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TOPOLOGIES ON IDEAL SPACES OF BANACH ALGEBRAS

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Abstract

The ideal space $Id(A)$ of a Banach algebra A can be studied as a bitopological space $(Id(A), \tau_u, \tau_n)$, where τ_u is the weakest topology for which all the norm functions $I \to \|a + I\|$ ($a \in A$, $I \in Id(A)$) are upper semi-continuous, and τ_n is the de Groot dual of τ_u. A survey is given of recent work in this area. When A is separable, $\tau_n \vee \tau_u$ is either a compact, metrizable topology, or it is neither Hausdorff nor first countable. It is shown that the Banach algebra consisting of separable Hilbert space with the zero multiplication exhibits the latter behaviour.

A basic idea of ring theory is to decompose an interesting ring into simpler quotient rings, and then to lift information about the quotient rings back to the original ring by reassembling the pieces, somehow. In its simplest form, this approach leads to sub-direct product decompositions. Given a ring R and a family X of two-sided ideals of R, there is a homomorphism $\rho : R \to \prod_{I \in X} R/I$ given by $\rho(r)(I) = r + I$ ($r \in R$, $I \in X$), and ρ is injective precisely when $\bigcap_{I \in X} I = \{0\}$. The problem of reassembling the pieces amounts to identifying those cross-sections in $\prod_{I \in X} R/I$ which lie in $\rho(R)$.

One possible way of identifying the image of R is to equip X with a topology, and hope that the cross-sections coming from

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elements of R are those which behave nicely with respect to the topology. The classic example here is the Gelfand theory for commutative Banach algebras. Let A be a unital, commutative Banach algebra, and take X to be the set of maximal ideals of A. Then $A/I \cong \mathbf{C}$ for each $I \in X$, by the Gelfand-Mazur theorem, so for $a \in A$, $\rho(a)$ is a complex-valued function on X. The weak topology on X induced by these functions is called the Gelfand topology, and is compact and Hausdorff. Thus ρ is a homomorphism (the Gelfand transform) from A into the algebra $C(X)$ of continuous, complex functions on the compact, Hausdorff space X, and if ρ is injective A is said to be *semisimple*. Even when A is semisimple ρ is not surjective, in general, nor is the norm on A usually equivalent to the supremum norm on $C(X)$, but in spite of these drawbacks the Gelfand transform is an indispensable tool in studying commutative Banach algebras. Furthermore, if A is a unital, commutative C*-algebra, ρ is a norm-preserving isomorphism onto $C(X)$, so the Gelfand transform works perfectly.

The success of the Gelfand theory for commutative C* algebras and Banach algebras leads, inevitably, to a search for a similar bundle representation in the non-commutative situation, and the two main questions that arise are,

(i) What set of ideals should be taken as the base-space X?

(ii) What topology should be used on the base-space?

In order to discuss these questions we shall introduce some notation. Let A be a Banach algebra, and let $Id(A)$ be the lattice of closed ideals of A (all ideals will be closed and two-sided). For $a \in A$ and $I \in Id(A)$, let $\|a + I\|$ denote the quotient norm of the element $a + I$ in the quotient Banach algebra A/I. The functions $Id(A) \to \mathbf{R} : I \mapsto \|a + I\|$ $(I \in Id(A),\ a \in A)$, are called *norm functions*. For non-commutative Banach algebras one can no longer expect the fibres A/I to be isomorphic to the complex numbers, or each other, so the simplest way to introduce topologies on the base-space is to use the norm functions. Thus let τ_l, τ_s, and τ_u be weakest topologies on $Id(A)$ such that all the norm functions are respectively lower semicontinuous, continuous, and upper semicontinuous.

Going back to the questions now, it turns out that neither of them has a unique answer for non-commutative C*-algebras. One cannot guarantee that any single bundle representation will have all the properties that one would wish. Thus the nicest topology of the three for bundle representations is obviously τ_s, because it is Hausdorff, and gives continuity of the norm functions, but it has the drawback that the natural base-space for τ_s is the set of minimal primal ideals of A, which is not always τ_s-compact [1; 4.8]. On the other hand, the natural base-space for τ_l is the set of primitive ideals of A, which is always τ_l-compact, if A is unital, but not always Hausdorff, and one loses the upper semicontinuity of the norm functions, in general. A third possibility for the base-space is the set of Glimm ideals of A, on which the natural topology is τ_u [6]. This time the base-space is τ_u-compact and Hausdorff, if A is unital, but the lower semicontinuity of the norm functions is usually lost, and also the connection with the primitive ideals of A. These are the three base-spaces that have been most commonly used, and in practice one has to work with whichever seems most appropriate, see e.g. [8], [11], [2], and other
references in [2]. In the best circumstances, however, some of the base-spaces coincide. If all three coincide the C*-algebra is said to be *central*, while if the spaces of Glimm and minimal primal ideals coincide the C*-algebra is said to be *quasi-standard* [2]. The class of quasi-standard C*-algebras is the largest class of C*-algebras for which a really satisfactory bundle representation can be found.

Before turning to consider Banach algebras, it will be helpful to look at the three topologies for C*-algebras a little more closely. The lattice $Id(A)$ of closed ideals of a C*-algebra A is a continuous lattice [10; I.1.20], and thus carries the three standard lattice topologies, lower, Lawson, and Scott, and these turn out to be precisely τ_l, τ_s, and τ_u [13]. Furthermore, it follows from the general theory of continuous lattices that τ_s is compact and Hausdorff on $Id(A)$, and that $(Id(A), \tau_u, \tau_l)$ is joincompact, where a bitopological space (X, τ, σ) is *joincompact* [18] if it satisfies the conditions:

(i) the topology $\tau \vee \sigma$ is compact and T_0,

(ii) for any $x, y \in X$, x is in the τ-closure of y if and only if y is in the σ-closure of x,

(iii) for any $x, y \in X$, if x is not in the τ-closure of y, then there exist disjoint sets $T \in \tau$ and $S \in \sigma$ such that $x \in T$ and $y \in S$ (this condition is called *pseudo-Hausdorffness*).

Conditions (ii) and (iii) imply that $\tau \vee \sigma$ is Hausdorff. Condition (ii) is the requirement that the specialization order of (X, σ) should be the reverse of the specialization order of (X, τ), where the *specialization order* $\leq$ on a topological space (X, τ) is given by $x \leq y$ if x is in the τ-closure of y, $(x, y \in X)$.

The importance of joincompact spaces is two-fold. On the one hand there is the natural duality principle: if (X, τ, σ) is joincompact, then clearly (X, σ, τ) is also joincompact. On the other hand there is a rigidity about joincompact spaces: if (X, τ, σ) is joincompact, then σ is necessarily the de Groot dual τ^{dG} of τ [18; 5.3], where given a topological space (X, τ) the *de Groot dual* τ^{dG} of τ is defined to be the topology on X generated by taking the τ-compact, upper subsets (in the specialization order) of X as a sub-base for the closed sets of τ^{dG}.

Let us return now to consider non-commutative Banach algebras. Algebraically these can be much more complicated than C*-algebras, and it is correspondingly more difficult to find an appropriate base-space. For semisimple Banach algebras one can still use the spaces of primitive or Glimm ideals, but these have the same disadvantages as before. So far no satisfactory analogue has been found for the space of minimal primal ideals. One of the aims of this work was to identify the classes of Banach algebras which correspond, from the point of view of bundle representations, to the central C*-algebras and quasi-standard C*-algebras mentioned above. For the former we propose in [20] the class of 'strongly central' Banach algebras (the name 'central' already being in use), but it is not yet clear what definition should be taken for 'quasi-standard' Banach algebras. The second question about topologies on base-spaces seems to be easier to answer, and from now on we shall restrict our attention to that.

Experience with C*-algebras suggests that useful topologies for bundle representations are likely to arise from a joincompact structure, or something akin to it. We begin by noticing that the lattice structure of $Id(A)$ is probably not going to be useful for defining topologies, in general. For instance, the lattice of closed ideals of a Banach algebra A is not always a continuous lattice, see Example 5, and even if $Id(A)$ is a continuous lattice, the lattice topologies might bear little relation to the norm functions, see Example 6. Thus for a general Banach algebra the best that one could hope for would be a joincompact structure on $Id(A)$, involving the norm functions, and respecting the natural order on $Id(A)$, but perhaps not otherwise involving the lattice structure.

The first instinct in this direction is to wonder whether $(Id(A), \tau_u, \tau_l)$ is always joincompact, but as Ferdinand Beckhoff has pointed out [3], this is not the case: the topology $\tau_s = \tau_u \vee \tau_l$ is always Hausdorff, but is not compact in general. There is a further problem, which is that τ_s might not be invariant under changing to an equivalent algebra norm on A, and since this is regarded as a minor operation in Banach algebra theory, it should not be allowed to affect the topology on the base-space of a bundle representation. The difficulty turns out to be with τ_l rather than with τ_u which is unaffected by a change to an equivalent norm. Beckhoff [3], [4] has suggested that instead of asking for lower semicontinuity, one should ask for *normality* of the norm functions: if (I_α) is a net of ideals converging to an ideal I in some topology, one requires that if $a \notin I$ then $\liminf \|a + I_\alpha\| > 0$. This property is invariant under changing to an equivalent norm.

Let us say that a topology τ has the *normality property* if every τ-convergent net has the normality property with respect to each of its limits. We would like to consider the weakest topology having the normality property, but there might not be any such topology. Instead we define a topology τ_n as follows: a set $C \subseteq Id(A)$ is τ_n-closed if and only if whenever it contains a net having the normality property with respect to an ideal I then $I \in C$. Any topology having the normality property is stronger than τ_n, but it is not clear whether τ_n has the normality property. Nevertheless, from the point of view of norm functions, τ_n would seem to be the appropriate replacement for τ_l in general Banach algebras.

Another possible approach suggests itself, however, which is to take τ_u as the basic topology, since it is invariant under changing to an equivalent norm. One is then led to consider τ_u^{dG}, since, by Kopperman's theorem [18; 5.3], if there is a topology σ such that $(Id(A), \tau_u, \sigma)$ is joincompact, then σ necessarily is τ_u^{dG}. Happily, these two approaches turn out to give the same result.

Proposition 1. [20; 2.5] *Let A be a Banach algebra. Then $\tau_n = \tau_u^{dG}$.*

Thus if one hopes for a joincompact structure on $Id(A)$, with topologies coming from the norm functions, then $(Id(A), \tau_u, \tau_n)$ is the natural candidate. Furthermore, it automatically satisfies conditions (i) and (ii) of joincompactness, so the only question concerns the pseudo-Hausdorffness. Let $\tau_r = \tau_u \vee \tau_n$. Then τ_r is compact and T_1, and we would like to know if it is Hausdorff. In fact, a dichotomy emerges: either τ_r is Hausdorff and both τ_r and τ_n have the normality property, or else all these desirable conditions fail.

Theorem 2. [20; 2.12] *Let A be a Banach algebra. The following are equivalent:*
(i) $(Id(A), \tau_n, \tau_u)$ is a joincompact space,
(ii) τ_r is Hausdorff,
(iii) τ_r has the normality property,
(iv) τ_n has the normality property.

For separable Banach algebras the dichotomy is even sharper: if τ_r is Hausdorff then it is metrizable—otherwise it is not even first countable.

Theorem 3. [20; 2.13] (based on [4; Theorem 6]) *Let A be a separable Banach algebra. Then the following are equivalent:*
(i) τ_r is Hausdorff,
(ii) τ_r is first countable,
(iii) τ_r is second countable,
(iv) τ_n is first countable,
(v) τ_n is second countable.

We shall give examples of both possibilities in a moment. But first we note that, irrespective of the dichotomy, we can answer the question about the correct topology for the base-space of a bundle representation. If a Banach algebra does have a representation as a bundle of Banach algebras, with good behaviour of the norm functions, then the topology on the base-space is necessarily τ_r. Furthermore, if the base-space is a 'horizontal slice' through $Id(A)$ then τ_r, τ_u, and τ_n all coincide on it.

Theorem 4. [20; 2.11, 4.7] *Let A be a Banach algebra.*

Suppose that τ is a compact topology on a subset $X \subset Id(A)$, such that τ-convergent nets have the normality property, and such that $\tau \supseteq \tau_u|_X$. Then τ is Hausdorff and $\tau = \tau_r|_X$. If, furthermore, $I \subseteq J \implies I = J$ for $I, J \in X$ then $\tau_r|_X = \tau_u|_X = \tau_n|_X$.

For example, if A is a unital, commutative Banach algebra then τ_r, τ_u, and τ_n all coincide, on the set of maximal ideals of A, with the Gelfand topology. If A is a C*-algebra then τ_r is equal to τ_s on $Id(A)$ (and τ_n of course is equal to τ_l).

Let us now give some examples to illustrate Theorems 2 and 3. The class of Banach algebras for which τ_r is Hausdorff on $Id(A)$ includes C*-algebras, TAF-algebras, commutative Banach algebras with spectral synthesis, various radical Banach algebras, and the Banach algebra $C^1[0,1]$ of continuously differentiable complex functions on $[0,1]$, see [20], [21], [7]. It follows from work of Beckhoff [4; Prop. 7] that finite-dimensional Banach algebras also belong to this class. On the other hand if A is a uniform algebra then τ_r is Hausdorff on $Id(A)$ if and only if A has spectral synthesis [7].

Example 5. Let A be the disc algebra, that is, the algebra of continuous functions on the unit disc which are analytic in the interior of the disc, equipped with the supremum norm. Then A is a uniform algebra without spectral synthesis, so τ_r is not Hausdorff on $Id(A)$. The ideal structure of A was completely determined by Beurling and Rudin, see [14], and it was shown by Lamoureux [19; 2.4] that there are elements of $Id(A)$ which are not an intersection of meet-irreducible ideals. This is impossible in a continuous lattice [10; 3.10], so $Id(A)$ is not continuous.

There are comparatively few Banach algebras for which a complete description of the closed ideals is known, but of course one does not need to know all the closed ideals, but only some suitable subset, in order to obtain a useful bundle representation.

One trivial way of manufacturing Banach algebras is to take an arbitrary Banach space B and define the product of any two elements of B is to be 0. Then B becomes a Banach algebra whose closed two-sided ideals are precisely the closed subspaces of B. Thus the dichotomy of Theorems 2 and 3 applies to the lattice of closed subspaces of a Banach space. We give an example to illustrate each side of the dichotomy.

Example 6. Let $A = \mathbf{C}^2$ with zero multiplication. Then A is finite-dimensional, so τ_r is Hausdorff on $Id(A)$, as we mentioned above. Clearly $Id(A)$ consists of $\{0\}$, A itself, and all the one-dimensional subspaces of A, which can be parametrized by the Riemann sphere S^2. The restriction of τ_r to the sphere is just the usual Euclidean topology, while A and $\{0\}$ are τ_r-isolated points.

It is curious to note that $Id(A)$ is a continuous lattice, but its Lawson topology is quite different from τ_r. Every point of $Id(A)$ is Lawson-isolated, except for $\{0\}$, so $(Id(A), \text{Lawson})$ is the one-point compactification of an uncountable discrete space.

One of the questions left open in [20] was whether τ_r is Hausdorff on $Id(H)$, when H is an infinite-dimensional Hilbert space with the zero multiplication. Here we are able to answer this question.

Let $\langle \cdot\,, \cdot \rangle$ denote the inner product on H, and let $B(H)$ be the algebra of bounded linear operators on H. Recall that the *strong operator topology* (SOT) is the weakest topology on $B(H)$ for which all the maps $T \mapsto \|Tx\|$ ($x \in H$, $T \in B(H)$) are continuous, while the *weak operator topology* (WOT) is the weakest topology on $B(H)$ for which all the maps $T \mapsto \langle Tx, y \rangle$ ($x, y \in H$, $T \in B(H)$) are continuous. Since $|\langle Tx, y\rangle| \leq$

$\|Tx\|\,\|y\|$, SOT is indeed stronger than WOT. A *projection* P in $B(H)$ is a self-adjoint operator such that $P^2 = P$. The strong and weak operator topology coincide on the set of projections, giving a topology which is Hausdorff but not compact [12; Q.115]. An operator $T \in B(H)$ is *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Example 7. *Let H be an infinite-dimensional Hilbert space, with the zero muliplication. Then $(Id(H), \tau_r)$ is not Hausdorff.*

Proof. The closed ideals of H are precisely the closed subspaces of H, as we observed above, and we can identify these with the projections in $B(H)$, as follows. Each closed subspace K is identified with the projection P_K onto its orthogonal complement $K^\perp$. Thus $K = \ker P_K$, and the norm functions $K \mapsto \|x + K\|$ are the maps $P_K \mapsto \|P_K x\|$, $x \in H$. With this identification, the τ_r-topology gives a compact topology on the set of projections of $B(H)$.

Next we note that the topology τ_r is weaker than the relative strong operator topology on the set of projections. To see this, suppose that (P_α) is an SOT-convergent net of projections with limit P, where P is a projection. Then $\|P_\alpha x\| \to \|Px\|$ for all $x \in H$, so (P_α) has the normality property with respect to P, and $P_\alpha \to P$ (τ_u). Thus $P_\alpha \to P$ (τ_r), as required.

Since H is infinite-dimensional, the closure of the set of projections of $B(H)$ in the weak operator topology is the set of positive operators $T \in B(H)$ such that $\|T\| \leq 1$ [12; Q.225]. In particular each operator of the form mI, where I is the identity operator on H, and $m \in \mathbf{R}$ with $0 \leq m \leq 1$, is in the WOT-closure of the set of projections. Let $m \in \mathbf{R}$ with $0 < m \leq 1$, and let (P_α) be a net of projections converging (WOT) to mI. Then any subnet of (P_α) has, by τ_r-compactness, a τ_r-convergent subnet (P_β) with limit P, say, where P is a projection. Let $x \in \ker P$. Then $\|P_\beta x\| \to 0$, by τ_u-convergence of (P_β) to P, so $\langle P_\beta x, x\rangle \leq \|P_\beta x\|\,\|x\| \to 0$.

But since (P_β) converges (WOT) to mI, $\langle P_\beta x, x\rangle \to m\langle x, x\rangle$. Hence $m\langle x, x\rangle = 0$, so $x = 0$. Thus $\ker P = \{0\}$, so $P = I$. It follows that the original net (P_α) converges to I (τ_r).

Now for each $n \in \mathbf{N}$, choose a net $(P_{\alpha(n)})_{\alpha(n)}$ converging (WOT) to $(1/n)I$, and hence converging (τ_r) to I, by the previous paragraph. Then $\lim_n \lim_{\alpha(n)} P_{\alpha(n)} = I$ in τ_r, while $\lim_n \lim_{\alpha(n)} P_{\alpha(n)} = \lim_n (1/n)I = 0$ in WOT. Thus by [15; §2, Theorem 4] there exists a net (P_γ) of projections converging to I (τ_r) and to 0 (WOT). But SOT and WOT agree on the set of projections, as we mentioned above, and 0 is a projection, so $P_\gamma \to 0$ (SOT). Since (SOT) is stronger than τ_r on the set of projections, it follows that $P_\gamma \to 0$ (τ_r). Thus τ_r is not Hausdorff on $Id(H)$. □

Note that, with (P_γ) as above, the net $(\ker P_\gamma)$ converges (τ_u) to everything in $Id(H)$, since it converges (τ_u) to $H = \ker 0$. On the other hand $(\ker P_\gamma)$ converges (τ_n) to everything in $Id(H)$, since it converges (τ_n) to $\{0\} = \ker I$. Thus $(\ker P_\gamma)$ converges (τ_r) to the whole of $Id(H)$.

Let us conclude by mentioning some other work on ideal spaces of Banach algebras. In [3] Beckhoff introduced the topology τ_∞ on $Id(A)$, which is compact, invariant under a change to an equivalent norm, and stronger than τ_n. It is closely related to the property of spectral synthesis [5], [21]. Since it is less often Hausdorff than τ_r [20], it is less suitable than τ_r as a topology on the base-space for bundle representations. Nevertheless, it seems to be an important topology.

In [9] Fell studied a topology on the set of equivalence classes of finite-dimensional, irreducible representations of a Banach algebra A. This induces a topology, which we call τ_F, on the set X_k of primitive ideals of A whose codimension is bounded by k^2 $(k \in \mathbf{N})$. Fell's work shows that τ_F is evidently the 'right' topology to use on X_k. For instance it is locally compact, and has the normality property and the Baire property. It was shown in [20; 5.7] that if A is separable and unital then τ_F and τ_n coincide on X_k.

Finally we should mention the work of Kitchen and Robbins on Banach bundle representations of Banach algebras, see e.g. [16], [17], but this is not closely related to the present work.

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