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SOME EQUIVALENT TOPOLOGIES ON HOMEOMORPHISM GROUPS

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Abstract

In 1993, it was shown [9] that the topology of quasi-uniform convergence which is induced on the collection of all self-homeomorphisms, $H(X)$, by the Pervin quasi-uniformity, is equivalent to the open-open topology on $H(X)$. We show that in fact all set-open topologies of a certain kind are equivalent to a topology of Pervin-type quasi-uniform convergence on $H(X)$.

0. Introduction

In 1948, L. Nachbin introduced the concept of a semi-uniform space [6]. The presently used terminology, quasi-uniform space, was provided by Á. Császár in 1960 [2]. It was V. S. Krishnan [4] in 1955 who first proved that every topological space has a compatible quasi-uniformity, however, the simplest proof of this was given by W. J. Pervin [8] in 1962. This proof consists of constructing a specific quasi-uniformity which is compatible with the given topology; this quasi-uniformity, P, is now called the Pervin quasi-uniformity.

In 1965, Naimpally and Murdeshwar [7] generalized the concept of uniform convergence which brought about the defini-

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tions of quasi-uniform convergence and the topology of quasi-uniform convergence on function spaces. The many interesting results that followed, usually involving groups of homeomorphisms, unfortunately involved the tedious and cumbersome notation needed when discussing the topology of quasi-uniform convergence.

K. Porter, in 1993 [9], proved that given a topological space, (X, T), the topology of Pervin quasi-uniform convergence on $H(X)$ is equivalent to the open-open topology, T_{oo}, on $H(X)$, which has as its subbasic open sets those that are in the form

$$(U, V) = \{h \in H(X) : h(U) \subset V\}$$

where the sets U and V are open in (X, T). Using the simple notation that the open-open topology affords, the theorems and proofs involving quasi-uniform convergence on groups of homeomorphisms are shorter and simpler.

In this paper, we show that any topology of Pervin-type quasi-uniform convergence on a subgroup G of $H(X)$ is equivalent to some set-open topology under some conditions on X, once again simplifying notation and proofs.

Throughout this paper we shall assume that (X, T) is a topological space.

1. Preliminaries

Let X be a non-empty set and let Q be a collection of subsets of $X \times X$ such that

(1) for all $U \in Q$, $\triangle = \{(x, x) \in X \times X : x \in X\} \subset U$, and

(2) for all $U \in Q$, if $U \subset V$ then $V \in Q$,

(3) for all $U, V \in Q$, $U \cap V \in Q$, and

(4) for all $U \in Q$, there exists some $W \in Q$ such that $W \circ W \subset U$ where $W \circ W = \{(p,q) \in X \times X : \text{ there exists some } r \in X \text{ with } (p,r),(r,q) \in W\}$ then Q is a *quasi-uniformity on* X.

A quasi-uniformity, Q, on X induces a topology, T_Q, on X, where for each $x \in X$, the set $\{U[x] : U \in Q\}$ is a neighborhood system at x where $U[x]$ is defined by $U[x] = \{y \in X : (x,y) \in U\}$.

A family, S of subsets of $X \times X$ which satisfies:

(i) for all $R \in S, \triangle \subset R$, and

(ii) for all $R \in S$, there exists $T \in S$ such that $T \circ T \subset R$, is a *subbasis* for a quasi-uniformity, Q, on X. This subbasis S generates a *basis*, B, for the quasi-uniformity, Q, where B is the collection of all finite intersections of elements of S. The basis, B, generates the quasi-uniformity $Q = \{U \subset X \times X : \hat{B} \subset U \text{ for some } \hat{B} \in B\}$.

Note what it means that every topological space (X,T) is quasi-uniformizable; given a topological space (X,T) there exists a quasi-uniformity, Q, which induces a topology T_Q on X which is equivalent to the original topology, i.e., $T = T_Q$. Pervin [8] actually constructed a specific quasi-uniformity which induces a compatible topology for a given topological space. His construction is as follows: Let (X,T) be a topological space. For $O \in T$ define

$$S_O = (O \times O) \cup ((X \setminus O) \times X).$$

One can show that for $O \in T, S_O \circ S_O = S_O$ and $\triangle \subset S_O$, hence, the collection $\{S_O : O \in T\}$ is a subbasis for a quasi-uniformity, P, on X, called the *Pervin quasi-uniformity*. Note that for $x \in X$, the neighborhoods of x in T_Q are

$$S_O[x] = \begin{cases} X & \text{if } x \notin O \\ O & \text{if } x \in O. \end{cases}$$

To generalize this idea, let $\mathcal{A}$ be any collection of open sets in (X,T). For each $O \in \mathcal{A}$, define, as above,

$S_O = (O \times O) \cup ((X \setminus O) \times X)$. Then, again, we have that for all $O \in \mathcal{A}$, $S_O \circ S_O = S_O$ and $\triangle \subset S_O$, and thus, the collection $\{S_O : O \in \mathcal{A}\}$ is a subbasis for a *Pervin-type quasi-uniformity*, $Q_{\mathcal{A}}$, on X. The topology induced on X by $Q_{\mathcal{A}}$ will be denoted by $T_{\mathcal{A}}$.

Let $G \subset H(X)$, then a collection $\mathcal{A} \subset P(X) = \{F : F \subset X\}$ is called a *G-invariant collection of sets* provided that for all $A \in \mathcal{A}$ and for all $g \in G$, $g(A) \in \mathcal{A}$. Some examples of $H(X)$-invariant collections of sets are the collections of all: *i*) open sets, *ii*) closed sets, *iii*) compact sets, *iv*) singleton sets, *v*) connected sets, *vi*) regular open sets, and *vii*) non-dense open sets.

If a G-invariant collection, $\mathcal{A}$, is also a basis for (X, T), then $\mathcal{A}$ will be called a *G-invariant basis.*

Theorem 1. *Let $G \subset H(X)$ and let $\mathcal{A}$ be a G-invariant basis for (X, T). Then $T = T_{\mathcal{A}}$ where $T_{\mathcal{A}}$ is the topology induced on X by the Pervin-type quasi-uniformity $Q_{\mathcal{A}}$.*

Proof. For each $A \in \mathcal{A}$, and each $x \in X$, it is true that
$S_A[x] = \begin{cases} X & \text{if } x \notin A \\ A & \text{if } x \in A. \end{cases}$
So, clearly it follows that $T = T_{\mathcal{A}}$. □

Let G be a subgroup of $H(X)$ and let $\mathcal{A}$ be a G-invariant basis for (X, T). For $A \in \mathcal{A}$, and $O \in T$, define the set

$$(A, O) = \{f \in G : f(A) \subset O\}.$$

Set $S_{\mathcal{A}O} = \{(A, O) : A \in \mathcal{A} \text{ and } O \in T\}$, then $S_{\mathcal{A}O}$ is a subbasis for a topology, $T_{\mathcal{A}O}$, on G. We call this type of topology a *set-open topology.*

Note that given a topological space , (X, T), the finest of all the set-open topologies on a subgroup G of $H(X)$ is the open-open topology [9] where $\mathcal{A} = T$.

Theorem 2. *If $\mathcal{A}$ is a G-invariant basis for (X, T) then the collection of all sets of the form (A, U), where A and U are both elements of $\mathcal{A}$, forms a subbasis for $(G, T_{\mathcal{A}O})$.*

Proof. If $h \in (A, O) \in (G, T_{\mathcal{A}O})$ then it follows that $h \in (A, h(A)) \subset (A, O)$. □

2. Topologies of Pervin-Type Quasi-Uniform Convergence

Let Q be a compatible quasi-uniformity for (X, T) and let G be a subgroup of $H(X)$. For $U \in Q$, define the set

$$W(U) = \{(f, g) \in G \times G : (f(x), g(x)) \in U \text{ for all } x \in X\}.$$

Then the collection $B = \{W(U) : U \in Q\}$ is a basis for a quasi-uniformity, Q^*, on G, called *the quasi-uniformity of quasi-uniform convergence with respect to Q* [7]. The topology, T_{Q^*}, induced by Q^* on G, is called *the topology of quasi-uniform convergence with respect to Q*. If $\mathcal{A}$ is a G-invariant basis for (X, T), then $T_{Q^*_{\mathcal{A}}}$ is called *a topology of Pervin-type quasi-uniform convergence.*

Theorem 3. *Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Assume $\mathcal{A}$ is a G-invariant basis for (X, T) and let $Q_{\mathcal{A}}$ be the Pervin-type quasi-uniformity induced on X. Then, the set-open topology, $T_{\mathcal{A}O}$, on G is equal to the topology of Pervin-type quasi-uniform convergence, $T_{Q^*_{\mathcal{A}}}$, on G.*

Proof. Let $W(U)[f]$ be a basic open set in $T_{Q^*_{\mathcal{A}}}$. So, $f \in W(U)[f]$. Now, $U \in Q_{\mathcal{A}}$, implies that there exists some basis element, $\bigcap_{i=1}^{n} S_{V_i} \subset U$ where $\{V_i : i = 1, 2, ..., n\}$ is a subfamily of $\mathcal{A}$. Clearly, $f \in W(\bigcap_{i=1}^{n} S_{V_i})[f]$. Since $\mathcal{A}$ is a G-invariant basis, $f^{-1}(V_i) \in \mathcal{A}$, for all $i = 1, 2, ..., n$. Thus, for each $i = 1, 2, ..., n$, $(f^{-1}(V_i), V_i) \in T_{\mathcal{A}O}$. This implies that $f \in \bigcap_{i=1}^{n}(f^{-1}(V_i), V_i) \in T_{\mathcal{A}O}$.

Claim: $\bigcap_{i=1}^{n}(f^{-1}(V_i), V_i) \subset W(\bigcap_{i=1}^{n} S_{V_i})[f]$: Assume that $g \in \bigcap_{i=1}^{n}(f^{-1}(V_i), V_i)$ then $g \circ f^{-1}(V_i) \subset V_i$ for all $i \in \{1, 2, 3, \ldots, n\}$. If $f(x) \in V_j$ for some $j \in \{1, 2, 3, \ldots, n\}$ and some $x \in X$, then $x \in f^{-1}(V_j)$, from which it follows that $g(x) \in V_j$. In this case, $(f(x), g(x) \in V_j \times V_j \subset S_{V_j}$. If $f(x) \notin V_j$ then $(f(x), g(x)) \in (X \setminus V_j) \times X \subset S_{V_j}$. Therefore, for any $x \in X$, $g \in W(\bigcap_{i=1}^{n} S_{V_i})[f]$. Hence, $T_{\mathcal{A}O} \subset T_{Q^*_{\mathcal{A}}}$.

Let (A, O) be a subbasic open set in $T_{\mathcal{A}O}$ and let $f \in (A, O)$. This means that $f(A) \subset O$ and $f(A) \in \mathcal{A}$, so $f \in W(S_{f(A)})[f]$. *Claim:* $W(S_{f(A)})[f] \subset (A, O)$: Let $g \in W(S_{f(A)})[f]$ and let $x \in A$. Then $f(x) \in f(A)$ so that $g(x) \in f(A)$. But $f(A) \subset O$, and hence, $g \in (A, O)$. Thus, $T_{Q^*_{\mathcal{A}}} \subset T_{\mathcal{A}O}$ and we are done. □

3. Some Properties

Let $(X, \mathcal{U})$ and $(Y, \mathcal{Q})$ be quasi-uniform spaces. A function $f : X \to Y$ is *quasi-uniformly continuous* if and only if for each $Q \in \mathcal{Q}$ there exists some $U \in \mathcal{U}$ such that if $(x, y) \in U$ then $(f(x), f(y)) \in Q$.

Theorem 4. *Let $(X, Q_{\mathcal{A}})$ be a Pervin-type quasi-uniform space and let G be a subgroup of $H(X)$ with the corresponding topology of Pervin-type quasi-uniform convergence, $T_{Q_{\mathcal{A}}}$. Then each $g \in G$ is quasi-uniformly continuous.*

Proof. Let $g \in G$ and $Q \in Q_{\mathcal{A}}$. Then there is a basic open set $\bigcap_{i=1}^{m} S_{O_i} \subset Q$ and $\bigcap_{i=1}^{m} S_{g^{-1}(O_i)} \in Q_{\mathcal{A}}$. If $(x, y) \in S_{g^{-1}(O_i)}$ for some $i \in \{1, 2, 3, \ldots, m\}$, then it must follow that either (a)

$(x, y) \in g^{-1}(O_i) \times g^{-1}(O_i)$ or (b) $(x, y) \in [X \setminus g^{-1}(O_i)] \times X$. In case (a), it follows that $(g(x), g(y)) \in O_i \times O_i \subset S_{O_i}$, and in case (b) we have $(g(x), g(y)) \in [X \setminus O_i] \times X \subset S_{O_i}$. Thus g is quasi-uniformly continuous on X. □

Let (X, T) and (Y, T') be topological spaces and let F be a collection of functions from X into Y. Suppose $\hat{T}$ is a topology on F, then we say that $\hat{T}$ is *admissible* on F provided the function, $E : (F, \hat{T}) \times (X, T) \longrightarrow (Y, T')$, defined by $E(f, x) = f(x)$ is continuous.

Theorem 5. *Let (X, T) be a topological space and G be a subgroup of $H(X)$. Assume $\mathcal{A}$ is a G-invariant basis for X. Then $T_{Q^*_{\mathcal{A}}}$ is admissible for G.*

Proof. On G, $T_{Q^*_{\mathcal{A}}} = T_{\mathcal{A}O}$. Let O be open in X and let $(h, p) \in E^{-1}(O)$. Then $h(p) \in O$. Since h is continuous, there exists $A \in \mathcal{A}$ such that $p \in A$ and $h(A) \subset O$ and $(h, p) \in (A, O) \times A$. It is then simple to show that $(A, O) \times A \subset E^{-1}(O)$ so that $T_{\mathcal{A}O}$ is admissible on G. □

Theorem 6. *Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Let $\mathcal{A}$ be a G-invariant basis for (X, T). Let T_{co} be the compact-open topology on G which has as a subbase the collection $S_{co} = \{(C, O)$: C is compact and $O \in T\}$ and $(C, O) = \{g \in G : g(C) \subset O\}$. Then $T_{co} \subset T_{\mathcal{A}O}$ on G.*

Proof. In [1], Arens proved that if T is admissible for $F \subset C(X, Y)$, then $T_{co} \subset T$ on F. □

Theorem 7. *Let (X, T) be a topological space, G a subgroup of $H(X)$, and $\mathcal{A}$ a G-invariant basis for X. If (X, T) is T_i, $i =$ 0, 1, 2, then $(H(X), T_{Q^*_{\mathcal{A}}})$ is T_i, $i = 0, 1, 2$, respectively.*

Proof. We will show the proof for the case $i = 2$. Assume (X, T) is T_2 and assume that $\mathcal{A}$ is a G-invariant basis for X.

Let $g, h \in (G, T_{\mathcal{A}\mathcal{O}})$ with $g \neq h$. Hence there exists $c \in X$ with $g(c) \neq h(c)$. But (X, T) is T_2, so there exists A, A' in $\mathcal{A}$ with $g(c) \in A$, $h(c) \in A'$ and $A \cap A' = \phi$. Then $g \in O_1 = (g^{-1}(A), A)$, $h \in O_2 = (h^{-1}(A'), A')$, $O_1 \cap O_2 = \phi$, and $O_1, O_2 \in T_{Q_{\mathcal{A}^*}}$; thus $(G, T_{Q_{\mathcal{A}^*}})$ is T_2. □

Let G be a group with group operation $\cdot$ and topology T^+. Then we say that G is a *quasi-topological group*[1] [5] provided the map $m : (G, T^+) \times (G, T^+) \to (G, T^+)$ defined by $m(f, g) = f \cdot g$ is continuous. If in addition, the map $\phi : (G, T^+) \to (G, T^+)$ defined by $\phi(f) = f^{-1}$ is continuous, then (G, T^+) is called a *topological group.*

Theorem 8. *Let (X, T) be a topological space and let $\mathcal{A}$ be a G-invariant basis for X. Then $(G, T_{Q^*_{\mathcal{A}}})$ is a quasi-topological group.*

Proof. Assume (A, O) is a subbasic open set in $T_{\mathcal{A}\mathcal{O}} = T_{Q^*_{\mathcal{A}}}$ and let $(g, h) \in m^{-1}((A, O))$. Then $g \circ h \in (A, O)$ or $g \circ h(A) \subset O$. Thus $h \in (A, h(A))$ and $g \in (h(A), O)$. But $(A, h(A))$ and $(h(A), O)$ are both subbasic open sets in $T_{\mathcal{A}\mathcal{O}}$; hence, $(g, h) \in (h(A), O) \times (A, h(A)) \in (G, T_{Q^*_{\mathcal{A}}}) \times (G, T_{Q^*_{\mathcal{A}}})$. If $(p, q) \in (h(A), O) \times (A, h(A))$, then $q(A) \subset h(A)$ and $p(h(A)) \subset O$ so that $p(q(A)) \subset p(h(A) \subset O$. Thus, $(p, q) \in m^{-1}((A, O))$ so $(G, T_{Q^*_{\mathcal{A}}})$ is a quasi-topological group. □

Recall that if $\mathcal{A}$ is the collection of all non-empty open sets, then $T_{Q^*_{\mathcal{A}}}$ is the topology of Pervin quasi-uniform convergence. However, it has been shown that $(H(X), T_{Q^*_{\mathcal{A}}})$ is not necessarily a topological group [9]. If, instead, we choose $\mathcal{A}$ to be the collection of all regular non-empty open sets, assuming that X is semi-regular, then $(G, T_{Q^*_{\mathcal{A}}})$ is a topological group since the hypotheses of the following theorem are satisfied.

[1] Also called *paratopological group* [3].

Theorem 9. *Let (X,T) be a topological space, G a subgroup of $H(X)$, and $\mathcal{A}$ a G-invariant basis for X. If $\mathcal{A}$ is a collection of regular open sets such that $\{\mathrm{int}(X \setminus V) : V \in \mathcal{A}\} \subset \mathcal{A}$ then $(G, T_{Q^*_{\mathcal{A}}})$ is a topological group.*

Proof. Assume that $\{int(X \setminus V) : V \in \mathcal{A}\} \subset \mathcal{A}$ and assume that for all $A \in \mathcal{A}$, A is a regular open set. We must show that $\phi : (G, T_{Q^*_{\mathcal{A}}}) \longrightarrow (G, T_{Q^*_{\mathcal{A}}})$, defined by $\phi(g) = g^{-1}$ is continuous. Note, for any set of the form (A, O), $\phi^{-1}(A, O) = ((X \setminus O), (X \setminus A))$. Let A and O be in $\mathcal{A}$. Assume $h \in \phi^{-1}((A, O)) = ((X \setminus O), (X \setminus A))$. So, $h(X \setminus O) \subset (X \setminus A)$ from which it follows that $h(int(X \setminus O)) \subset int(X \setminus A)$. Since $\mathcal{A}$ is a collection of regular open sets, $(int(X \setminus O), int(X \setminus A)) \subset (X \setminus O, X \setminus A)$. But $(int(X \setminus O), int(X \setminus A)) \in T_{\mathcal{A}O}$, so $(X \setminus O, X \setminus A) \in T_{\mathcal{A}O}$. Therefore, $(G, T_{Q^*_{\mathcal{A}}}) = (G, T_{\mathcal{A}O})$ is a topological group. □

P. Fletcher defined a *Pervin space* [3] to be a topological space, (X,T), in which for each finite collection, $\mathcal{A}$, of open sets in X, there exists some $h \in H(X)$ such that $h \neq e$, the identity map, and $h(U) \subset U$ for all $U \in \mathcal{A}$. P. Fletcher [3] also proved that $H(X)$ with the topology of Pervin quasi-uniform convergence is not discrete if and only if (X,T) is a Pervin space. K. Porter [9], using the notation of the open-open topology simplified the proof.

Now since it is true that $T_{co} \subset T_{\mathcal{A},O} = T_{Q^*_{\mathcal{A}}} \subset T_{oo}$ on G for any G-invariant basis $\mathcal{A}$ on X, it follows that if T_{oo} is not discrete, then no other Pervin-type topology of quasi-uniform convergence is discrete. However, if T_{oo} is discrete, what other conditions will insure that the Pervin-type topology of quasi-uniform convergence is not discrete? To this end we make the following definition. Let (X,T) be a topological space and let G be a subgroup of $H(X)$. Assume $\mathcal{A}$ is a G-invariant basis for (X,T). Then (X,T) is called a *$(\mathcal{A},G)$-Pervin space* provided that for each finite collection, $\mathcal{O}$, of sets from $\mathcal{A}$, there exists some $g \in G$ such that $g \neq e$ and $g(U) \subset U$ for all $U \in \mathcal{O}$.

Theorem 10. *Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Assume $\mathcal{A}$ is a G-invariant basis for (X, T). Then $(G, T_{\mathcal{A}O})$ is not discrete if and only if (X, T) is an $(\mathcal{A}, G)$-Pervin space.*

Proof. First, assume that (X, T) is an $(\mathcal{A}, G)$-Pervin space. Let W be a basic open set in $T_{\mathcal{A}O}$ which contains e; i.e. $W = \bigcap_{i=1}^{n}(O_i, U_i)$ where $O_i \subset U_i$ for each $i = 1, 2, 3, ..., n$ and O_i and U_i are open in X. $\{O_i : i = 1, 2, 3, ..., n\}$ is a finite collection of open sets from $\mathcal{A}$, and X is an $(\mathcal{A}, G)$-Pervin space, hence, there exists some $g \in G$ such that $g \neq e$ and $g(O_i) \subset O_i \subset U_i$ for all i. So, $g \in W$ and $g \neq e$. Since (G, T_{oo}) is a quasi-topological group, $(G, T_{\mathcal{A}O})$ is not a discrete space.

Now assume that $(G, T_{\mathcal{A}O})$ is not discrete. Let V be a finite collection of open sets in $\mathcal{A}$. Let $O = \bigcap_{U \in V}(U, U)$. Then, O is a basic open set in $(G, T_{\mathcal{A}O})$ which is not a discrete space. Hence, there exists $h \in O$ with $h \neq e$. So, (X, T) is Pervin. □

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