an ultrafilter on the Boolean algebra $\mathscr{P}\aleph_n$ such that $X \in F_n$ for every $X \subseteq \aleph_n$ for which $|\aleph_n \backslash X| < \aleph_n$. Define $f(a) = \{n \in \omega : a \cap \aleph_n \in F_n\}$ for every $a \in A$. It is easy to see that f is a homomorphism from A onto $\mathrm{Finco}\omega$.

For λ regular limit (meaning that it is weakly inaccessible), we can give a complete description of the cellularity homomorphism relation. For this we need another lemma. This lemma is proved like Lemma 3.

Lemma 6. *Suppose that* κ *and* λ *are cardinals,* λ *is weakly inaccessible,* $2^\mu < 2^{<\lambda}$ *for all* $\mu < \lambda$, *and* $\lambda \leq \kappa$. *Let* $A = \langle[\kappa]^{<\lambda}\rangle_{\mathscr{P}\kappa}$. *Let* I *be an ideal on* A, *and assume that* $|A/I| = 2^{<\lambda}$. *Then*

(i) $\forall a \in I(|a| < \lambda)$.

(ii) Suppose that $\mathscr{A} \subseteq A$, $\forall a \in \mathscr{A}(|a| < \lambda)$, $\langle[a]_I : a \in \mathscr{A}\rangle$ *is pairwise disjoint, and* $\mathscr{A}$ *is maximal with these properties. Then* $\sum_{a\in\mathscr{A}}[a]_I = 1$.

(iii) Continuing (ii), $|A/I| \leq \left|[\bigcup\mathscr{A}]^{<\lambda}\right|$.

(iv) $\mathrm{c}(A/I) \geq \lambda$.

Proof. Only (iv) requires additional scrutiny. If $\mathrm{c}(A/I) < \lambda$, then $|\mathscr{A}| < \lambda$, so by the regularity of λ, $|\bigcup\mathscr{A}| < \lambda$. But then $\left|[\bigcup\mathscr{A}]^{<\lambda}\right| = |\mathscr{P}(\bigcup\mathscr{A})| < 2^{<\lambda}$, contradiction. □

Theorem 7. *Suppose that λ is uncountable and weakly inaccessible and $\lambda \leq \kappa$. Let $A = \langle [\kappa]^{<\lambda} \rangle_{\mathscr{P}\kappa}$. Define*

$$\begin{aligned} S &= \{(\mu, \nu) : \omega \leq \mu \leq \nu < 2^{<\lambda}, \nu^\omega = \nu\}; \\ T &= \{(\mu, \mu^{<\lambda}) : 2^{<\lambda} < \mu \leq \kappa\}; \\ U &= \{(\mu, \kappa^{<\lambda}) : 2^{<\lambda} < \mu, \mu^{<\lambda} = \kappa^{<\lambda}, \kappa < \mu\}; \\ V &= \{(\mu, 2^{<\lambda}) : \omega \leq \mu \leq 2^{<\lambda}\}; \\ W &= \{(\mu, 2^{<\lambda}) : \lambda \leq \mu \leq 2^{<\lambda}\}. \end{aligned}$$

Then

(i) If $2^\rho = 2^{<\lambda}$ for some $\rho < \lambda$, then $\mathrm{c_{Hr}}(A) = S \cup T \cup U \cup V$;

(ii) If $2^\rho < 2^{<\lambda}$ for all $\rho < \lambda$, then $\mathrm{c_{Hr}}(A) = S \cup T \cup U \cup W$.

(iii) If λ is strongly inaccessible, then $\mathrm{c_{Hr}}(A) = S \cup T \cup U \cup \{(\lambda, \lambda)\}$.

Proof. The proof that $S \cup T \cup U \subseteq \mathrm{c_{Hr}}(A) \subseteq S \cup T \cup U \cup V$ is very similar to the proof for Theorem 4. For example, to show that $U \subseteq \mathrm{c_{Hr}}(A)$, take μ such that $2^{<\lambda} < \mu$, $\mu^{<\lambda} = \kappa^{<\lambda}$, and $\kappa < \mu$. Then $2^{<\lambda} < \kappa$ by an argument like that in the proof of Theorem 4. Since $\kappa < \mu \leq \kappa^{<\lambda}$, choose ρ so that $\kappa < \kappa^\rho$ and $\rho < \lambda$, and then proceed as before.

Now suppose that $\rho < \lambda$ and $2^\rho = 2^{<\lambda}$. The mapping $a \mapsto a \cap \rho$ gives a homomorphism from A onto $\mathscr{P}\rho$. Then the argument at the beginning of the proof of Theorem 4 shows that $(\mu, 2^{<\lambda}) \in \mathrm{c_{Hr}}(A)$ for all $\mu \in [\omega, 2^{<\lambda}]$. This proves (i).

Next, suppose that $2^\rho < 2^{<\lambda}$ for all $\rho < \lambda$, and that $\lambda \leq \mu < 2^{<\lambda}$. Then there is a $\rho < \lambda$ such that $\mu < 2^\rho$. Write $\lambda = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$, $|\Gamma_0| = \lambda$, and $|\Gamma_1| = \rho$. By Theorem 4 there is a homomorphism f of $\mathscr{P}\Gamma_1$ onto an algebra of size 2^ρ and cellularity μ. Let $g(a) = (a \cap \Gamma_0, f(a \cap \Gamma_1))$ for all $a \in A$. The image of g has size $2^{<\lambda}$ and cellularity μ.

To get a homomorphic image of size and cellularity $2^{<\lambda}$ we have to modify this argument. Let M be the set of all infinite cardinals less than λ, and let $\langle \Gamma_\alpha : \alpha \in M \rangle$ be a partition of λ with $|\Gamma_\alpha| = \alpha$ for all $\alpha \in M$. For each $\alpha \in M$ let f_α be a homomorphism of $\mathscr{P}\Gamma_\alpha$ onto an algebra of size and cellularity 2^α. Then let $g(a)_\alpha = f_\alpha(a \cap \Gamma_\alpha)$ for all $a \in A$. Then the image of g is as desired.

That no other pairs are in $c_{\mathrm{Hr}}(A)$ follows from Lemma 6. Thus (ii) holds.

(iii) is a clear consequence of (ii). □

For the next result we need a standard Boolean algebraic fact:

Proposition 8. *Suppose that A is κ-complete, and I is a κ-complete maximal ideal in A. Suppose that $f : I \longrightarrow B$ preserves $(<\kappa)$-joins, $(<\kappa)$-meets, and 0. Then f extends to a unique κ-complete homomorphism $f^+ : A \longrightarrow B$. Moreover, f^+ is one-one iff $\forall x \in I[f(x) = 0 \Rightarrow x = 0]$ and $\forall x \in I[f(x) \neq 1]$.*

Proof. The following definition of f^+ is forced upon us:

$$f^+(a) = \begin{cases} f(a) & \text{if } a \in I, \\ -f(-a) & \text{if } a \notin I. \end{cases}$$

Then f^+ preserves $-$, since if $a \in I$, then $f^+(-a) = -f(a) = -f^+(a)$, and if $a \notin I$, then $f^+(-a) = f(-a) = --f(-a) = -f^+(a)$.

Now we show that f^+ preserves $(<\kappa)$-joins. So, let $\sum_{\xi<\alpha} a_\xi$ be given, with $\alpha < \kappa$. If $\forall \xi < \alpha[a_\xi \in I]$, then

$$f^+\left(\sum_{\xi<\alpha} a_\xi\right) = f\left(\sum_{\xi<\alpha} a_\xi\right) = \sum_{\xi<\alpha} f(a_\xi) = \sum_{\xi<\alpha} f^+(a_\xi).$$

Now suppose that $\exists \xi < \alpha[a_\xi \notin I]$. Let $\Gamma = \{\xi < \alpha : a_\xi \in I\}$.

Then

$$\sum_{\xi\in\Gamma} a_\xi + \left(-\sum_{\xi<\alpha} a_\xi\right) = \sum_{\xi\in\Gamma} a_\xi + \left(-\left(\sum_{\xi\in\Gamma} a_\xi + \sum_{\xi\in\alpha\setminus\Gamma} a_\xi\right)\right)$$
$$= \sum_{\xi\in\Gamma} a_\xi + \left(-\sum_{\xi\in\Gamma} a_\xi \cdot - \sum_{\xi\in\alpha\setminus\Gamma} a_\xi\right)$$
$$= \sum_{\xi\in\Gamma} a_\xi + \left(-\sum_{\xi\in\alpha\setminus\Gamma} a_\xi\right).$$

Using this,

$$\sum_{\xi<\alpha} f^+(a_\xi) + \left(-f^+\left(\sum_{\xi<\alpha} a_\xi\right)\right)$$
$$= \sum_{\xi\in\Gamma} f(\alpha_\xi) + \sum_{\xi\in\alpha\setminus\Gamma} -f(-a_\xi) + f\left(-\sum_{\xi<\alpha} a_\xi\right)$$
$$= f\left(\sum_{\xi\in\Gamma} a_\xi + \left(-\sum_{\xi<\alpha} a_\xi\right)\right) + \sum_{\xi\in\alpha\setminus\Gamma} -f(-a_\xi)$$
$$= f\left(\sum_{\xi\in\Gamma} a_\xi + \left(-\sum_{\xi\in\alpha\setminus\Gamma} a_\xi\right)\right) + \sum_{\xi\in\alpha\setminus\Gamma} -f(-a_\xi)$$

$$= f\left(\sum_{\xi\in\Gamma} a_\xi\right) + f\left(-\sum_{\xi\in\alpha\setminus\Gamma} a_\xi\right) + \sum_{\xi\in\alpha\setminus\Gamma} -f(-a_\xi)$$
$$= f\left(\sum_{\xi\in\Gamma} a_\xi\right) + \prod_{\xi\in\alpha\setminus\Gamma} f(-a_\xi) + \left(-\prod_{\xi\in\alpha\setminus\Gamma} f(-a_\xi)\right)$$
$$= 1.$$

And if $\xi \in \Gamma$, then

$$\begin{aligned} f^+(a_\xi) \cdot -f^+\left(\sum_{\eta<\alpha} a_\eta\right) &= f(a_\xi) \cdot f\left(-\sum_{\eta<\alpha} a_\eta\right) \\ &= f\left(a_\xi \cdot -\sum_{\eta<\alpha} a_\eta\right) \\ &= f(0) = 0. \end{aligned}$$

If $\xi \in \alpha \backslash \Gamma$, then

$$f^+(a_\xi) \cdot -f^+\left(\sum_{\eta<\alpha} a_\eta\right) = -f(-a_\xi) \cdot f\left(-\sum_{\eta<\alpha} a_\eta\right).$$

Now $a_\xi \leq \sum_{\eta<\alpha} a_\eta$, so $-\sum_{\eta<\alpha} a_\eta \leq -a_\xi$, hence $f\left(-\sum_{\eta<\alpha} a_\eta\right) \leq f(-a_\xi)$, so $-f(-a_\xi) \cdot f\left(-\sum_{\eta<\alpha} a_\eta\right) = 0$. So we have proved that $f^+\left(\sum_{\xi<\alpha} a_\xi\right) = \sum_{\xi<\alpha} f(a_\xi)$. So f is a κ-homomorphism.

Concerning the final statement, the direction $\Rightarrow$ is clear. Now suppose the indicated condition holds, and $f^+(a) = 0$. If $a \in I$, then $f(a) = f^+(a) = 0$, so $a = 0$. If $a \notin I$, then $f^+(a) = -f(-a) = 0$, so $f(-a) = 1$ and $-a \in I$, contradiction. □

Lemma 9. *Suppose that $\kappa < \lambda$, κ is regular, $\mathscr{A} \subseteq [\kappa]^\kappa$ is almost disjoint, and $|\mathscr{A}| = \lambda$. Let A be the κ-complete subalgebra of $\mathscr{P}\kappa$ generated by $\mathscr{A} \cup \{\{\xi\} : \xi < \kappa\}$. Then $A/[\kappa]^{<\kappa} \cong \langle[\lambda]^{<\kappa}\rangle_{\mathscr{P}\lambda}$.*

Proof. Let $\langle X_\alpha : \alpha < \lambda\rangle$ be a one-one enumeration of $\mathscr{A}$. Set $I = [\kappa]^{<\kappa}$. For each $\Gamma \in [\lambda]^{<\kappa}$ let $f(\Gamma) = \left[\bigcup_{\alpha\in\Gamma} X_\alpha\right]_I$. Clearly f preserves $(<\kappa)$-joins, and $f(0) = 0$. It also preserves $(<\kappa)$-meets. For, suppose that $\Gamma_\alpha \in [\lambda]^{<\kappa}$ for all $\alpha < \gamma$, where $\gamma < \kappa$. Let $\Delta = \bigcup_{\alpha<\gamma} \Gamma_\alpha$. So $|\Delta| < \kappa$ since κ is regular. Let

P be the set of all nonconstant $g \in \prod_{\alpha<\gamma} \Gamma_\alpha$. Then

$$\bigcap_{\alpha<\gamma} \bigcup_{\xi\in\Gamma_\alpha} X_\xi = \bigcup_{g\in\prod_{\alpha<\gamma}\Gamma_\alpha} \bigcap_{\alpha<\gamma} X_{g(\alpha)}$$
$$= \bigcup_{\xi\in\bigcap_{\alpha<\gamma}\Gamma_\alpha} X_\xi \cup \bigcup_{g\in P} \bigcap_{\alpha<\gamma} X_{g(\alpha)}.$$

Now

$$\bigcup_{g\in P} \bigcap_{\alpha<\gamma} X_{g(\alpha)} \subseteq \bigcup\{X_\alpha \cap X_\beta : \alpha, \beta \in \Delta, \alpha \neq \beta\},$$

and the latter set has size less than κ. This shows that f preserves $(<\kappa)$-meets.

Hence by Proposition 8, f extends to a κ-homomorphism from $\langle[\lambda]^{<\kappa}\rangle_{\mathscr{P}\lambda}$ into A/I. by the same proposition it is clear that f is one-one. Since $f[[\lambda]^{<\kappa}]$ generates A/I as a κ-complete algebra, f maps onto A/I. □

Theorem 10. (GCH) *Let $\mathscr{A} \subseteq [\kappa^+]^{\kappa^+}$ be κ^+-ad, with $|\mathscr{A}| = \kappa^{++}$. Let A be the κ^+-complete subalgebra of $\mathscr{P}\kappa^+$ generated by $\mathscr{A} \cup \{\{\alpha\} : \alpha < \kappa^+\}$. Then*

$c_{\mathrm{Hr}}A = \{(\mu,\nu) : \omega \leq \mu \leq \nu \leq \kappa^+, \mathrm{cf}\nu > \omega\} \cup \{(\kappa^+,\kappa^{++}),(\kappa^{++},\kappa^{++})\}$.

Proof. Let $\langle X_\alpha : \alpha < \kappa^{++}\rangle$ be a one-one enumeration of $\mathscr{A}$. Let $I = [\kappa^+]^{\leq\kappa}$. Then by Lemma 9,

(1) $A/I \cong \langle[\kappa^{++}]^{\leq\kappa}\rangle_{\mathscr{P}\kappa^{++}}$.

Hence by Corollary 5, $c_{\mathrm{Hr}}A$ contains the set of the theorem. Suppose that $(\mu,\nu) \in c_{\mathrm{Hr}}A$, with (μ,ν) not in the indicated set. Then $\nu = \kappa^{++}$ and $\mu \leq \kappa$. So A has an independent subset $\mathscr{F}$ of size κ^{++}. Since $|I| = \kappa^+$, we may assume that the members of $\mathscr{F}$ are pairwise inequivalent modulo I, each nonzero modulo I. By the proof of (1), for each $a \in \mathscr{F}$ we can choose a $\Gamma_a \in [\kappa^{++}]^\kappa$ such that $[a]_I = \left[\bigcup_{\alpha\in\Gamma_a} X_\alpha\right]_I$. Then there is a $\Delta \in [\mathscr{F}]^{\kappa^{++}}$ such that $\langle \Gamma_a : a \in \Delta \rangle$ is a Δ-system. Let a, b, c be distinct members of Δ. Then $[a \cdot b \cdot -c]_I = 0$, i.e., $|a \cdot b \cdot -c| \leq \kappa$. Hence

$$\langle a \cdot b \cdot -c \cdot d : d \in \Delta \backslash \{a, b, c\} \rangle$$

is a system of κ^{++} independent subsets of $a \cdot b \cdot -c$, which contradicts GCH. □

Taking $\kappa = \omega$ in this theorem we get, under GCH, a BA A such that

$$\mathrm{c_{Hr}} A = \{(\omega, \omega_1), (\omega_1, \omega_1), (\omega_1, \omega_2), (\omega_2, \omega_2)\}.$$

This solves Problem 8(iii) of Monk [6] positively.

For the next result we need a fact about one of the standard ways of forcing a large mad family. This fact was observed by Richard Laver, and we thank him for allowing us to include the proof of the fact here.

Theorem 11. *In a model of ZFC+GCH, suppose that κ and λ are infinite cardinals, κ regular, $\kappa < \lambda$. Then there is an extension preserving cofinalities and cardinalities in which there is a system $\langle A_\alpha : \alpha < \lambda \rangle$ of almost disjoint members of $[\kappa]^\kappa$ with the following property:*

(*) if $X \in [\kappa]^\kappa$ and $|X \cap A_\alpha| = \kappa$ for κ many $\alpha < \lambda$, then $|X \cap A_\alpha| = \kappa$ for co-κ^+ many $\alpha < \lambda$.

Proof. Let $\mathbb{P}$ be the set of all functions f such that there exist an $F \in [\lambda]^{<\kappa}$ and a $\nu < \kappa$ such that $f : F \times \nu \to 2$. For $f \in \mathbb{P}$ we let F_f and ν_f be the F, ν mentioned, with $F_f = 0 = \nu_f$ if $f = 0$. We write $f \leq g$ iff $g \subseteq f$ and for any distinct $\alpha, \beta \in F_g$ and any $\iota \in \nu_f \backslash \nu_g$, $f(\alpha, \iota) = 0$ or $f(\beta, \iota) = 0$. Clearly

(1) $(\mathbb{P}, \leq)$ is κ-closed and satisfies the κ^+-chain condition. Consequently, forcing with $(\mathbb{P}, \leq)$ preserves cofinalities and cardinals.

(2) For any $\alpha < \lambda$, the set $\{f \in \mathbb{P} : \alpha \in F_f\}$ is dense.

In fact, given $g \in \mathbb{P}$, if $\alpha \notin F_g$, let $F_f = F_g \cup \{\alpha\}$, $\nu_f = \nu_g$, and let f extend g with $f(\alpha, \iota) = 0$ for all $\iota < \nu_g$. Clearly this proves (2).

Now let G be generic for $(\mathbb{P}, \leq)$ over the ground model. We then set, for any $\alpha < \lambda$,

$$A_\alpha = \{\iota < \kappa : \exists g \in G(\alpha \in F_g, \iota < \nu_g, g(\alpha, \iota) = 1)\}$$
$$\Gamma_\alpha = \{(\hat{\iota}, g) : \alpha \in F_g, \iota < \nu_g, g(\alpha, \iota) = 1\}.$$

Thus $\Gamma_\alpha^G = A_\alpha$.

(3) For each $\alpha < \lambda$, $|A_\alpha| = \kappa$.

In fact, it suffices to show that for any $\mu < \kappa$ the following set is dense:

$$\{g \in \mathbb{P} : \alpha \in F_g \text{ and } \exists \xi \in \kappa \setminus \mu(\xi < \nu_g \text{ and } g(\alpha, \xi) = 1)\}.$$

To prove this, let $f \in \mathbb{P}$. By (2) we may assume that $\alpha \in F_f$. Now let $f \subseteq g$, $F_f = F_g$, $\nu_g = \max(\nu_f + 1, \mu + 2)$, $\xi = \max(\nu_f, \mu+1)$, with $g(\beta, \iota) = 0$ if $\nu_f \leq \iota$ and $\beta \neq \alpha$, $g(\alpha, \iota) = 0$ if $\iota \neq \xi$, and $g(\alpha, \xi) = 1$. Clearly $g \in \mathbb{P}$ and $g \leq f$, as desired in (3).

(4) $|A_\alpha \cap A_\beta| < \kappa$ for $\alpha \neq \beta$.

In fact, by (2) choose $g \in G$ such that $\alpha, \beta \in F_g$. Then, we claim, $A_\alpha \cap A_\beta = \{\iota < \nu_g : g(\alpha, \iota) = 1 = g(\beta, \iota)\}$, which will prove (4). Clearly $\supseteq$ holds. Now suppose that $\iota \in A_\alpha \cap A_\beta$. Then there is an $f \in G$ such that $f \leq g$, $\iota < \nu_f$ and $f(\alpha, \iota) = 1 = f(\beta, \iota)$. From the definition of $\leq$ it follows that $\iota < \nu_g$, and hence $f(\alpha, \iota) = g(\alpha, \iota)$ and $f(\beta, \iota) = g(\beta, \iota)$, as desired.

Now suppose that $X \in [\kappa]^\kappa$ and $|X \cap A_\alpha| = \kappa$ for κ many α's. Let τ be a name for X. Choose $p \in G$ so that

(5) $p \Vdash \forall H \in [\lambda]^{<\kappa}(|\tau \setminus \bigcup_{\alpha \in H} \Gamma_\alpha| = \kappa)$.

Now we claim

(6) There is a $C \in [\lambda]^{\leq\kappa}$ such that $F_p \subseteq C$ and for all q, μ, H, if $q \in \mathbb{P}$, $F_q \subseteq C$, $q \leq p$, $\mu < \kappa$, and $H \in [C]^{<\kappa}$, then there is a $q' \leq q$ such that $F_{q'} \subseteq C$ and there is a $\xi \in \kappa\backslash\mu$ such that $q' \Vdash \xi \in \tau\backslash\bigcup_{\beta\in H}\Gamma_\beta$.

For we construct $\langle C_\alpha : \alpha < \kappa\rangle$ by induction. Let $C_0 = F_p$. For α limit, let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. Now suppose that C_α has been constructed, with $|C_\alpha| \leq \kappa$. For q, μ, H such that $q \in \mathbb{P}$, $q \leq p$, $F_q \subseteq C_\alpha$, $\mu < \kappa$, and $H \in [C_\alpha]^{<\kappa}$, there exist a $q' = q'(q, \mu, H)$ and a $\xi \in \kappa\backslash\mu$ such that $q' \leq q$ and $q' \Vdash \xi \in \tau\backslash\bigcup_{\beta\in H}\Gamma_\beta$. Let

$$C_{\alpha+1} = C_\alpha \cup \bigcup\{F_{q'(q,\mu,H)} : q, \mu, H \text{ as above}\}.$$

Let $C = \bigcup_{\alpha<\kappa} C_\alpha$. Clearly C is as desired in (6).

Now take any $\alpha \in \lambda\backslash C$ and $\mu < \kappa$. We finish the proof by showing

(7) $\{q : q \Vdash \exists\xi \in \kappa\backslash\mu(\xi \in \tau \cap \Gamma_\alpha)\}$ is dense below p.

To show this, let $r \leq p$ be arbitrary. By (2), we may assume that $\alpha \in F_r$. Let $s = r \restriction (C \times \nu_r)$. By (6), choose $q' \leq s$ and $\xi > \max(\mu, \nu_r)$ such that $F_{q'} \subseteq C$ and $q' \Vdash \xi \in \tau\backslash\bigcup_{\beta\in F_s}\Gamma_\beta$. Now let $F_q = F_{q'} \cup F_r$, $\nu_q = \max(\nu_{q'}, \xi + 1)$, and for any $\beta \in F_q$ and $\iota < \nu_q$ let

$$q(\beta, \iota) = \begin{cases} q'(\beta, \iota) & \text{if } \beta \in F_{q'} \text{ and } \iota < \nu_{q'}, \\ r(\beta, \iota) & \text{if } \beta \in F_r\backslash F_{q'} \text{ and } i < \nu_r, \\ 1 & \text{if } \beta = \alpha \text{ and } \iota = \xi, \\ 0 & \text{in all other cases.} \end{cases}$$

Clearly $q \in \mathbb{P}$. Since $q(\alpha, \xi) = 1$, we have $q \Vdash \xi \in \Gamma_\alpha$.

(8) $q \leq q'$.

In fact, clearly $q' \subseteq q$. Now suppose that β and γ are distinct members of $F_{q'}$ and $\iota \in \nu_q\backslash\nu_{q'}$. Then by definition we have $q(\beta, \iota) = 0$ or $q(\gamma, \iota) = 0$, as desired; so (8) holds.

So it remains only to prove

(9) $q \leq r$.

For this, first note that $F_r = (F_r \cap C) \cup (F_r \backslash C) \subseteq F_q$. And $\nu_r \leq \nu_{q'} \leq \nu_q$. Now suppose that $\beta \in F_r$ and $\iota < \nu_r$. If $\beta \in C$, then $r(\beta, \iota) = s(\beta, \iota) = q'(\beta, \iota) = q(\beta, \iota)$. If $\beta \notin C$, then directly from the definition, $q(\beta, \iota) = r(\beta, \iota)$. All of this shows that $r \subseteq q$.

Now suppose that β and γ are distinct members of F_r and $\iota \in \nu_q \backslash \nu_r$. To finish the proof we want to show that $q(\beta, \iota) = 0$ or $q(\gamma, \iota) = 0$.

Case 1. $\beta, \gamma \in C$ and $\iota < \nu_{q'}$. Then $\beta, \gamma \in C \cap F_r = F_s \subseteq F_{q'}$, so $q(\beta, \iota) = q'(\beta, \iota)$ and $q(\gamma, \iota) = q'(\gamma, \iota)$. Also, $\iota \in \nu_{q'} \backslash \nu_s$ since $\nu_s = \nu_r$. So $q'(\beta, \iota) = 0$ or $q'(\gamma, \iota) = 0$.

Case 2. $\beta \in C$, $\iota \geq \nu_{q'}$. So $q(\beta, \iota) = 0$.

Case 3. $\gamma \in C$, $\iota \geq \nu_{q'}$. So $q(\gamma, \iota) = 0$.

Case 4. $\beta \notin C$, $\nu_r \leq \iota$, $\beta \neq \alpha$ or $\iota \neq \xi$. Then $q(\beta, \iota) = 0$.

Case 5. $\gamma \notin C$, $\nu_r \leq \iota$, $\gamma \neq \alpha$ or $\iota \neq \xi$. Then $q(\gamma, \iota) = 0$.

Case 6. $\beta \in C$, $\iota = \xi$, $\nu_r \leq \iota < \nu_{q'}$. Then $q(\beta, \iota) = q'(\beta, \xi) = 0$ since $q' \Vdash \xi \notin \Gamma_\beta$.

Case 7. $\gamma \in C$, $\iota = \xi$, $\nu_r \leq \iota < \nu_{q'}$. Then $q(\gamma, \iota) = q'(\gamma, \xi) = 0$ since $q' \Vdash \xi \notin \Gamma_\gamma$.

Case 8. None of the above. So not both of β, γ are in C, by Cases 1,2. Suppose one is in C, the other not; say $\beta \in C$, $\gamma \notin C$. Since $\iota \geq \nu_r$, it follows that $\gamma = \alpha$ and $\iota = \xi$. Then $q(\beta, \iota) = 0$, either because $\xi < \nu_{q'}$ and $q' \Vdash \xi \notin \Gamma_\beta$, or because $\xi \geq \nu_{q'}$ and the definition of q. So, suppose that $\beta, \gamma \notin C$. Then one of Cases 4,5 must hold, contradiction. □

Theorem 12. *Let $\langle A_\alpha : \alpha < \kappa \rangle$ be a system of infinite almost disjoint subsets of ω such that $\kappa > \omega$ and*

(*) For every infinite subset X of ω, if $\{\alpha < \kappa : X \cap A_\alpha\}$ is infinite, then it is cocountable.

Let A be the subalgebra of $\mathscr{P}\omega$ generated by

$$\{A_\alpha : \alpha < \kappa\} \cup \{\{i\} : i < \omega\}.$$

Then $\mathrm{c_{Hr}}A = \{(\omega, \kappa)\} \cup \{(\mu, \mu) : \omega \leq \mu \leq \kappa\}$.

Proof. $A/\text{fin} \cong \text{Finco}\kappa$, so $\supseteq$ holds. Now suppose that $(\mu,\nu) \in \mathrm{c}_{\mathrm{Hr}}A$, $\omega \le \mu < \nu \le \kappa$, and $(\mu,\nu) \neq (\omega,\kappa)$; we want to get a contradiction. Let I be an ideal of A such that $|A/I| = \nu$ and $\mathrm{c}(A/I) = \mu$. Let $b = \{i < \omega : \{i\} \in I\}$.

(1) $\Gamma \overset{\text{def}}{=} \{\alpha < \kappa : A_\alpha \backslash b \text{ is infinite}\}$ is infinite.

For, suppose that Γ is finite. Let ρ be regular, with $\mu < \rho \le \nu$; we are going to show that A/I has a disjoint family of size ρ, contradiction. Now there is a $\Delta \in [\kappa]^\rho$ such that for all $\alpha \in \Delta$, $A_\alpha/I \neq 0$ and $A_\alpha\backslash b$ is finite, and for all distinct $\alpha, \beta \in \Delta$, $A_\alpha/I \neq A_\beta/I$. Let $\Omega \in [\Delta]^\rho$ be such that $\langle A_\alpha\backslash b : \alpha \in \Omega\rangle$ is a Δ-system, say with kernel K. Now if $(A_\alpha\backslash K)/I = 0$, then $A_\alpha/I \le K/I$, and $(A/I) \restriction (K/I)$ is finite. So wlog, $(A_\alpha\backslash K)/I \neq 0$ for all $\alpha \in \Omega$. Now if α, β are distinct members of Ω, then $((A_\alpha \cap A_\beta)\backslash b)\backslash K = 0$, so $(A_\alpha \cap A_\beta)\backslash K = (A_\alpha \cap A_\beta \cap b)\backslash K$. But $A_\alpha \cap A_\beta \cap b \in I$ since $A_\alpha \cap A_\beta$ is finite, so $(A_\alpha \cap A_\beta)\backslash K \in I$. Thus $\langle (A_\alpha\backslash K)/I : \alpha \in \Omega\rangle$ is a system of ρ disjoint elements, contradiction. This proves (1).

So from (*) it follows that Γ is cocountable. Now the map $\alpha \mapsto A_\alpha\backslash b$ for $\alpha \in \Gamma$ is one-one. For any $x \in A$ let $g(x) = (x/I, x\backslash b)$. This is a homomorphism. If $x \in I$, then $x\backslash b = 0$, and so $g(x) = (0,0)$. And if $g(x) = (0,0)$, then $x \in I$. So the image of g is isomorphic to A/I. It follows that $|A/I| = \kappa$. Hence $\omega < \mu$. Let $\langle c_\alpha/I : \alpha < \omega_1\rangle$ be a system of nonzero pairwise disjoint elements. Since there are only countably many finite subsets of ω, wlog each c_α is infinite. In fact, we may assume that each c_α has the form

$$A_\beta \cdot -A_{\gamma_1} \cdot \ldots \cdot -A_{\gamma_m} \cdot -F,$$

where F is finite and each $\gamma_i \neq \beta$. This can be written as

$$A_\beta \cdot -(A_\beta \cdot A_{\gamma_1}) \cdot \ldots \cdot -(A_\beta \cdot A_{\gamma_m}) \cdot -F,$$

and each $A_\beta \cdot A_{\gamma_i}$ is finite. So wlog $m = 0$. Thus we may assume that we have a pairwise disjoint system $\langle (A_\alpha \cdot -F_\alpha)/I : \alpha \in \Delta\rangle$ of nonzero elements, each F_α finite, $\Delta \in [\kappa]^{\omega_1}$.

Now we have $A_\alpha \backslash b$ infinite for all α in a cocountable subset Δ' of Δ. So $(A_\alpha \backslash F_\alpha)\backslash b$ is infinite for each $\alpha \in \Delta'$. Now for $\alpha \neq \beta$ the set $A_\alpha \cdot -F_\alpha \cdot A_\beta \cdot -F_\beta$ is in I and hence is a subset of b. So $\langle (A_\alpha \backslash F_\alpha)\backslash b : \alpha \in \Delta' \rangle$ is a system of ω_1 pairwise disjoint subsets of ω, contradiction. □

Theorem 13. *Suppose that* $(\kappa^+, \kappa^{++}) \in \mathrm{c_{Hr}} A$ *and* $(\kappa, \kappa^{++}) \notin \mathrm{c_{Hr}} A$. *Then* $(\kappa^+, \kappa^+) \in \mathrm{c_{Hr}} A$.

Proof. We work in the Stone space X of A. We may assume that X has cellularity κ^+ and weight κ^{++}. Take points one apiece from a pairwise disjoint family of κ^+ open sets. If their closure has exactly κ^+ clopen sets, we are done, otherwise the closure has κ^{++} clopen sets, and we may assume without loss of generality that the closure is all of X. Thus X has isolated points $\{x_\alpha : \alpha < \kappa^+\}$, listed without repetitions, and they are dense in X. For all $\alpha \in [\kappa, \kappa^+)$ let $X_\alpha = \mathrm{cl}\{x_\beta : \beta < \alpha\}$. Thus X_α is a Boolean space with κ isolated points, which are dense in X_α. So by the hypothesis of the theorem, $|\mathrm{clop} X_\alpha| \leq \kappa^+$.

Case 1. $Y \stackrel{\text{def}}{=} \bigcup_{\alpha \in [\kappa, \kappa^+)} X_\alpha$ is closed. Then $\bigcup_{\alpha \in [\kappa,\kappa^+)} \mathrm{clop} X_\alpha$ is a network for Y. Hence Y has weight κ^+. Since $\{x_\alpha : \alpha < \kappa^+\}$ is its set of isolated points, and this set is dense in Y, the conclusion of the theorem holds.

Case 2. Y is not closed. Let $g \in \mathrm{cl} Y \backslash Y$. Then $g \notin \mathrm{cl} Z$ for all $Z \in [Y]^\kappa$, so the tightness of Y is at least κ^+. Let $\langle y_\alpha : \alpha < \kappa^+ \rangle$ be a convergent free sequence (by Juhász, Szentmiklossy [3]). Say it converges to z. Let $Z = \mathrm{cl}\{y_\alpha : \alpha < \kappa^+\}$. Note that each y_α is isolated in Z, and the y_α's are dense in Z. So it suffices to show that Z has weight κ^+. Let $W_\alpha = \mathrm{cl}\{y_\alpha : \beta < \alpha\}$ for all $\alpha \in [\kappa, \kappa^+)$. Thus W_α is clopen in Z by freeness. Clearly $\bigcap_{\alpha \in [\kappa,\kappa^+)} (Z \backslash W_\alpha) = \{z\}$. So $\{Z \backslash W_\alpha : \alpha \in [\kappa, \kappa^+)\}$ is a neighborhood basis for z. Now by hypothesis, each W_α has weight at most κ^+; let $\mathscr{B}_\alpha$ be a base for W_α with $|\mathscr{B}_\alpha| \leq \kappa^+$. Then

$$\bigcup_{\alpha \in [\kappa,\kappa^+)} \mathscr{B}_\alpha \cup \{Z \backslash W_\alpha : \alpha < \kappa^+\}$$

is a network for Z, so Z has weight κ^+, as desired. □

This proof generalizes to give the following result:

If $\kappa^+ < \nu$, *cof*$\nu \neq \kappa^+$, $(\kappa^+, \nu) \in \mathrm{c}_{Hr}A$, *and* $(\kappa, \nu) \notin \mathrm{c}_{Hr}A$, *then* $(\kappa^+, \mu) \in \mathrm{c}_{Hr}A$ *for some* $\mu < \nu$.

Problem. *Is it necessary to assume that* $\mathrm{cof}\nu \neq \kappa^+$ *in the foregoing result?* Finally, a result on c_{Sr}:

Theorem 14. *For every infinite cardinal* κ, *and every BA* A, *if* $\mathrm{c}A \geq \kappa^{++}$ *and* $(\kappa, \kappa^{++}) \in \mathrm{c}_{\mathrm{Sr}}A$, *then* $(\kappa^+, \kappa^{++}) \in \mathrm{c}_{\mathrm{Sr}}A$.

Proof. Suppose not. Let B be a subalgebra of size κ^{++} with cellularity κ.

(1) There is an $a \in A$ such that $B \restriction a$, which by definition is $\{b \cdot a : b \in B\}$, has cellularity κ^{++}.

To see this, let X be pairwise disjoint of size κ^+. Then $\langle B \cup X\rangle$ is of size κ^{++} and has cellularity greater than κ, so its cellularity is κ^{++}; let Y be a pairwise disjoint subset of size κ^{++}. We may assume that each element $y \in Y$ has the form $y = b_y \cdot a_y$ with $b_y \in B$ and $a_y \in \langle X\rangle$. Since $|X| < \kappa^{++}$, we may in fact suppose that each a_y is equal to some element a, as desired in (1).

Choose such an a, and let $X \in [B]^{\kappa^{++}}$ be such that $\langle x \cdot a : x \in X\rangle$ is a system of nonzero pairwise disjoint elements. Let Y be a subset of X of size κ^+, and let

$$C = \langle\{x \cdot a : x \in Y\} \cup \{x \cdot -a : x \in X \backslash Y\}\rangle.$$

Now define $x \equiv y$ iff $x, y \in X\backslash Y$ and $x \cdot -a = y \cdot -a$. Then

(2) Every $\equiv$-class has size at most κ.

For, suppose that $|x/\equiv| > \kappa$. For any $y \in (x/\equiv)\backslash\{x\}$ we have

$$\begin{aligned} y \cdot -x &= y \cdot -x \cdot a + y \cdot -x \cdot -a \\ &= y \cdot a \cdot -(x \cdot a) + x \cdot -x \cdot -a \\ &= y \cdot a. \end{aligned}$$

This means that B has a pairwise disjoint subset of size greater than κ, contradiction. So (2) holds.

From (2) it follows that $|C| = \kappa^{++}$. Thus we must have $cC = \kappa^{++}$. Hence by the argument for (1), there is a $d \in \langle\{x \cdot a : x \in Y\}\rangle$ and a

$$Z \in [\langle\{x \cdot -a : x \in X\backslash Y\}\rangle]^{\kappa^{++}}$$

such that $\langle z \cdot d : z \in Z\rangle$ is a system of nonzero pairwise disjoint elements. We may assume that each $z \in Z$ has the form

$$\begin{aligned} &x_{z,0} \cdot -a \cdot \ldots \cdot x_{z,m-1} \cdot -a \\ &\cdot (-y_{z,0} + a) \cdot \ldots \cdot (-y_{z,n-1} + a), \end{aligned}$$

where each $x_{z,i}$ and $y_{z,j}$ is in $X\backslash Y$, and m and n do not depend on z.

Now since $\langle\{x \cdot a : x \in Y\}\rangle$ is isomorphic to $\mathrm{Finco}\kappa^+$, there are two cases.

Case 1. $d = \sum_{x\in F} x \cdot a$ for some finite $F \subseteq Y$. Then we may assume that in fact $d = x \cdot a$ for some $x \in Y$. In this case we have $m = 0$, and then each $z \cdot d$ is just equal to d, contradiction.

Case 2. $d = -\sum_{x\in F}(x \cdot a)$ for some finite $F \subseteq Y$. Thus $d = -a + a \cdot -\sum_{x\in F} x$. If $m = 0$, then each $z \cdot d$ is $\geq a \cdot -\sum_{x\in F} x$, so these elements are not disjoint, contradiction. Thus $m > 0$. Hence $z \cdot d = z$ for each $z \in Z$. For each $z \in Z$ write $e_z = x_{z,0} \cdot \ldots \cdot x_{z,m-1}$ and $c_z = e_z \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1}$. Define $z \cong w$ iff $z, w \in Z$ and $e_z = e_w$. If $z \ncong w$, then

$$c_z \cdot c_w = c_z \cdot c_w \cdot a + c_z \cdot c_w \cdot -a = z \cdot w = 0.$$

Since $c_z \in B$ for each $z \in Z$, it follows that there are at most κ $\cong$-classes. So, some $\cong$-class has κ^{++} members. Thus we may assume that all of the e_z's are the same. Thus for any $z \in Z$ we have

$$\begin{aligned} z &= x_0 \cdot \ldots \cdot x_{m-1} \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1} \cdot -a, \\ c_z &= x_0 \cdot \ldots \cdot x_{m-1} \cdot -y_{z,0} \cdot \ldots \cdot -y_{z,n-1}. \end{aligned}$$

Note that $c_z \cdot a = x_0 \cdot \ldots \cdot x_{m-1} \cdot a$. So if $z \neq w$, then

$$\begin{aligned} c_z \cdot -c_w &= c_z \cdot -c_w \cdot a + c_z \cdot -c_w \cdot -a \\ &= c_z \cdot a \cdot -(c_w \cdot a) + c_z \cdot -a \cdot -(c_w \cdot -a) \\ &= z \cdot -w = z. \end{aligned}$$

So if we fix $w \in Z$, then $\langle c_z \cdot -c_w : z \in Z \backslash \{w\} \rangle$ is a system of κ^{++} nonzero pairwise disjoint elements of B, a contradiction. □

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