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COMPLEXITY SPACES: LIFTING & DIRECTEDNESS

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Abstract

The theory of complexity spaces has been introduced in [Sch95] as part of the development of a topological foundation for Complexity Analysis.

The topological study of these spaces has been continued in the context of the theory of upper weightable spaces ([Sch96]), while the specific properties of total boundedness and Smyth completeness have been analyzed in [RS96].

Here we introduce a technique of "lifting", which allows one to extend an upper weightable space, and hence a complexity space, by a maximum. This leads to a characterization of the upper weightable spaces as the weightable spaces which have a weightable *directed* extension.

We motivate the property of directedness from a complexity theoretic point of view, which leads to the study of the particular class of weightable directed spaces.

Weightable directed spaces are shown to be non metrizable and their weighting functions are analyzed. These weighting functions are shown to be upper weightings among which there is a "fading" weighting.

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Finally we show that the process of lifting of a weighted directed space of fading weight, in particular of a complexity space, does not essentially alter the topology of the original space.

1. Introduction

The weightable quasi-metric spaces, or the equivalent partial metric spaces, have been introduced by Matthews in the context of the study of denotational semantics of dataflow networks (e.g. [Kah74] and [Mat94]). Recently the research on partial metrics has been continued by Heckmann ([Hec96]).

The topological study of the weightable quasi-metric spaces has been the subject of [KV94] and also of the survey paper "Nonsymmetric Topology" ([Kün93]).

We recall that upper weightable quasi-metric spaces have been introduced in [Sch96]as part of the development of a topological foundation for Complexity Analysis, which involves the theory of complexity spaces (e.g. [Sch95]). As shown in [Sch96], upper weightable spaces are in a sense more tractable than the general class of weightable spaces. This is for instance illustrated by the characterization of weightable optimal join semilattices obtained in [Sch96], while the general characterization of weightable spaces remains open ([Kün93]).

The study of complexity spaces has been continued in [RS96] where it has been shown that the complexity analysis of algorithms can be carried out based on the dual of a complexity space. This has the advantage that the theory becomes mathematically more elegant but on the other hand complexity theoretic interpretations become less straightforward (cf. [RS96]).

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We recall that the use of the dual results in a reversal of the ordering such that the maximum "$\top$" of a complexity space (intuitively the complexity of a program which is undefined on all inputs) becomes the minimum "$\perp$" of the dual complexity space, which is more in accordance with a traditional domain theoretic point of view.

The question arises whether the domain theoretic construction of lifting of a partial order (this is extension by a minimum "$\perp$") can be carried out in the context of the theory of dual complexity spaces. This turns out to be the case.

However, since we will provide computer science motivations throughout the paper, we choose to work with complexity spaces rather than their dual. We will show that every upper weightable space has an extension by a maximum, referred to as the "lifting" of the space.

This extension technique is used to show that upper weighted quasi-metric spaces are exactly those weighted spaces which have a weighted *directed* extension.

The notion of directedness is motivated from a complexity theoretic point of view, which leads to a directedness requirement on complexity spaces and to a study of weightable directed spaces in general.

Weightable directed spaces are shown to be strongly upper weighted, in the sense that every weighting function is an upper weighting.

The notion of fading weight is introduced and motivated in the theory of complexity spaces and we show that every upper weightable space possesses a fading weighting.

In [Sch95], an intuitive motivation has been given for the fact that complexity spaces are essentially non symmetric. Here we show that weightable directed spaces are not metrizable, which provides a theoretical justification of the above intuition.

Finally, the topology of the lifting of a weighted directed space of fading weight is characterized in terms of the topology of the underlying space. The result illustrates that the process of lifting of a complexity space is essentially harmless, as is the process of lifting of a traditional domain.

The topological study of the weightable directed spaces is continued in [Sch97] where a (partial) solution of Problem 7 of [Kün93] is presented in the context of the theory of optimal join semilattices (cf. also [Sch96]).

2. Background

The following notation is used throughout: $\mathcal{N}$ denotes the set of natural numbers, $\mathcal{R}$ denotes the set of real numbers, $\mathcal{R}^+ = (0, \infty)$, $\mathcal{R}_0^+ = [0, \infty)$, while $\overline{\mathcal{R}} = \mathcal{R} \cup \{-\infty, \infty\}, \overline{\mathcal{R}^+} = \mathcal{R}^+ \cup \{\infty\}$ and $\overline{\mathcal{R}_0^+} = \mathcal{R}_0^+ \cup \{\infty\}$.

A quasi-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ and (2) $d(x, z) \leq d(x, y) + d(y, z)$.

The *associated partial order* $\leq_d$ of a quasi-metric d is defined by $x \leq_d y$ iff $d(x, y) = 0$. A quasi-metric space *has a maximum* when its associated partial order has a maximum.

A quasi-metric d on a set X is *order convex* iff $\forall x, y, z \in X.\, x \geq_d y \geq_d z \Rightarrow d(x, y) + d(y, z) = d(x, z)$.

The *conjugate* d^{-1} of a quasi-metric d is defined to be the function $d^{-1}(x, y)$
$= d(y, x)$, which is again a quasi-metric (e.g. [FL82]). The conjugate of a quasi-metric space (X, d) is the quasi-metric space (X, d^{-1}). The *metric* d^* induced by a quasi-metric d is defined by $d^*(x, y) = max\{d(x, y), d(y, x)\}$.

A partial order $(X, \leq)$ is *directed* iff $\forall x, y \in X\, \exists z \in X.\, z \geq x$ and $z \geq y$.

A partial order $(X, \leq)$ is *linear* iff $\forall x, y \in X.\, x \leq y$ or $y \leq x$.

For any function $f: A \to B$ and for any set $X \subseteq A$, $f|X$ indicates the *restriction* of f to the set X. We also say that a function f *extends* a function g when g is the restriction of f to a subset of the domain of f. A *subspace* of a quasi-metric space

(X, d) is a pair $(Y, d|Y^2)$, where $Y \subseteq X$. When no confusion can arise, we denote a subspace $(Y, d|Y^2)$ of (X, d) by (Y, d). In this case we also refer to the quasi-metric space (X, d) as an *extension* of the quasi-metric space (Y, d) and we also write that (X, d) *extends* (Y, d). A quasi-metric space (X, d) is *extendible by a maximum* iff there exists an extension (X', d') of (X, d) such that $X' = X \cup \{x_0\}$ and x_0 is the maximum of the extension. The notion "extendible by a weightless point" is defined in a similar way.

We remark that not every quasi-metric space is extendible by a maximum and thus the process of lifting in the context of quasi-metric spaces is not as trivial as the ordinary domain theoretic notion of lifting!

Consider for instance the right distance space, that is the space $(\mathcal{R}_0^+, d)$ where $d(x, y) = x - y$ when $x > y$ and $d(x, y) = 0$ otherwise.

This space is not extendible by a maximum. We argue by contradiction. If an extension would exist, say (X_0, d_0) with a maximum x_0 then, by the Monotonicity Lemma mentioned in the paper, we have that $\forall x \in \mathcal{R}_0^+ . \, d_0(x_0, 0) \geq d_0(x, 0) = x$ and thus $d_0(x_0, 0) = \infty$, a contradiction.

We recall ([Sch96], Lemma 5) that quasi-metrics satisfy the following property, which we refer to as "*the Monotonicity Lemma*": if (X, d) is a quasi-metric space then $\forall x, y, z \in X$. $(x' \leq_d x$ and $y' \geq_d y) \Rightarrow d(x', y') \leq d(x, y)$.

We discuss a few examples of quasi-metric spaces.

The function $d_1 : \mathcal{R}^2 \to \mathcal{R}_0^+$, defined by $d_1(x, y) = y - x$ when $x < y$ and $d_1(x, y) = 0$ otherwise, and its conjugate are quasi-metrics. We refer to d_1 as the "left distance" and to its conjugate as the "right distance". These quasi-metrics are the nonsymmetric versions of the standard metric m on the reals, where $\forall x, y \in \mathcal{R} . \, m(x, y) = |x - y|$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x, y \in \mathcal{R}.\, x \leq_{d_1^{-1}} y \Leftrightarrow x \leq y$, while for the left distance we have $\forall x, y \in \mathcal{R}.\, x \leq_{d_1} y \Leftrightarrow x \geq y$.

The function $d_2 : (\overline{\mathcal{R}} - \{0\})^2 \to \mathcal{R}_0^+$, defined by $d_2(x, y) = \frac{1}{y} - \frac{1}{x}$ when $y < x$ and 0 otherwise, and its conjugate are quasi-metrics.

The complexity space (C, d_C) has been introduced in [Sch95] (cf. also [Sch96] and [RS96]). Here

$$C = \{f : \omega \to (0, +\infty] | \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty\}$$

and d_C is the quasi-metric on C defined by

$$d_C(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(\frac{1}{g(n)} - \frac{1}{f(n)}) \vee 0]$$

whenever $f, g \in C$. Any subspace of (C, d_C) is also called a complexity space (cf. [Sch95]).

We recall that the complexity space has a maximum $\top$, which is the function with constant value ∞.

A quasi-metric space (X, d) is *weightable* iff there exists a function $w : X \to \mathcal{R}_0^+$ such that $\forall x, y \in X.\, d(x, y) + w(x) = d(y, x) + w(y)$. The function w is called a *weighting function*, $w(x)$ is the *weight* of x and the quasi-metric d is *weightable by the function* w. A *weighted* space is a triple (X, d, w) where (X, d) is a quasi-metric space weightable by the function w. A *weightless point* of a weighted quasi-metric space is a point of zero weight.

Examples. The quasi-metric space $(\mathcal{R}_0^+, d_1)$ is weightable by the identity function, $w_1(x) = x$. The quasi-metric space $(\overline{\mathcal{R}^+}, d_2)$ is weightable by the function $w_2(x) = \frac{1}{x}$. The complexity space (C, d_C) is weightable by the function w_C where $\forall f \in C.\, w_C(f) = \sum_n \frac{2^{-n}}{f(n)}$.

We recall that the conjugate quasi-metric space $(\mathcal{R}_0^+, d_1^{-1})$ is *not* weightable ([Sch96]). For more information on conjugates of weightable spaces we refer the reader to [KV94].

An *extension* of a weighted space (X, d, w) is a weighted space (X', d', w') such that the quasi-metric space (X', d') is an extension of the quasi-metric space (X, d) and such that $w'|X$ coincides with w.

3. Upper Weightable Spaces

Definition 1. *If (X, d) is a quasi-metric space then (X, d) is upper weightable iff there exists a weighting function w for (X, d) such that $\forall x, y \in X.\, d(x, y) \leq w(y)$. We refer to such a function w as an upper weighting function. A weighted space (X, d, w) is upper weighted iff w is an upper weighting function. An upper weightable space is strongly upper weighted iff all of its weighting functions are upper weighting functions.*

We recall a motivation behind the notion of an upper weighted space ([Sch96]).

The following property of weighted spaces (X, d, w) is an easy consequence of the weighting equality:

$$\forall x, y \in X.\, d(x, y) \leq w(y) \Leftrightarrow d(y, x) \leq w(x)$$

or equivalently:

$$\forall x, y \in X.\, d(x, y) \geq w(y) \Leftrightarrow d(y, x) \geq w(x).$$

The spaces which arise as "extreme cases" with respect to these equivalent properties are the upper weightable spaces, satisfying the inequality $\forall x, y \in X.\, d(x, y) \leq w(y)$, and the lower weightable spaces, satisfying the inequality $\forall x, y \in X.\, d(x, y) \geq w(y)$.

The class of lower weightable spaces is easily characterized to coincide with the class of the metric spaces ([Sch96]). So the upper weightable spaces can be interpreted to form a class of spaces which in a sense is "orthogonal" to the class of the metric spaces inside the class of the weightable spaces. These spaces also provide a convenient framework for the study of the complexity spaces ([Sch96]).

Examples. the quasi-metric space $(\mathcal{R}_0^+, d_1)$ is upper weightable by the weighting function w_1, the quasi-metric space $(\mathcal{R}^+, d_2)$ is upper weightable by the weighting function w_2 and the complexity space (C, d_C) is upper weightable by the weighting function w_C.

We remark that the notion of an upper weighted space (X, d, w) is equivalent, by the Correspondence Theorem, to a partial metric space (X, p) such that

$$\forall x, y \in X.\, p(x, y) \leq p(x, x) + p(y, y).$$

We discuss a complexity theoretic interpretation of the upper weighting of the complexity space $(\mathcal{C}, d_C, w_C)$. Since $d(\top, y) = w_C(y)$, the weighting expresses the distance from the maximum $\top$.

Since the complexity distance intuitively measures improvements in the complexity of a program (cf. [Sch95]), the weight of a program essentially represents the maximal improvement possible obtained by replacing any program by the given program. This follows from the fact that (by the monotonicity lemma): $\forall x, y \in \mathcal{C}.\, d_C(x, y) \leq d_C(\top, y) = w_C(y)$.

We will show that in a sense the weight of an upper weighted space can always be expressed as the distance from a maximum (Theorem 11).

The following subsection studies the properties of the maximum "$\top$" of a complexity space in the more general context of the theory of weightable spaces. In particular the closely related notions of weightless points and maxima are discussed.

3.1. Maxima and Weightless Points

We recall a result of [Sch96] (Lemma 8), where the following notation is used: if (X,d) is a quasi-metric space and $x_0 \in X$ then the function f_{x_0} is defined by $\forall x \in X.\, f_{x_0}(x) = d(x_0, x)$.

The result implies that weightable spaces with a maximum essentially have a unique weighting function which expresses the distance from the maximum.

Proposition 2. *If (X,d) is a weightable quasi-metric space with a maximum x_0 then its weighting functions are exactly the functions $f_{x_0} + c$ where $c \geq 0$.*

Corollary 3. *Every weightable quasi-metric space with a maximum is strongly upper weighted.*

Proof. If (X,d) is weightable by a weighting function w and has a maximum x_0 then by the Monotonicity Lemma we have that $\forall x, y \in X.\, d(x,y) \leq d(x_0, y) = f_{x_0}(y)$ and thus the result follows since, by Proposition 2, $w = f_{x_0} + c$ for some $c \geq 0$. □

Remark. Proposition 2 implies that every weightable quasi-metric space with a maximum x_0 is upper weightable by the weighting function f_{x_0} for which the maximum x_0 is weightless.

We analyze spaces with a weighting function which expresses the distance from a given point in more detail.

Definition 4. *A quasi-metric space (X,d) is (upper) weightable with respect to a point $x_0 \in X$ iff there exists a(n upper) weighting function w for the space, which satisfies*

$$\forall x \in X.\, w(x) = f_{x_0}(x).$$

In that case, we say that the quasi-metric space (X,d,w) is (upper) weighted with respect to the point x_0.

We will show (cf. the remark following Proposition 6) that the notions of "weighted with respect to a point" and "upper weighted with respect to a point" are equivalent. Hence in the following examples we only discuss spaces weighted with respect to a point.

Examples. The quasi-metric space $(\mathcal{R}_0^+, d_1)$ is weightable with respect to the point 0 via the weighting function w_1. The quasi-metric space $(\overline{\mathcal{R}^+}, d_2)$ is weightable with respect to ∞ via the weighting function w_2. The complexity space $(\mathcal{C}, d_C)$ is weightable with respect to $\top$ via the weighting function w_C.

The remark following Corollary 3 can be reformulated as follows.

Lemma 5. *Every weightable space with a maximum is weightable with respect to this maximum.*

Weighted spaces with a weightless maximum can be characterized in the following way.

Proposition 6. *A weighted space (X, d, w) is weighted with respect to a point x_0 iff x_0 is a weightless maximum of the space (X, d, w).*

Proof. We assume that (X, d, w) is a weighted space.

We show that when (X, d, w) is weighted with respect to the point x_0, this point is a weightless maximum. The fact that x_0 is weightless follows immediately since $w(x_0) = d(x_0, x_0) = 0$. The point x_0 is a maximum since by weightedness with respect to x_0, we have that $\forall x.\, d(x_0, x) + d(x, x_0) = d(x_0, x_0) + d(x_0, x)$ and thus $\forall x.\, d(x, x_0) = 0$, which is equivalent to $\forall x.\, x \leq_d x_0$.

Conversely, we show that when x_0 is a weightless maximum, the space (X, d, w) is weighted with respect to x_0.

If x_0 is a weightless maximum then we have by weightedness that $\forall x \in X.\, w(x) + d(x, x_0) = w(x_0) + d(x_0, x)$ and thus $\forall x \in X.\, w(x) = d(x_0, x)$. $\square$

Proposition 6 implies that the notions of "weighted with respect to a point" and "upper weighted with respect to a point" are equivalent. It suffices to verify that every space (X, d) weighted with respect to a point x_0 is upper weighted with respect to this point.

Let (X, d, w) be a space which is weighted with respect to a point x_0. By Proposition 6, the point x_0 is a maximum and thus the result follows by Corollary 3.

We remark that for weighted spaces the notions of a maximum and of a weightless point do not necessarily coincide.

A weighted space can have a maximum which is not a weightless point, a weightless point which is not a maximum or neither possess a weightless point nor a maximum, as the following counterexamples show.

1) A weighted space $([a, b], d_1, w_1)$, where $a > 0$, has a maximum a which is not weightless.

2) Any non trivial metric space equipped with the constant zero weighting.

3) A weighted space $((a, b], d_1, w_1)$, where $a > 0$, has neither a maximum nor a weightless point.

This also implies that Proposition 6 can not be sharpened to state that a weighted space with a maximum x_0 (or alternatively with a weightless point x_0) necessarily is weighted with respect to x_0.

Corollary 7. *Every space weightable with respect to a point is strongly upper weighted.*

Proof. Immediate by Proposition 6 and by Corollary 3. □

In particular every space weighted with respect to a point is upper weighted. As might be expected the converse does not hold. It is easy to verify that example 1) above provides an upper weighted space which is not (upper) weighted with respect to a point.

The following lemma provides a sufficient condition for the converse to hold.

Lemma 8. *An upper weighted quasi-metric space with a weightless point is (upper) weighted with respect to this point.*

Proof. Assume that (X, d, w) is an upper weighted quasi-metric space with a weightless point, say x_0. Since $w(x_0) = 0$, we have that $\forall x \in X.\, d(x, x_0) \leq w(x_0) = 0$, and thus $\forall x \in X.\, d(x, x_0) = 0$ or equivalently $\forall x \in X.\, x \leq_d x_0$. So x_0 is the weightless maximum of the weighted space (X, d, w). Hence, by Proposition 6, the space is weighted with respect to x_0. □

Corollary 9. *A weightless point of an upper weighted quasi-metric space is a maximum.*

Proof. Immediate by Lemma 8 and Proposition 6. □

Remark. The fact that a maximum of an upper weighted quasi-metric space is not necessarily weightless is illustrated by example 1) above. We will consider a sufficient condition for the converse of Corollary 9 to hold in Subsection 3.3, based on the notion of fading weight.

In the next section we show that upper weighted spaces are extendible by a weightless point and hence by a maximum (Theorem 11).

3.2. Lifting

The following theorem establishes the close relationship between upper weighted spaces and spaces upper weighted with respect to a point, based on a lifting technique. The technique allows one to extend an upper weighted space by a weightless point such that the extension is upper weighted with respect to this point.

As the weightless point necessarily is a maximum, lifting provides a way to extend a complexity space by a maximum $\top$. Complexity spaces without a maximum arise for instance in the complexity analysis of programs which are required to be total (such as the sorting algorithms considered in [Sch95]). We will illustrate further on that, as might be expected, lifting is a harmless operation.

A natural question arises as to the comparison of the process of lifting of an upper weighted space and the extension of directed spaces by a maximum as discussed in [Sch96]. For more information on this question, we refer the reader to [Sch97].

Definition 10. *A lifting of an upper weighted space (X, d, w) is the triple (X_0, d_0, w_0) obtained in the following way:*

If (X, d, w) has a weightless point then (X_0, d_0, w_0) is defined to be (X, d, w),
otherwise (X_0, d_0, w_0) is defined by:

> $X_0 = X \cup \{x_0\}$, *where* $x_0 \notin X$.
> *The function* w_0 *extends the function* w *by:* $w_0(x_0) = 0$.
> *The function* d_0 *extends the function* d *by:*
> $\forall x \in X_0.\, d_0(x, x_0) = 0$ *and* $d_0(x_0, x) = w_0(x)$.

We also refer to the quasi-metric space (X_0, d_0) as the lifting of the quasi-metric space (X, d).

Theorem 11. *If (X, d, w) is an upper weighted quasi-metric space then its lifting (X_0, d_0, w_0) is a T_0 quasi-metric space which is upper weighted with respect to x_0.*

Proof. Let (X, d, w) be an upper weighted quasi-metric space and let (X_0, d_0, w_0) be its lifting.

If (X, d, w) has a weightless point, say x_0, then since the space is upper weighted, by Lemma 8, the space is upper weighted with respect to x_0 and the result follows.

Otherwise, let x_0 be a point not in X and let (X_0, d_0, w_0) be the lifting corresponding to x_0, obtained as in Definition 10. We verify that (X_0, d_0, w_0) is a T_0 quasi-metric space which is upper weighed with respect to x_0.

I) d_0 is a quasi-metric on X_0.

1) $d_0(x_0, x_0) = 0$ and $\forall x \in X. d_0(x, x) = d(x, x) = 0$.

2) $d_0(x, z) \leq d_0(x, y) + d_0(y, z)$
Case 1: $z = x_0$: trivial since $d_0(x, x_0) = 0$:
Case 2: $z \neq x_0$:

a) $x, y \neq x_0$: immediate since d_0 extends d.

b) $x = x_0$: we must show that $d_0(x_0, z) \leq d_0(x_0, y) + d_0(y, z)$, or equivalently $w_0(z) \leq w_0(y) + d_0(y, z)$.
Since $z \neq x_0$, we have $w_0(z) = w(z)$.
So we need to verify that $w(z) \leq w_0(y) + d_0(y, z)$.
If $y = x_0$, then $d_0(y, z) = w_0(z)$, and thus the inequality holds.
Otherwise, $y \neq x_0$, and the inequality holds by the weightedness of d with respect to w.

c) $y = x_0$: we must show that $d_0(x, z) \leq d_0(x, x_0) + d_0(x_0, z)$, or equivalently that $d_0(x, z) \leq w_0(z) = w(z)$.
If $x = x_0$ then the inequality holds.
Otherwise the inequality holds by upper weightedness of d with respect to w.

II) (X_0, d_0, w_0) is upper weighted with respect to x_0.
It suffices to verify that (X_0, d_0, w_0) is weighted with respect to x_0, that is we need to verify that $\forall x, y \in X_0.\ d_0(x, y) + d_0(x_0, x) = d_0(y, x) + d_0(x_0, y)$.

1) $x, y \neq x_0$: this case follows by the fact that d is weighted with respect to w.
2) $x = x_0$: the equality reduces to $w_0(y) = w_0(y)$.
3) $y = x_0$: the equality reduces to $w_0(x) = w_0(x)$.

III) (X_0, d_0) is T_0.
We need to verify the antisymmetry of the preorder $\leq_{d_0}$. So we need to verify whether $\forall x, y \in X_0. (x \leq_{d_0} y$ and $y \leq_{d_0} x) \Rightarrow (x = y)$.
Let $x, y \in X_0$ such that $x \leq_{d_0} y$ and $y \leq_{d_0} x$.

1) $x, y \neq x_0$: the case follows immediately.
2) $x = x_0$: if $y = x_0$ then $x = y$. So it suffices to consider the case where $y \neq x_0$.
We have in particular that $x_0 \leq_{d_0} y$. Thus $d_0(x_0, y) = 0$ and hence $w(y) = 0$.
So we obtain that y is a weightless point. However this case has been excluded at the beginning of the proof. So the case where $y \neq x_0$ can not arise.
3) $y = x_0$: similar. □

As an application of the technique, upper weighted spaces are characterized as those weighted spaces which have a directed weighted extension.

Proposition 12. *A weighted directed space is upper weighted.*

Proof. Let (X, d, w) be a weighted directed space. Then, if $x, y \in X$, we have that $\exists z \geq_d x, y$ and thus $w(y) - w(z) = d(z, y) \geq d(x, y)$, by weightedness and by the Monotonicity Lemma, which implies that $d(x, y) \leq w(y)$. So the space (X, d, w) is upper weighted. □

Theorem 13. *A weighted space is upper weighted iff it has a directed weighted extension.*

Proof. By Theorem 11, any upper weighted space has a lifting which is an extension by a weightless point such that the extension is weighted with respect to this point. Hence the point is a maximum and thus the extension is directed.

To show the converse, let (X, d, w) be a weighted space which has a directed weighted extension, say (X', d', w'). By Proposition 12, the space (X', d', w') is upper weighted and hence, since upper weightedness is a hereditary property, the space (X, d, w) is upper weighted. □

We have shown that any upper weighted space has an extension which is upper weighted with respect to a point. Hence upper weighted spaces essentially correspond to weighted spaces (X, d, w) with the property
$(*)\, \forall x, y \in X.\, d(x, y) \leq d(x_0, y) = w(y)$.

In other words, the weighting function of an upper weighted space essentially expresses, modulo an extension, the distance from a maximum. Thus property $(*)$ generalizes the interpretation of the weight of a program in the context of the theory of complexity spaces (cf. the remark preceding Subsection 3.1).

By Proposition 2 and Corollary 3, every weightable quasi-metric space with a maximum is strongly upper weighted and all of its weighting functions are "translations" of a given weighting. Also, by Theorem 11, every upper weightable space is extendible by a maximum. So the question arises whether every upper weightable space is strongly upper weighted and has weightings which are uniquely determined by a given weighting function.

Clearly, by Theorem 11 and by Corollary 7, every upper weightable space has a strongly upper weighted extension. However the original upper weightable space may not be strongly upper weighted.

This is illustrated by a metric space (X, d), where d is bounded, say by a constant K, and where X has at least two elements. Such a space is upper weightable by the function

w defined by $\forall x \in X. w(x) = K$. However the space is not strongly upper weighted since it is weightable by the weighting with constant value 0, which is not an upper weighting.

The next section introduces and motivates the notion of fading weight and the final section discusses some properties of weightable directed spaces. The study of these spaces is continued in more detail in [Sch97].

3.3. Fading Weight

Definition 14. *A weighted quasi-metric space is of fading weight iff it has points of arbitrarily small weight. In that case we also say that the weighting function is fading.*

Examples. The spaces $(\mathcal{R}^+, d_1, w_1)$, $(\mathcal{R}^+, d_2, w_2)$, the complexity space (C, d_C, w_C) and the Baire space $(\mathcal{N}^{<\omega}, b, w_b)$ are weighted spaces of fading weight.

A computational interpretation of the notion of fading weight can be given in the context of the theory of complexity spaces. We recall that complexity spaces typically consist of complexity functions of programs which compute a given partial recursive function ([Sch95]). In this context the following property holds: any program can be replaced by a program with larger complexity, in the pointwise ordering on complexity functions, which computes the same partial recursive function as the original program. This property is obvious since in general one can always create programs of larger complexity which compute a given problem.

Since, for a given complexity measure C, the weight of a program P with complexity function C_P is $w(C_P) = \sum_n \frac{1}{C_{P(n)}} \frac{1}{2^n}$, we obtain immediately that any program can be replaced by a program of smaller weight and hence it is reasonable to require complexity spaces to be of fading weight. This assumption will be made implicitly in the remainder of the paper.

We show that for upper weighted spaces of fading weight the notions of a weightless point and of a maximum coincide (cf. the remark made at the end of Subsection 3.1).

Lemma 15. *Given an upper weighted space of fading weight, then a point of the space is weightless iff it is a maximum.*

Proof. Note that by Corollary 9 it suffices to show that a maximum of a an upper weighted space of fading weight is weightless. Assume that x_0 is the maximum of an upper weighted space (X, d, w) of fading weight. By weightedness we know that $\forall x \in X.\, d(x_0, x) + w(x_0) = d(x, x_0) + w(x)$, and thus $\forall x \in X.\, w(x_0) = w(x) - d(x_0, x) \leq w(x)$. So we have that $\forall x \in X.\, w(x_0) \leq w(x)$ and thus, since the space has points of arbitrarily small weight, we obtain that $w(x_0) = 0$. □

Remark 1. Since the lifting of an upper weighted space has a weightless maximum and using Theorem 11, we observe that the lifting of an upper weighted space of fading weight is an upper weighted space of fading weight.

Remark 2. Every weightable space has a fading weighting. This observation has been made in [KV94]. For the convenience of the reader we recall the argument. Let (X, d) be a weightable space, say with a weighting function w. Let $L = \inf\{w(x) | x \in X\}$. It is easy to verify that $w - L$, defined by $\forall x \in X.\, (w - L)(x) = w(x) - L$, is a fading weighting function for the quasi-metric space (X, d).

We remark that an upper weightable space need not have a fading *upper* weighting. Indeed, consider the example of a bounded metric space as discussed above. It is easy to verify that the weighting functions of a metric space are exactly the constant functions. So the only fading weighting is the function with constant value 0, which is not an upper weighting.

In what follows we focus on weightable *directed* spaces, for which problems of this kind do not occur (cf. Proposition 16).

Though we regard the study of weightable directed spaces as of independent theoretical interest (cf. [Sch97]), it is interesting that the directedness condition can be motivated in the context of the theory of complexity spaces.

We need to show that for any two programs P_1 and P_2 and for any given complexity measure C, there exists a program C_P such that C_P dominates C_{P_1} and C_{P_2} in the pointwise order.

It suffices to construct P such that on any given input x, P will call P_1 and execute P_1 on x, after which P calls P_2 and executes P_2 on input x. For instance in an imperative language, with assignments of a term t to a variable z indicated as usual by "$z := t$", we can define such a program by pseudo-code as follows: $\underline{\textit{begin}}\ z_1 := P_1(x); z_2 := P_2(x)\ \underline{\textit{end}}$.

3.4. Weightable Directed Spaces

Proposition 16. *Every weightable directed space is strongly upper weighted. A weightable directed space has a fading upper weighting function.*

Proof. Let (X, d) be a weightable directed space.

If w is a weighting function for (X, d) then w is an upper weighting function by Proposition 12. So by Theorem 11, (X, d, w) has an extension (X_0, d_0, w_0), say by a weightless point x_0, which is weighted with respect to this point. Since $\forall x \in X.\, w_0(x) = d(x_0, x)$, we obtain, by the Monotonicity Lemma, that the function w_0 is decreasing.

Hence (X, d, w) is upper weighted by a strictly decreasing function since strict decreasingness is a hereditary property. Since this holds for all weighting functions of (X, d), the quasi-metric space (X, d) is strongly upper weighted and all of its weighting functions are strictly decreasing.

Finally we remark that by Remark 2, the space has a fading weighting function which is an upper weighting since the space is strongly upper weighted. $\square$

We will show that directed spaces are not metrizable. Also, a characterization of the topology of the lifting of upper weighted spaces will be given in terms of the topology of the original space.

Proposition 17. *If (X, d) is a directed space such that $X \neq \emptyset$, then the quasi-metric space (X, d) is a Hausdorff space iff X is a singleton.*

Proof. The implication "$\Leftarrow$" holds trivially.

Let (X, d) by a directed Hausdorff space. We observe that $x \leq y$ implies that $y \in B_\epsilon[x]$ for all $\epsilon > 0$. If $x, y \in X$ then there is $z \in X$ with $x \leq z$ and $y \leq z$. By the above observation, the point z belongs to all neighbourhoods of both x and y. Since (X, d) is Hausdorrff, this implies that $x = y$. □

Corollary 18. *Directed spaces (X, d), such that X is not a singleton, are not metrizable.*

Proof. Note that if a directed space (X, d) would be metrizable it would be a Hausdorff space and thus X would be a singleton. □

In particular we obtain that the complexity space (C, d_C) is not metrizable, which justifies the intuitive motivation for the non symmetry of the complexity space discussed in [Sch95].

The following proposition provides a characterization of the topology of the lifting of weighted directed spaces of fading weight, which essentially states that the lifting of such a space does not significantly alter its topological structure. In the specific context of complexity spaces this result illustrates in particular that the extension of such a space by the maximum $\top$ does not essentially change the topology of the space. This situation is intuitively similar to traditional domain theory where the extension of a domain by a minimum $\perp$ ("lifting") is a harmless operation (e.g. [DP91]).

Proposition 19. *If (X, d, w) is an upper weighted space and if (X_0, d_0, w_0) is its lifting, say with a maximum x_0, then*
a) $\mathcal{T}_{d_0} = \{O \cup \{x_0\} \mid O \in \mathcal{T}_d; O \neq \emptyset\} \cup \{\emptyset\}$, *in case w is fading,*
b) $\mathcal{T}_{d_0} = \{O \cup \{x_0\} \mid O \in \mathcal{T}_d\} \cup \{\emptyset\}$, *otherwise.*

Proof. Let (X, d, w) be an upper weighted space.

By Theorem 11, the space (X_0, d_0, w_0) obtained via the Lifting Theorem is an upper weighted space, where w_0 is fading when w is fading.

In case w is not fading, it is easy to see that the set $\{x_0\}$ is open and thus $\mathcal{T}_{d_0} = \{O \cup \{x_0\} \mid O \in \mathcal{T}_d\} \cup \{\emptyset\}$.

We continue to discuss the case where w is fading.

In case (X, d, w) has a weightless point, lifting does not change the space. In that case every nonempty open set contains the weightless point (which is the maximum), so $\mathcal{T}_{d_0}$ coincides with $\mathcal{T}_d$.

In case (X, d, w) does not have a weightless point, it suffices to show that $\mathcal{B}_0 = \{(B_\epsilon[x]) \cup \{x_0\} \mid x \in X, \epsilon > 0\}$, where $B_\epsilon[x] = \{y \in X \mid d(x, y) < \epsilon\}$, is a base for $\mathcal{T}_{d_0}$.

Indeed, if this is the case then, under the condition that $O' \neq \emptyset$, we have that $O' \in \mathcal{T}_{d_0} \Leftrightarrow O' = \cup_{i \in I}(B_{\epsilon_i}[x_i] \cup \{x_0\}) \Leftrightarrow O' = (\cup_{i \in I} B_{\epsilon_i}[x_i]) \cup \{x_0\} \Leftrightarrow (O' = O \cup \{x_0\}$ for some $O \in \mathcal{T}_d, O \neq \emptyset)$.

We show that $\mathcal{B}_0$ is a base for $\mathcal{T}_{d_0}$. Let $O' \in \mathcal{T}_{d_0}$ such that $O' \neq \emptyset$ and let $x' \in O'$.

Case 1: $x' = x_0$.

Since $O' \in \mathcal{T}_{d_0}$, there exists an $\epsilon > 0$ such that $B'_\epsilon[x_0] = \{y \in X_0 \mid d_0(x_0, y) < \epsilon\} \subseteq O'$. Pick $y \in X$ such that $w(y) < \frac{\epsilon}{2}$. Then for $0 < \delta < \frac{\epsilon}{2}$ we show that $x_0 \in B_\delta[y] \cup \{x_0\} \subseteq O'$, where $B_\delta[y] = \{z \mid d(y, z) < \delta\}$. It suffices to show that $B_\delta[y] \subseteq O'$ since $x_0 \in O'$. If $z \in B_\delta[y]$ then $d(y, z) < \delta$. Thus $d_0(x_0, z) \leq d_0(x_0, y) + d_0(y, z) = d_0(x_0, y) + d(y, z) = w_0(y) + d(y, z) = w(y) + d(y, z) < \frac{\epsilon}{2} + \delta < \epsilon$. So we obtain that $z \in B'_\epsilon[x_0]$ and thus $B_\delta[y] \cup \{x_0\} \subseteq B'_\epsilon[x_0] \subseteq O'$. So $x_0 \in B_\delta[y] \cup \{x_0\} \subseteq O'$.

Case 2: $x' \in X$.

Again, since $O' \in \mathcal{T}_{d_0}$, there exists an $\epsilon > 0$ such that $B'_\epsilon[x'] = \{y | d_0(x', y) < \epsilon\} \subseteq O'$. Note that $B_\epsilon[x'] \subseteq B'_\epsilon[x']$ since d coincides with d_0 on X. We also have that $x_0 \in B'_\epsilon[x']$ since x_0 is the maximum of $\leq_{d_0}$ and thus $d_0(x', x_0) = 0$. So $x' \in B_\epsilon[x'] \cup \{x_0\} \subseteq B'_\epsilon[x']$. $\square$

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