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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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CLOSED IDEALS IN $C(X)$ AND Φ-ALGEBRAS

Ross Stokke

Abstract

Given a topological space X, the ring $C(X)$ of continuous real-valued functions on X is endowed with the uniform metric. The closed ideals of $C(X)$ in this metric are of interest, and a new, purely algebraic characterization of these ideals is provided. The result is applied to describe the real maximal ideals of $C(X)$, and to characterize several types of topological spaces. A Φ-algebra is an archimedian lattice-ordered algebra closely related to $C(X)$. z-ideals in Φ-algebras are examined, and as an application to this study, several conditions equivalent to regularity in a Φ-algebra are obtained. A uniform metric may also be placed upon a Φ-algebra, and we give necessary and sufficient conditions to ensure that an ideal of a Φ-algebra is closed. Moreover, for two broad classes of Φ-algebras we show that these conditions are equivalent, thus generalizing our characterization from the $C(X)$ case.

1. Closed Ideals in $C(X)$

Beginning with a (Tychonoff) space X, the *uniform metric* ρ on $C(X)$ is defined by

$$\rho(f,g) = sup\{|f(x) - g(x)| \wedge 1 : x \in X\}, \qquad (f,g \in C(X)).$$

In the sequel, all topological properties of $C(X)$ will be with respect to the uniform topology. As in [GJ], the convention that all ideals are proper will be followed.

If K is a compact space then the uniform topology on $C(K)$ coincides with the supremum norm topology on $C(K)$, in which case the closed ideals of $C(K)$ are precisely the intersections of maximal ideals of $C(K)$, [GJ, 4O]. In the case of an arbitrary space X, a characterization of the closed ideals of $C(X)$ was obtained by Nanzetta and Plank in [NP], however their description, (unlike the one quoted above for closed ideals of $C(K)$, where K is compact), is highly non-algebraic in nature. In this section, we present a new, purely algebraic characterization of the closed ideals of $C(X)$ and apply the result to describe the real maximal ideals of $C(X)$ and to characterize pseudocompact spaces. Where there is overlap with [NP], the proofs given are independent of, and perhaps simpler than those found in that paper.

The principal tool to be used is the concept of strongly divisible ideals, which was introduced in [A1], and used there to characterize Lindelöf spaces.

Definition 1.1. *An ideal I of a commutative ring R is called strongly divisible if for every countable subset $\{a_n : n \in \mathbf{N}\}$ of I there is an $a \in I$ and a subset $\{b_n : n \in \mathbf{N}\}$ of R such that for each $n \in \mathbf{N}$, $ab_n = a_n$.*

One immediately observes that any principal ideal is strongly divisible, and any countably generated strongly divisible ideal is principal. The first part of the following theorem is by F. Azarpanah and is found in [A1]. Its proof is not long and so for the sake of completeness we choose to include it.

Theorem 1.2. *Let X be a Tychonoff space.*
1. [Azarpanah] If I is a z-ideal of $C(X)$ such that $Z[I]$ is closed under countable intersection, then I is strongly divisible.
2. If I is strongly divisible, then $Z[I]$ is closed under countable intersection.

Proof. 1. Let $(f_n) \subset I$. Because $C(X)$ is closed under uniform convergence, $g = \sum_{n=1}^{\infty} 2^{-n} f_n^{2/3}(1 + f_n^{2/3})^{-1}$ belongs to $C(X)$, and it is clear that $Z(g) = \bigcap_{n=1}^{\infty} Z(f_n)$. $Z[I]$ is closed under countable intersection, and therefore $Z(g) \in Z[I]$; it follows that g belongs to the z-ideal I. But for each n, $g \geq 2^{-n}\frac{f_n^{2/3}}{1+f_n^{2/3}}$, and therefore $|f_n| \leq |2^n(1+f_n^{2/3})g|^{3/2}$. By [GJ, 1D], each f_n is a multiple of $2^n(1+f_n^{2/3})g$, hence each f_n is a multiple of g. Therefore I is strongly divisible.

2. Let $(f_n) \subset I$, so that $\{Z(f_n) : n \in \mathbf{N}\}$ is an arbitrary subset of $Z[I]$. I is strongly divisible, so take $g \in I$ such that for each n, g divides f_n. It is clear then that $Z(g) \subset \bigcap_{n=1}^{\infty} Z(f_n)$. But $Z[I]$ is a z-filter containing $Z(g)$, so $\bigcap_{n=1}^{\infty} Z(f_n) \in Z[I]$. □

Corollary 1.3. *1. A maximal ideal of $C(X)$ is real if and only if it is strongly divisible.*
2.A space X is pseudocompact if and only if every maximal ideal of $C(X)$ is strongly divisible.

Proof. A maximal ideal M of $C(X)$ is real if and only if $Z[M]$ is closed under countable intersection, [GJ, 5.14], if and only if M is strongly divisible (by the above theorem). 2. X is pseudocompact if and only if $\upsilon X = \beta X$ if and only if every maximal ideal is real. □

If A is a subset of $C(X)$ then $\overline{A}$ will denote its (uniform) closure in $C(X)$.

Theorem 1.4. *The following are equivalent for an ideal I of $C(X)$.*
1. I is closed (in the uniform topology on $C(X)$).
2. I is a strongly divisible z-ideal.

Proof. 1. $\Rightarrow$ 2. Let I be a closed ideal of $C(X)$. Suppose $f \in C(X)$ and $g \in I$ are such that $Z(f) = Z(g)$. For every positive integer n, let $h_n = [(f - 1/n) \vee 0] + [(f + 1/n) \wedge 0)]$. Then for each n, $Z(h_n) = f^{\leftarrow}[-1/n, 1/n]$, so $Z(g) = Z(f) \subset intZ(h_n)$. By [GJ, 1D], each h_n is a multiple of g, hence each $h_n \in I$. But $|f - h_n| \leq 2/n \to 0$ as $n \to \infty$, so h_n converges to

f uniformly (i.e. h_n converges to f in the uniform topology). Since I is closed it follows that $f \in I$, proving that I is a z-ideal. It remains to prove that I is strongly divisible.

Let (f_n) be a countable subset of I. By (1.2) , to prove strong divisibility it suffices to show that $\bigcap_{n=1}^{\infty} Z(f_n) \in Z[I]$. Because $C(X)$ is closed under uniform convergence, $g = \sum_1^{\infty} |f_n| \wedge 2^{-n}$ belongs to $C(X)$. If for each $n \in \mathbf{N}$, $g_n = \sum_{k=1}^{n} |f_k| \wedge 2^{-k}$, then it is clear that the sequence (g_n) converges to g uniformly. But for each n, $Z(g_n) = \bigcap_{k=1}^{n} Z(f_k) \in Z[I]$, and I is a z-ideal, so each $g_n \in I$. I is closed, so $g \in I$ and therefore $\bigcap_{n=1}^{\infty} Z(f_n) = Z(g) \in Z[I]$.

2. $\Rightarrow$ 1. Let $\bar{I}$ denote the closure of I in $C(X)$ and let $f \in \bar{I}$. Then for every positive integer n, take $f_n \in I$ such that $|f - f_n| \leq 1/n$. If $x \in \bigcap_{n=1}^{\infty} Z(f_n)$, then for each n, $|f(x)| = |f(x) - f_n(x)| \leq 1/n$ and therefore $x \in Z(f)$. Hence $\bigcap_{n=1}^{\infty} Z(f_n) \subset Z(f)$. By (1.2), $\bigcap_{n=1}^{\infty} Z(f_n) \in Z[I]$ and thus $Z(f) \in Z[I]$. But I is a z-ideal, so $f \in I$. Thus $I = \bar{I}$ and I is closed. □

Recall that [GJ, 4A] asserts that the following algebraic condition is necessary and sufficient for an ideal I in $C(X)$ to be a z-ideal:

Given $f \in C(X)$, if there exists $g \in I$ such that f belongs to every maximal ideal containing g, then $f \in I$.

Thus, as claimed, 1.4 is algebraic in character.

The following extends Corollary 4.3 of [NP].

Corollary 1.5. *The following are equivalent for an ideal I of $C(X)$.*

1. *I is real.*
2. *I is a closed maximal ideal of $C(X)$.*
3. *I is a maximal closed ideal of $C(X)$.*
4. *I is a maximal strongly divisible ideal of $C(X)$.*
5. *I is a strongly divisible maximal ideal of $C(X)$.*

Proof. The equivalence of 1., 2., and 5. is clear from (1.3) and (1.4); that 2. implies 3. is obvious.

3. $\Rightarrow$ 4. If $I \subset J$, where J is strongly divisible, then $Z^{\leftarrow}[Z[J]]$ is a strongly divisible z-ideal (by 1.2), hence closed by (1.4), with $I \subset J \subset Z^{\leftarrow}[Z[J]]$. By 3. $I = Z^{\leftarrow}[Z[J]]$, hence $I = J$. Therefore I is a maximal strongly divisible ideal of $C(X)$.
4. $\Rightarrow$ 3. Were $I \subset J$, with J closed then J is strongly divisible by (1.4), hence $I = J$ by 4.
3. $\Rightarrow$ 5. Supposing I is not maximal, take M a maximal ideal of $C(X)$ with I properly contained in M. If $f \in M \backslash I$, then we claim that

$$\mathcal{B} = \{Z(f) \bigcap Z(g) : g \in I\}$$

is a base for a z-filter $\mathcal{A}$ on X that is closed under countable intersection. To see this, observe that $\mathcal{A} \subset Z[M]$, $Z[M]$ a z-filter so $\emptyset \notin \mathcal{A}$. I is closed, therefore strongly divisible, hence by (1.2) $Z[I]$ is closed under countable intersection; it follows that $\mathcal{B}$ is closed under countable intersection, whence $\mathcal{A}$ is closed under countable intersection.
From (1.2) $Z^{\leftarrow}[\mathcal{A}]$ is a strongly divisible z-ideal, hence a closed ideal of $C(X)$ containing I. But $f \in Z^{\leftarrow}[\mathcal{A}]$, $f \notin I$, so this containment is proper, a contradiction to 3. □

That an ideal is merely strongly divisible does not alone guarantee that it is closed, that is, strongly divisible ideals need not be z-ideals. For example, if $\mathbf{i}$ denotes the identity funtion on $\mathbf{R}$, then the principal, (hence strongly divisible) ideal $(\mathbf{i})$ is not a z-ideal in $C(\mathbf{R})$; see [GJ, 2.4]. The following is a slight extension of Theorem 2.1 of [NP]. A shorter though less interesting proof of their final implication is given.

Corollary 1.6. *The following are equivalent:*
1. X is pseudocompact.
2. The closure of any ideal in $C(X)$ is an ideal.
3. Every ideal in $C(X)$ is contained in a strongly divisible ideal.
4. Every ideal in $C(X)$ is contained in a closed ideal.

Proof. 1. $\Rightarrow$ 2. If X is pseudocompact, then the map $C(\beta X) \to C(X) : f \to f|X$ is an isometric isomorphism. Since βX is compact, by [GJ,2M] closures of ideals of $C(\beta X)$ are ideals. 2. follows.

2. $\Rightarrow$ 3. Clear from (1.4).

3. $\Rightarrow$ 4. Let I be a strongly divisible ideal of $C(X)$. Then $Z[I]$ is closed under countable intersection by (1.2). But $J = Z^{\leftarrow}[Z[I]]$ is a z-ideal, with $Z[J] = Z[I]$ closed under countable intersection. Hence J is a strongly divisible z-ideal, therefore closed, and $I \subset J$. Thus 4. follows from 3.

4. $\Rightarrow$ 1. By 4., every maximal ideal is closed, therefore real by (1.5). Hence $vX = \beta X$ and X is pseudocompact. □

We now attempt to determine which prime ideals of $C(X)$ are closed in the uniform topology.

Lemma 1.7. *For all $p \in \beta X$, $\overline{O^p} = \overline{M^p}$.*

Proof. It is enough to show that $M^p \subset \overline{O^p}$, so let $f \in M^p$ and let $\varepsilon > 0$. Let $g = [(f - \varepsilon) \vee 0] + [(f + \varepsilon) \wedge 0]$. Then $Z(g) = f^{\leftarrow}[-\varepsilon, \varepsilon]$. Now by [GJ,7D] $f^*(p) = 0$, where $f^* : \beta X \to \mathbf{R}^*$ is the (unique) continuous function from βX into $\mathbf{R}^*$, (the one point compactification of $\mathbf{R}$), such that $f^*|X = f$. By [GJ, 7.12], $Z(g) \in Z[O^p]$ and therefore, since O^p is a z-ideal, $g \in O^p$. But $|f - g| < \varepsilon$, hence $f \in \overline{O^p}$. □

[GJ, 7.15] asserts that for every prime ideal P of $C(X)$, there is a unique $p \in \beta X$ such that $O^p \subset P \subset M^p$. It follows from the lemma that no non-maximal prime ideal of $C(X)$ is closed. However, as noted earlier, the closed maximal ideals of $C(X)$ are precisely the real ideals of $C(X)$, i.e. M^p is closed if and only if $p \in vX$. Hence we have

Corollary 1.8. *Let X be a Tychonoff space.*
1. If $p \in \beta X$ then $\overline{O^p} = M^p$ if and only if $p \in vX$.
2. Let P be a prime ideal of $C(X)$, M^p the unique maximal ideal containing P. Then $\overline{P}$ is a (necessarily maximal) ideal if and only if $p \in vX$.

A space X is pseudocompact if and only if $\beta X = \upsilon X$. Hence we get the following extension of (1.6):
A (Tychonoff) space X is pseudocompact if and only if the closure of any prime ideal of $C(X)$ is a (maximal) ideal.

If $p \in \upsilon X$ and $O^p \neq M^p$, then O^p is a proper dense subset of M^p that fails to be strongly divisible by (1.4). Thus even if the closure of an ideal is an ideal, it may fail to be strongly divisible.

2. Spaces X For Which Every Countably Generated Ideal of $C(X)$ is Principal

As an application to the above, in this section it is shown that those spaces with the property of the preceding title are precisely the finite (Tychonoff) spaces. The result is perhaps surprising given that *F-spaces* - spaces X for which every finitely generated ideal in $C(X)$ is principal- may be non-discrete and of uncountably infinite cardinality, for example $\beta\mathbf{N}\backslash\mathbf{N}$, [GJ, 14.27]. Our theorem hinges upon the following observation.

Proposition 2.1. *Let R be a commutative ring. Then every ideal in R is strongly divisible if and only if every countably generated ideal of R is principal.*

Proof. That strongly divisible, countably generated ideals are principal is obvious. For the converse let I be an ideal of R and let $\{a_n\}$ be any countable subset of I. Then $J = (a_n : n = 1, 2, ...)$ the ideal generated by $\{a_n\}$ is principal, say $J = (a)$, where $a \in R$. But a divides each a_n, and since $J \subset I$, $a \in I$. Hence I is strongly divisible. □

A result of De Marco's, [D], states that if X is compact, then the ideal $I = \bigcap_{p \in S} O_p$, where S is a zero-set of X, is countably generated. Recall that a topological space is a *P-space* if each of its zero-sets is open.

Theorem 2.2. *Every countably generated ideal of $C(X)$ is principal if and only if X is finite.*

Proof. If every countably generated ideal of $C(X)$ is principal, then by (2.1), every ideal of $C(X)$ is strongly divisible, and consequently, X is pseudocompact (1.6). Suppose for now that X is compact and let S be a zero-set of X. By the lemma, the ideal $I = \bigcap_{p \in S} O_p$ is countably generated and therefore principal by hypothesis, so take $f \in I$ such that $(f) = I$. Clearly then, $Z(f) = S$, but since $f \in I$, it is also true that $S \subset intZ(f)$. Hence $S = Z(f)$ is open and therefore X is a P-space. But compact P-spaces are finite [GJ, 4K], so X is finite. If X is assumed only to be pseudocompact, then $C(X)$ is ring isomorphic to $C(\beta X)$, so every countably generated ideal of $C(\beta X)$ is principal, and by the above argument βX is finite, whence X is finite.

Conversely suppose X is finite, and let $I = (f_n : n = 1, 2, ...)$ be the countably generated, (proper, non-trivial) ideal of $C(X)$ generated by $\{f_n : n = 1, 2, ...\}$. By the blanket assumption that all spaces are Tychonoff, X is discrete (and compact). Therefore the zero-set $S = \bigcap Z[I] = \bigcap_n Z(f_n)$ is a non-empty proper subset of X, and so the characteristic function, call it f, on $X \backslash S$ is a non-unit in $C(X)$. Now $Z(f) = S$, so for each n, $Z(f) \subset Z(f_n)$, and clearly $ff_n = f_n$, hence $(f) \supset I$. On the other hand, it is easy to see that for some n, $\bigcap_{k=1}^{n} Z(f_k) = S$. Define g to be $f_1^2 + ... + f_n^2$; then $g \in I$, and $Z(g) = Z(f)$. By the discreteness of X, [GJ, 1D] implies that f is a multiple of g. It follows that $(f) = I$. □

Corollary 2.3. *Every ideal of $C(X)$ is closed if and only if X is finite.*

Proof. Closed ideals are strongly divisible, which together with (2.1) and (2.5), provides necessity. If X is finite, then by the above every countably generated ideal of $C(X)$ is principal, hence every ideal of $C(X)$ is strongly divisible. As a disrete space, X is also a P-space, therefore every ideal of $C(X)$ is also a z-ideal, [GJ, 4J]. Thus every ideal of $C(X)$ is closed, by (1.4). □

3. Weakly Lindelöf Spaces

In F. Azarpanah's paper [A1], the notion of strongly divisible ideals is used to characterize (Tychonoff) Lindelöf spaces as those spaces X such that every strongly divisible ideal of $C(X)$ is fixed. Here we give a similar characterization for weakly Lindelöf spaces.

Recall that a space X is *weakly Lindelöf* if every open cover of X contains a countable subfamily whose union is dense in X, [PW]. A collection $\mathcal{C}$ of subsets of a space X will be said to have the *strong countable intersection property* (SCIP) if the intersection of any countable subfamily of $\mathcal{C}$ has non-empty interior. A subset of a topological space is *regular closed* if it is the closure of an open set.

Theorem 3.1. *The following are equivalent for a topological space X*
(1) X is weakly Lindelöf.
(2) If $\mathcal{C}$ is a family of closed subsets of X with SCIP then $\bigcap \mathcal{C} \neq \emptyset$.
(3) If $\mathcal{C}$ is a family of regular closed subsets of X with SCIP then $\bigcap \mathcal{C} \neq \emptyset$.
(4) If $\mathcal{U}$ is any family of open subsets of X with SCIP then $\bigcap\{cl(U) : U \in \mathcal{U}\} \neq \emptyset$.
(5) If $\mathcal{C}$ is a family of basic closed subsets of X with SCIP then $\bigcap \mathcal{C} \neq \emptyset$.

Proof. We prove only the implication (4) $\Rightarrow$ (1). It is routine to verify that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), and that (2) $\Leftrightarrow$ (5) is immediate.

(4) $\Rightarrow$ (1) Supposing (1) is false let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X such that the union of no countable subfamily

of $\mathcal{U}$ is dense in X. Let $\mathcal{D} = \{D \subset A : D \text{ is countable}\}$. For each $D \in \mathcal{D}$ let $V_D = X \backslash cl[\bigcup_{\alpha \in D} U_\alpha]$. Then each V_D is open and by the hypothesis on $\mathcal{U}$, each $V_D \neq \emptyset$. Let $\mathcal{V} = \{V_D : D \in \mathcal{D}\}$. Then $\mathcal{V}$ has SCIP. For if (D_n) is a countable subfamily of $\mathcal{D}$, then $D = \bigcup D_n \in \mathcal{D}$ and it is routine to verify that $\bigcap_n V_{D_n} \supset V_D$. But V_D is open, non-empty and therefore $int(\bigcap_n V_{D_n}) \neq \emptyset$, hence $\mathcal{V}$ does indeed have SCIP.

But

$$\begin{aligned}
\bigcap cl(V) : V \in \mathcal{V} &= \bigcap cl(V_D) : D \in \mathcal{D} \\
&= \bigcap_{D \in \mathcal{D}} cl[X \backslash cl(\bigcup_{\alpha \in D} U_\alpha)] \\
&= \bigcap_{D \in \mathcal{D}} X \backslash int[cl(\bigcup_{\alpha \in D} U_\alpha)] \subset \bigcap_{D \in \mathcal{D}} (X \backslash \bigcup_{\alpha \in D} U_\alpha) \\
&= X \backslash (\bigcup_{D \in \mathcal{D}} \bigcup_{\alpha \in D} U_\alpha) \\
&= X \backslash \bigcup_{\alpha \in A} U_\alpha = X \backslash X = \emptyset,
\end{aligned}$$

since $\mathcal{U}$ covers X. Therefore (4) is false. □

Definition 3.2. *A strongly divisible ideal I of a commutative ring R comprised entirely of divisors of zero will be called neighborhood strongly divisible, or simply nsd.*

Recall that a member f of $C(X)$ is a divisor of zero if and only if $Z(f)$ has non-empty interior. Thus an ideal I of $C(X)$ is nsd if given any countable subfamily (f_n) of I there is a $g \in I$ and $(h_n) \subset C(X)$ such that for each n, $f_n = gh_n$, and $Z(g)$ has non-empty interior. The proof of the following lemma is similar to that of (1.2).

Lemma 3.3. *Let X be Tychonoff, I a z-ideal of $C(X)$. Then I is nsd if and only if $Z[I]$ has SCIP and is closed under countable intersection.*

Theorem 3.4. *If X is a (Tychonoff) space, then X is weakly Lindelöf if and only if every nsd ideal of $C(X)$ is fixed.*

Proof. Suppose X is weakly Lindelöf, I a nsd ideal of $C(X)$. Then, given any countable subset (f_n) of I there is a $g \in I$ and $(h_n) \subset C(X)$ such that for each n, $f_n = gh_n$, and $intZ(g) \neq \emptyset$. Therefore, $intZ(g) \subset Z(g) \subset \bigcap_n Z(f_n)$, hence $int[\bigcap_n Z(f_n)] \neq \emptyset$. Therefore $Z[I]$ has SCIP, so by (3.1), $\bigcap Z[I] \neq \emptyset$, that is I is fixed.

Conversely let $\mathcal{A}$ be a collection of zero-sets (basic closed in Tychonoff X), of X with SCIP. Let $\mathcal{B}$ be the collection of all countable intersections of members of $\mathcal{A}$. Then $\mathcal{B}$ is a base for a z-filter $\mathcal{F}$ on X which has SCIP and is closed under countable intersection. Therefore $Z^{\leftarrow}[\mathcal{F}]$ is a z-ideal and $Z[Z^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$ has SCIP, and is closed under countable intersection. By the lemma, $Z^{\leftarrow}[\mathcal{F}]$ is nsd. Hence, by 2., $Z^{\leftarrow}[\mathcal{F}]$ is fixed, so $\bigcap \mathcal{F} \neq \emptyset$. But $\mathcal{A} \subset \mathcal{F}$ so $\bigcap \mathcal{A} \neq \emptyset$, and X is weakly Lindelöf. □

As observed by Professor Henriksen, a corollary to the above is the following theorem which is part of [RW, 5.11].

Corollary 3.5. *Weakly Lindelöf almost P-spaces are Lindelöf.*

Proof. Zero-sets of almost P-spaces have non-empty interior by definition. The result follows immediately from the above and the Azarpanah characterization of Lindelöf spaces, which was quoted at the beginning of this section. □

4. Φ-algebra Preliminaries

A Φ-algebra is an archimedian lattice-ordered algebra over the field $\mathbf{R}$ of real numbers in which 1 is a weak-order unit. In this section we describe some of the basic results on Φ-algebras to be used in this paper. Details of this brief survey, as well as any undefined concepts and notation, may be found without exception in the paper [HJ], by M. Henriksen and D. Johnson. An essentially complete history of the subject may be found in [H2].

Let $\gamma\mathbf{R} = \mathbf{R} \cup \{\pm\infty\}$ denote the two-point compactification of the real field $\mathbf{R}$. For a compact space X, let $D(X)$ denote the set of all continuous functions $f : X \to \gamma\mathbf{R}$ for which

$$\mathcal{R}(f) = \{x \in X : f(x) \in \mathbf{R}\}$$

is a dense (necessarily open) subset of X. The elements of $D(X)$ are called *extended (real-valued) functions.* Beginning with functions $f, g \in D(X)$, and $\lambda \in \mathbf{R}$, the functions λf, $f \wedge g$, and $f \vee g$ defined pointwise are clearly also in $D(X)$. If there are functions $h, k \in D(X)$ satisfying

$$h(x) = f(x) + g(x), \qquad k(x) = f(x)g(x)$$

for all $x \in \mathcal{R}(f) \bigcap \mathcal{R}(g)$, then h and k are called the sum and product of f and g, and we write $h = f + g$, $k = fg$. Note that since $\mathcal{R}(f) \cap \mathcal{R}(g)$ is dense in X these operations are uniquely defined.

An ideal I of a Φ-algebra A is called an *l-ideal*, or an *absolutely convex ideal*, if $x \in I$ and $|y| \leq |x|$, implies $y \in I$. The collection of all maximal l-ideals of A, denoted $\mathcal{M}(A)$, is endowed with the hull-kernel topology and will be referred to as the *maximal l-ideal space of A.*

Theorem 4.1. [HJ, 5.3] **(Henriksen-Johnson Representation Theorem)** *Every Φ-algebra A is isomorphic to a sub-Φ-algebra $\overline{A}$ of $D(\mathcal{M}(A))$. Moreover, if $\mathcal{S}$, $\mathcal{T}$ are disjoint closed subsets of $\mathcal{M}(A)$, then there is an $\overline{a} \in \overline{A}$ such that $\overline{a}[\mathcal{S}] = 0$, $\overline{a}[\mathcal{T}] = 1$, and $0 \leq \overline{a} \leq 1$.*

Henceforth, wherever it is convenient to do so, a Φ-algebra A will be identified with its isomorphic copy, the algebra $\overline{A} \subset D(\mathcal{M}(A))$ of extended functions.

We presently outline some of the theory derived in [HJ] from the representation theorem. The first result is theorem 2.5 of [HJ].

Theorem 4.2. *A subset M of a Φ-algebra A is a maximal l-ideal of A if and only if there is a unique $x \in \mathcal{M}(A)$ such that*

$$M = M_x = \{a \in A : (ab)(x) = 0 \text{ for all } b \in A\}.$$

In light of the above, an element of the maximal l-ideal space $\mathcal{M}(A)$ will be written as 'M', or 'M_x' if we wish to view it as a maximal l-ideal, and simply 'x' if we wish to view it as a point of the topological space $\mathcal{M}(A)$.

If M is a maximal l-ideal of A, then the totally ordered algebra A/M contains the real field $\mathbf{R}$ as a subfield via the embedding map $\mathbf{R} \to A/M : r \mapsto M(r \cdot 1)$. M is called real if $A/M = \mathbf{R}$, otherwise M is called hyper-real. An immediate consequence of (4.2), which is stated for reference is

Corollary 4.3. *If $x \in \mathcal{M}(A)$, then M_x is real if and only if $M_x = \{a \in A : a(x) = 0\}$.*

Let A be a Φ-algebra. The *uniform metric* ρ on A is defined by

$$\rho(a,b) = inf\{r \in \mathbf{R} : |a-b| \wedge 1 \leq r\}, \qquad (a,b \in A),$$

where, as usual, $r \cdot 1$ and r are identified. Henceforth all references to topological properties of a Φ-algebra will be with respect to its *uniform topology.*

If A is complete with respect to ρ, then A is said to be *uniformly closed.* Well-known examples of uniformly closed Φ-algebras are $\mathbf{R}$ and $C(X)$, where X is any topological space. Indeed many Φ-algebras of interest, such as the respective algebras of Baire and Lebesgue functions on the real line are uniformly closed, and consequently the properties of uniformly closed Φ-algebras are of especial interest. We outline some of these properties now; the first is [HJ, 3.2] and [HJ, 3.7].

Theorem 4.4. *A Φ-algebra A is uniformly closed if and only if A^* and $C(\mathcal{M}(A))$ are isomorphic.*

A Φ-algebra A is *closed under bounded inversion* if $a \in A$, $a \geq 1$, implies $1/a \in A$. The principal ideal of a member a of Φ-algebra A will be denoted (a), thus $(a) = \{ab : b \in A\}$. The smallest l-ideal of A containing a will be denoted $(a)_l$ and will be called the *l-principal ideal of A generated by a*, (as in [H]). It is easy to see that for any $a \in A$, $(a)_l = \{c \in A : |c| \leq |ab| \textit{ for some } b \in A\}$. The following is [HJ, 3.3] and [HJ, 3.4].

Theorem 4.5. *(1) Every uniformly closed Φ-algebra is closed under bounded inversion.*
(2) If A is a Φ-algebra closed under bounded inversion, then for $a \in A$, $(a)_l = A$ if and only if $1/a \in A$.

5. z-Ideals in Φ-Algebras

The main objective in our study of Φ-algebras is to generalize the characterization of closed ideals of $C(X)$ as found in section one, to a Φ-algebra setting. The notion of a z-ideal in $C(X)$ was central to our earlier theorem as it will be when we attempt to describe the closed l-ideals of Φ-algebras. Because z-ideals are crucial to the study of rings of continuous functions, we choose now to examine their role in Φ-algebras. Using z-ideals we expand upon some of the results found in [HJ], thus illustrating the use of z-ideals in this more general context. We note that that z-ideals in the even more general settings of commutative rings and partially-ordered rings were studied by G. Mason in [M]. There, some of the results obtained in this section may also be found, though we remark that our notation is rather different from Mason's. Due to the fact that we shall remain only in the world of Φ- algebras and therefore have access to the Henriksen-Johnson representation theorem, in those places where there is overlap with [M], the results that follow are sometimes slightly stronger than those found there. Unless explicitly stated otherwise, in the remainder of this paper, A will denote a Φ-algebra.

For each $f \in A$, let $\mathcal{M}(f) = \{x \in \mathcal{M}(A) : f \in M_x\}$.

Definition 5.1. *An ideal I of a Φ-algebra A will be called a z-ideal if $\mathcal{M}(f) \supset \mathcal{M}(g)$, $f \in A$ and $g \in I$ implies $f \in I$.*

Note that since $\mathcal{M}(f) \supset \mathcal{M}(g)$ if and only if $\mathcal{M}(f) = \mathcal{M}(fg)$, '$\supset$' may without loss of generality be replaced by '$=$' in the above definition. As noted in section one, if X is a topological space, then the above definition of z-ideal agrees with the usual notion of a z-ideal in $C(X)$, [GJ, 4A].

By the definition of the hull-kernel topology, the collection $\mathcal{M}[A] = \{\mathcal{M}(f) : f \in A\}$ is a base for the closed subsets of $\mathcal{M}(A)$. Similarly the zero-sets of a (completely regular) toplogical space X comprise a closed base for X. In this section, the similarities between the $\mathcal{M}[A]$ and the zero-sets of a topological space will be illustrated; indeed, if X were a compact topological space, and $\mathcal{M}(C(X))$ the maximal ideal space of $C(X)$, then the sets $\mathcal{M}(f)$, $(f \in C(X))$, are precisely the zero-sets of $C(X)$, (up to the equivalence of X and $\mathcal{M}(C(X))$).

The following notation is employed. If S is a subset of A, then the collection of subsets of $\mathcal{M}(A)$, $\{\mathcal{M}(f) : f \in S\}$, will be denoted by $\mathcal{M}[S]$. Hence $\mathcal{M}[A] = \{\mathcal{M}(f) : f \in A\}$ and $\mathcal{M}(A) = \{M : M \textit{ is a maximal } l - \textit{ideal of } A\}$; square and round brackets distinguish the difference. If $\mathcal{S}$ is a subset of $\mathcal{M}[A]$, then $\mathcal{M}^{\leftarrow}[\mathcal{S}] = \{f \in A : \mathcal{M}(f) \in \mathcal{S}\}$. Using this notation we see that

$$I \textit{ is a } z - \textit{ideal in } A \textit{ if and only if } I = \mathcal{M}^{\leftarrow}[\mathcal{M}[I]].$$

The proof of the following is not difficult.

Proposition 5.2. *$\mathcal{M}[A]$ forms a lattice under set containment.*

What follows is a sequence of statements which parallel the results of [GJ, chapter 2], stated in the context of Φ-algebras. In most cases the proofs may be constructed just as in [GJ], and therefore are omitted. Details may be found in [S]. A filter on the lattice $\mathcal{M}[A]$, will be called a *z-filter*.

Proposition 5.3. *a) If I is an l-ideal of A, then $\mathcal{M}[I]$ is a z-filter.*
b) If $\mathcal{F}$ is a z-filter on $\mathcal{M}[A]$, then $\mathcal{M}^{\leftarrow}[\mathcal{F}]$ is a z-ideal of A.

Each member of $\mathcal{M}(A)$ is an l-ideal, so by part (2) of [GJ, 5.3], for each $f \in A$, $\mathcal{M}(f) = \mathcal{M}(|f|)$. It follows that *every z-ideal of A is an l-ideal.*

An ultrafilter on the lattice $\mathcal{M}[A]$ will be called a *z-ultrafilter.* It is an easy corollary of the above proposition that the maximal l-ideals of A and the z-ultrafilters of A are in one-to-one corespondence via $\mathcal{M}$-imaging. Indeed the maximal l-ideals of A are precisely

$$M_x = \{f \in A : f \in M_x\} = \{f \in A : x \in \mathcal{M}(f)\}, (x \in \mathcal{M}(A)),$$

so the z-ultrafilters of A are precisely

$$\mathcal{U}_x = \mathcal{M}[M_x] = \{\mathcal{M}(f) : x \in \mathcal{M}(f)\} \qquad (x \in \mathcal{M}(A)).$$

Lemma 5.4. *If $h, g \in A$, $x \in \mathcal{M}(h)$, and $h(x) \geq g(x) \geq 0$, then $x \in \mathcal{M}(g)$.*

Proof. If $x \in \mathcal{M}(h)$, and $k \in A^+$, then $(hk)(x) = 0$, and $(hk)(x) \geq (gk)(x)$, hence $(gk)(x) = 0$. It follows from (4.2) that $g \in M_x$, and therefore $x \in \mathcal{M}(g)$. □

Theorem 5.5. *Let A be a Φ-algebra, I a z-ideal of A. Then the following are equivalent.*
1. I is prime.
2. I contains a prime ideal.
3. For all $g, h \in A$, if $gh = 0$, then $g \in I$, or $h \in I$.
4. For every $f \in A$, there is a member of $\mathcal{M}[I]$ on which f does not change sign.

Recall that if A is a commutative ring with unity, I an ideal of A, and $\mathcal{P}$ the collection of all prime ideals of A containing I, then $\bigcap \mathcal{P} = \{f \in A : f^n \in I \text{ for some } n = 0, 1, 2, ...\}$, [GJ, 0.18]. Moreover, [HJ, 1.5] says that if I is an l-ideal of A disjoint from a multiplicative system T of A, (that is $T \subset A$ is closed under multiplication, $1 \in T$, and $0 \notin T$), then I is contained in a prime l-ideal of A disjoint from T.

Proposition 5.6. *Every z-ideal is the intersection of the collection of all prime l-ideals containing it.*

Proof. Let I be a z-ideal of Φ-algebra A, $\mathcal{P}$, $\mathcal{P}'$ respectively the collections of all prime ideals and prime l-ideals containing I, $J = \bigcap \mathcal{P}$, $J' = \bigcap \mathcal{P}'$. Suppose $f \in J$. Then for some power of n, $f^n \in I$. But maximal l-ideals are prime, so $\mathcal{M}(f) = \mathcal{M}(f^n)$, and since I is a z-ideal, $f \in I$. Hence $I = J$. Now certainly $J \subset J'$. If $f \notin J$, then $T = \{f^n : n = 0, 1, 2, ...\}$ is a multiplicative system disjoint from I, hence $f \notin J'$. Therefore $J' \subset J$, hence $J' = I$. □

Using the above, just as in [GJ], the following statements can be proved. A z-filter $\mathcal{F}$ will be called prime if whenever the union of two sets from $\mathcal{M}[A]$ belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

Proposition 5.7. *(1) Intersections of z-ideals are z-ideals.*
(2) Every prime l-ideal is contained in a unique maximal l-ideal.
(3) If P is a prime l-ideal in A, then $\mathcal{M}[P]$ is a prime z-filter.
(4) If $\mathcal{F}$ is a prime z-filter, then $\mathcal{M}^{\leftarrow}[\mathcal{F}]$ is a prime z-ideal.
(5) Every prime z-filter is contained in a unique z-ultrafilter.

Recall from [HJ], the l-ideal

$$N_x = \{f \in A : f \textit{ vanishes on a neighborhood of } x\}$$

$$= \{f \in A : x \in \textit{int}\mathcal{M}(f)\}.$$

It follows easily from the second part of the Henriksen-Johnson Representation Theorem that $\bigcap \mathcal{M}[N_x] = \{x\}$, and so N_x is contained in the unique maximal l-ideal M_x.

Proposition 5.8. *For each $x \in \mathcal{M}(A)$,*
1. N_x is a z-ideal.
2. $f \in N_x$ if and only if $fg = 0$ for some $g \notin M_x$.

Proof. 1. is clear.
2. If $f \in N_x$, then $x \in int\mathcal{M}(f)$. By the Henriksen-Johnson representation theorem, there is a $g \in A$, $0 \leq g \leq 1$ with $g[\mathcal{M}(A)\backslash int\mathcal{M}(f)] = 0$ and $g(x) = 1$. Then $g \notin M_x$ and $fg = 0$. Conversely suppose that $g \notin M_x$ and $fg = 0$. Then $x \notin \mathcal{M}(g)$, yet $\mathcal{M}(g) \bigcup \mathcal{M}(f) = \mathcal{M}(fg) = \mathcal{M}(A)$. Therefore $\mathcal{M}(A)\backslash\mathcal{M}(g)$ is an open set containing x that is contained in $\mathcal{M}(f)$, hence $x \in int\mathcal{M}(f)$. Therefore $f \in N_x$. □

Theorem 5.9. *An l-ideal I of A is contained in a unique maximal l-ideal M_x if and only if $I \supset N_x$.*

Proof. Since M_x is the unique maximal l-ideal containing N_x, sufficiency is clear. Conversely suppose that M_x is the unique maximal l-ideal containing I and let $f \in N_x$. Take $g \notin M_x$ such that $fg = 0$. Since $g \notin M_x$, $(I, g)_l = A$, where $(I, g)_l$ denotes the smallest l-ideal containing both I and g. Therefore there is an $h \in I$, and $s \in A^+$ such that $1 \leq h + s|g|$. Therefore $h \geq 1 - s|g|$, hence $h|f| \geq |f| - s|g||f| = |f| - s|gf| = |f|$. But $h|f| \in I$, and I is an l-ideal, so $|f| \in I$, hence $f \in I$. □

The following corollary is [HJ, 2.10].

Corollary 5.10. *Let P be a prime l-ideal of Φ-algebra A. Then there is a unique $x \in \mathcal{M}(A)$ such that $N_x \subset P \subset M_x$, and N_x is the intersection of all prime l-ideals containing it.*

Proof. N_x is a z-ideal, hence the last statement. It has already been shown that every prime l-ideal is contained in a unique maximal l-ideal, so by the above theorem the result follows. □

We conclude this section by generalizing the results of section one, concerning prime ideals and closure in $C(X)$ to arbitrary Φ-algebras.

Proposition 5.11. *For every $x \in \mathcal{M}(A)$, $\overline{N_x} = \overline{M_x}$.*

Proof. Let $x \in \mathcal{M}(A)$. Since $N_x \subset M_x$, $\overline{N_x} \subset \overline{M_x}$. To prove the reverse containment let $a \in M_x$ and let $\varepsilon > 0$. Then $a(x) = 0$ and since $a : \mathcal{M}(A) \to \gamma\mathbf{R}$ is continuous, $U = a^{\leftarrow}[(-\varepsilon, \varepsilon)]$ is an open neighborhood of x in $\mathcal{M}(A)$. Let $b = [(a-\varepsilon) \vee 0] + [(a+\varepsilon) \wedge 0]$. Then $b \in A$, $|b - a| \leq 2\varepsilon$ and $b[U] = \{0\}$. Hence $b \in N_x$, and so $a \in \overline{N_x}$. Therefore $M_x \subset \overline{N_x}$, whence $\overline{M_x} \subset \overline{N_x}$. □

Now if P is a prime ideal l-ideal of A, then there is a unique $x \in \mathcal{M}(A)$ such that $N_x \subset P \subset M_x$. By the proposition $\overline{N_x} = \overline{P} = \overline{M_x}$, from which it follows that non-maximal prime l-ideals of A are never closed. Suppose however that A is a uniformly closed Φ-algebra. Then by [P, 2.6] the closed maximal l-ideals of A are precisely the real l-ideals of A. Thus if M_x is a real l-ideal of A with $P \subset M_x$, then $\overline{N_x} = \overline{P} = M_x$. Hence we have

Corollary 5.12. *Let A be a uniformly closed Φ-algebra.*
1. If M_x is a real l-ideal of A, then $\overline{N_x} = M_x$.
2. If P is a prime l-ideal of A, then $\overline{P}$ is a (necessarily maximal) l-ideal if and only if the unique maximal l-ideal M containing P is real; in this case $\overline{P} = M$.

6. An Application: P-algebras

A Φ-algebra A will be called a $P - algebra$ if every prime l-ideal of A is a maximal l-ideal. By the definition of a P-space, [GJ, 4J], $C(X)$ is a P-algebra if and only if X is a P-space.

Theorem 6.1. *The following are equivalent for a Φ-algebra A.*
1. A is a P-algebra.
2. Every z-ideal is an intersection of maximal l-ideals.
3. For each $x \in \mathcal{M}(A)$, $N_x = M_x$.
4. For each $f \in A$, $\mathcal{M}(f)$ is open.

Proof. 1. $\Rightarrow$ 2. Every z-ideal is an intersection of prime l-ideals.
2. $\Rightarrow$ 3. N_x is a z-ideal contained in precisely one maximal l-ideal, namely M_x. It follows from 2. that $N_x = M_x$.
3. $\Rightarrow$ 4. Let $f \in A$. If $\mathcal{M}(f) = \emptyset$, then $\mathcal{M}(f)$ is open, otherwise take $x \in \mathcal{M}(f)$. Then $f \in M_x = N_x$, whence $x \in int(\mathcal{M}(f))$. Therefore $\mathcal{M}(f) = int(\mathcal{M}(f))$.
4. $\Rightarrow$ 3. $M_x = \{f \in A : x \in \mathcal{M}(f)\} = \{f \in A : x \in int\mathcal{M}(f)\} = N_x$.
3. $\Rightarrow$ 1. If P is a prime l-ideal of A, then there is an $x \in \mathcal{M}(A)$ such that $N_x \subset P \subset M_x$; by 3., $P = M_x$. □

In the event that A is uniformly closed we may say more. Recall that a (commutative) algebra A is called *regular* if for every $f \in A$ there is a $g \in A$ such that $f = gf^2$. The following generalizes [GJ, 14.29].

Theorem 6.2. *If A is a uniformly closed Φ-algebra, then the following are equivalent.*
(1) A is a P-algebra.
(2) For every $x \in \mathcal{M}(A)$, $N_x = M_x$.
(3) For every $f \in A$, $\mathcal{M}(f)$ is open.
(4) Every l-ideal is a z-ideal and every l-principal ideal is principal.
(5) Every ideal of A is a z- ideal.
(6) For every $f, g \in A$, the ideal (f, g) is the principal ideal $(f^2 + g^2)$.
(7) A is a regular Φ-algebra.
(8) Every prime ideal in A is a maximal ideal.

Proof. (This proof is as suggested by the referee. A proof which uses the Henriksen-Johnson representation theorem may be found in [S]).
The equivalence of (1), (2) and (3) was established in (6.1). The equivalence of (4) through (7) is essentially [M, 1.2]. It is well-known that a commutative ring is regular if and only if

every prime ideal is maximal. Hence (1) implies (7) holds in Φ-algebras in which
(*) every prime ideal is an l-ideal.
In [L] it is shown that an f-ring in which $0 \leq x \leq y^2$ implies x is a multiple of y satisfies (*). In particular this holds in a uniformly closed Φ-algebra. That (8) implies (1) is clear. □

Corollary 6.3. *If A is a uniformly closed P-algebra, then every ideal of A is an l-ideal.*

Evidently, amongst uniformly closed Φ-algebras, the P-algebras are precisely the regular Φ-algebras, and as such interesting examples abound. As mentioned earlier, $C(X)$, where X is a P-space, is one such example. Others include the Baire functions on $\mathbf{R}$ and the Lebesgue measurable functions on $\mathbf{R}$, each of which if desired may be taken modulo its ideal of functions that vanish almost everywhere, [HJ, 3.10]; yet another example of a uniformly closed regular Φ-algebra is the epimorphic hull of $C(X)$ which is examined in the preprint [RW], by Raphael, and Woods. B. Brainerd in the late fifties studied regular F-rings, (F-rings are uniformly closed Φ-algebras, see [P] for references), and, as they pertain to z-ideals, regular rings were studied by Mason in [M]. The following is due to the referee.

Example. *(1) implies (7) need not hold in Φ-algebras that fail to be uniformly closed.*

Let $\alpha N = N \cup \{\infty\}$ denote the one-point compactification of the discrete space of positive integers, and let $A = \{f \in C(N)$: there is an $n_f \in N$ such that f is a polynomial when restricted to $[n_f, \infty)\}$. Using (6.1), it is easy to see that the prime ideals of A are the maximal ideals M_n for $n \in N$ and the prime l-ideal $O_\infty = \{f \in A : Z(f)$ contains a tail of $N\}$. Clearly O_∞ is a maximal l-ideal, so A is a P-algebra that is not regular.

7. z-Ideals, Z-ideals, and Strong Divisibility

At this point we attempt to generalize the characterization (1.4) of closed ideals of $C(X)$ to various types of Φ-algebras. To begin we consider another extension of the notion of z-ideal in the context of a Φ-algebra.

Let A be a Φ-algebra. Let $\mathcal{R}(A)$ denote the space of real maximal l-ideals of A, and for each $a \in A$, let $\mathcal{S}(a) = \{M \in \mathcal{R}(A) : a \in M\}$. As before, let $\mathcal{M}(a) = \{M \in \mathcal{M}(A) : a \in M\}$.

Definition 7.1. *Let A be a Φ-algebra, and let I be an ideal of A. I will be called a Z-ideal if*
$(\mathcal{S}(b) \supset \mathcal{S}(a)$ *and* $a \in I) \Rightarrow b \in I$.

Clearly then, every Z-ideal is a z-ideal. It is a well-known result that if X is a topological space, then the above definition of z-ideal agrees with the usual notion of a z-ideal in $C(X)$, [GJ, 4A]. In fact:

Proposition 7.2. *If X is a topological space, and I is an ideal of $C(X)$, then I is a z-ideal if and only if I is a Z-ideal.*

Proof. Suppose I is a z-ideal of $C(X)$ and suppose $\mathcal{S}(g) \supset \mathcal{S}(f)$, $f \in I$. Let $p \in Z(f)$. Then $f \in M_p$ and M_p is real. Therefore $g \in M_p$, whence $p \in Z(g)$. Therefore $Z(f) \subset Z(g)$, and $Z(f) \in Z[I]$, $Z[I]$ a z-filter. It follows that $Z(g) \in Z[I]$. But I is a z-ideal in $C(X)$ and therefore $g \in I$, showing that I is a Z-ideal. □

Recall from [HJ] that a Φ-algebra A is called an *algebra of real-valued functions* if its space of real maximal ideals $\mathcal{R}(A)$ is dense in $\mathcal{M}(A)$. Equivalently A is a Φ-algebra of real-valued functions if the intersection of its real ideals is $\{0\}$; in this case A can be embedded as a sub-Φ-algebra of $C(\mathcal{R}(A))$. We remark that the algebra of Lebesgue measurable functions modulo the ideal of functions that vanish almost everywhere is a Φ-algebra that has no real (maximal) ideals. A Φ-algebra of real-valued functions A is *closed under inversion* if, for all $a \in A$, $a^{\leftarrow}(0) \bigcap \mathcal{R}(A) = \emptyset$ implies $(a)_l = A$, where $(a)_l$ denotes the smallest l-ideal in A containing a. Note that by (4.3), given $a \in A$, and $x \in \mathcal{M}(A)$, $x \in \mathcal{S}(a)$ if and only if M_x is real and $a(x) = 0$. Thus $\mathcal{S}(a) = a^{\leftarrow}(0) \cap \mathcal{R}(A)$. The following is [HJ, 4.6].

7.3. *If A is a Φ-algebra of real-valued functions which is closed under inversion, then for each $x \in \mathcal{M}(A)$,*

$$M_x = \{a \in A : x \in \overline{\mathcal{S}(a)}\}.$$

Theorem 7.4. *Let A be a Φ-algebra of real-valued functions. Then A is closed under inversion if and only if z-ideals in A are Z-ideals.*

Proof. Suppose I is a z-ideal and $\mathcal{S}(b) \supset \mathcal{S}(a)$, $a \in I$. Let $x \in \mathcal{M}(A)$ and suppose $a \in M_x$. Then $x \in \overline{\mathcal{S}(a)}$ hence $x \in \overline{\mathcal{S}}(b)$. Thus $b \in M_x$, showing that $\mathcal{M}(a) \subset \mathcal{M}(b)$. But $a \in I$ and I is a z-ideal, so $b \in I$. This shows that I is a Z-ideal.
For the converse, supposing that A is not closed under inversion, we may take $a \in A$ such that $\mathcal{S}(a) = \emptyset$, yet $(a)_l \neq A$. Thus a maximal l-ideal M_x may be found such that $(a)_l \subset M_x$. Therefore, if $b \in A \backslash M$,then $\mathcal{S}(b) \supset \mathcal{S}(a)$, $a \in M_x$, yet $b \notin M_x$. It follows that the z-ideal M_x is not a Z-ideal. □

Example. *A uniformly closed Φ-algebra of real-valued functions in which z-ideals need not be Z-ideals.*

An example of Henriksen and Johnson [HJ, 4.5], does the job. Let $A = \{f \in C(\mathbf{R}^+) : lim_{x \to \infty} f(x)e^{-ax} = 0 \; for \; all \; real \; a > 0\}$. Then A is a uniformly closed Φ-algebra of real-valued functions. A^* and $C(\mathbf{R}^+)$ are isomorphic, so $\mathcal{M}(A) = \mathcal{M}(A^*) = \mathcal{M}(C^*(\mathbf{R}^+)) = \beta\mathbf{R}^+$. Take $g(x) = e^{-x}$, $(x \in \mathbf{R}^+)$. Clearly then $g \in A$. Moreover, $g^{\leftarrow}(0) = \beta\mathbf{R}^+ \backslash \mathbf{R}^+ = \mathcal{M}(A) \backslash \mathcal{R}(A)$, therefore, $\mathcal{M}(g) \subset \mathcal{M}(A) \backslash \mathcal{R}(A)$, and hence

$\mathcal{S}(g) = \emptyset$. But $1/g \notin A$, so $(g) \neq A$, and therefore there is a maximal l-ideal M containing g. M is then a z-ideal but is not a Z-ideal. For take any $h \notin M$. Then $\mathcal{S}(h) \supset \mathcal{S}(g)$, $g \in M$, but $h \notin M$. □

To prove the next few theorems, the following definitions and results from the papers [H] by M. Henriksen and [P] by D. Plank are needed.

In [H] an *ideal set* is defined to be a closed subset Δ of the space $\mathcal{M}(A)$ of maximal l-ideals of Φ-algebra A such that whenever $a \in A$ and $a[\Delta] = 0$, then $(ab)[\Delta] = 0$ for all $b \in A$. It is shown that:

[H, 2.4] An l-ideal I of a uniformly closed Φ-algebra A is closed if and only if there is an ideal set Δ of $M(A)$ such that $I = \{a \in A : a[\Delta] = 0\}$.

[P, 2.6] If A is a uniformly closed Φ-algebra, then a maximal l-ideal of A is closed if and only if it is real.

[P, 3.7] If I is either a closed or maximal ideal in the uniformly closed Φ-algebra A, then I is an l-ideal.

Recall also from section four, that if A is uniformly closed then its sub-Φ-algebra of bounded elements A^* is l-isomorphic to $C(\mathcal{M}(A))$, and as such $C(\mathcal{M}(A))$ may be regarded as a sub-Φ-algebra of A.

These results are used in the proofs that follow.

Theorem 7.5. *(1) Closed ideals of uniformly closed Φ-algebras are strongly divisible z-ideals.*
(2) Any strongly divisible Z-ideal of a Φ-algebra A is closed in A. (Here A is not assumed to be uniformly closed).

Proof. (1) Let I be a closed ideal of uniformly closed Φ-algebra A. Let $I^* = I \bigcap A^*$. Then I^* is a closed ideal of $A^* = C(\mathcal{M}(A))$, and therefore is strongly divisible by (1.4). Let (a_n) be a countable subset of I. A is closed under bounded inversion, so for each n, $(1 + |a_n|)^{-1}$ exists in A. But each $\frac{a_n}{1+|a_n|} \in A^* \bigcap I = I^*$, so by the strong divisibility of I^* there is a $b \in I^*$, and $(b_n) \subset A^*$ such that for each n, $b \cdot b_n = \frac{a_n}{1+|a_n|}$. Hence $b \in I$ and for each n, $b \cdot b_n \cdot (1+|a_n|) = a_n$, proving that I is strongly divisible. As a closed ideal of uniformly closed A, I is an l-ideal, [P, 3.7], so let Δ be an ideal set such that $I = \{a \in A : a[\Delta] = 0\}$. Suppose $\mathcal{M}(b) \supset \mathcal{M}(a)$ and $a \in I$. If $x \in \Delta$ then for all $c \in A$, $(ac)(x) = 0$, hence $a \in M_x$, by (4.2). Therefore $b \in M_x$, hence $b(x) = 0$. Thus $b[\Delta] = 0$ and therefore $b \in I$. This proves that I is a z-ideal.

(2) Let I be a strongly divisible Z-ideal of Φ-algebra A, let $b \in \overline{I}$. Take (b_n) a countable subset of I such that for each $n \in \mathbf{N}$, $|b - b_n| < 1/n$. Then by the strong divisibility of I, there is an $a \in I$ and $(a_n) \subset A$ such that for each n, $a \cdot a_n = b_n$. Suppose $a \in M_x$, where M_x is a real ideal of A. Then $a(x) = 0$ and therefore for each $n \in \mathbf{N}$, $a \cdot a_n(x) = 0$. Therefore for all n, $1/n > |a \cdot a_n(x) - b(x)| = |b(x)|$, hence $|b(x)| = 0$; therefore $b \in M_x$ by (4.3). This shows that $\mathcal{S}(a) \subset \mathcal{S}(b)$. But $a \in I$ and I is a Z-ideal, so $b \in I$. Therefore $I = \overline{I}$, hence I is closed. □

One might ask whether or not strongly divisible z-ideals in Φ-algebras are always closed. Certainly an argument similar to the one given above will not answer the question in the affirmative; however, if we assume that A is a P-algebra, then the full converse to (8.6, part one)) holds.

Theorem 7.6. *If A is a P-algebra, then strongly divisible z-ideals in A are closed.*

Proof. Letting I be a strongly divisible z-ideal in A, choose $b \in \overline{I}$. As in the proof of (8.6, part two), we may find $a \in I$ and $(a_n) \subset A$ such that for each $n \in \mathbf{N}$, $|a \cdot a_n - b| < 1/n$. Taking $x \in \mathcal{M}(a)$, for each n, $(a \cdot a_n)(x) = 0$, hence $b(x) = 0$, and therefore $\mathcal{M}(a) \subset b^{\leftarrow}(0)$. But A is a P-algebra, so by (6.1), $\mathcal{M}(a)$ is open, hence $\mathcal{M}(a) \subset int(b^{\leftarrow}(0))$, and since $int(b^{\leftarrow}(0)) \subset \mathcal{M}(b)$, we have $\mathcal{M}(a) \subset \mathcal{M}(b)$. [We note that were A not assumed to be a P-algebra, $\mathcal{M}(a) \subset \mathcal{M}(b)$ could not be shown in this manner]. But I is a z-ideal in A and $a \in I$, hence $b \in I$, proving that $I = \overline{I}$. □

As a consequence of the above we get the following characterization of the closed ideals in two broad classes of Φ-algbras.

Corollary 7.7. *Let A be a uniformly closed Φ-algebra that is either*
(i) a Φ-algebra of real-valued functions that is closed under inversion, or
(ii) regular.
Then
1. An l-ideal I of A is closed if and only if it is a strongly divisible z-ideal.
2. A maximal l-ideal of A is real if and only if it is strongly divisible.

The Henriksen-Johnson representation theorem has been used to characterize $C(X)$, for various homeomorphism classes of spaces X, algebraically within the class of Φ-algebras. Characterizations of several such classes of (realcompact) Tychonoff spaces may be found in [HJ] and many other papers. Nevertheless a longstanding open problem concerning Φ-algebras asks, 'find an internal characterization of $C(X)$ for X a Tychonoff space within the class of Φ-algebras', [H2, problem 5]. The following is a reformulation of D. Plank's theorem, [P, 3.6] which phrased in terms of strongly divisible rather than closed ideals, perhaps seems more algebraic in character. The bracketed z is meant to indicate that it is optional.

Theorem 7.8. *A non-trivial Φ-algebra A is l-isomorphic to $C(X)$ for some Lindelöf space X if and only if*
(i) A is uniformly closed,
(ii) every z-ideal in A is a Z-ideal, and
(iii) every strongly divisible $(z-)$ideal of A is contained in a strongly divisible maximal l-ideal of A.

Recall that $C(X)$ is (l)-isomorphic to $C(\upsilon X)$, and X is weakly Lindelöf if and only if υX is weakly Lindelöf. Hence a

Φ-algebra A is l-isomorphic to $C(X)$ for some weakly Lindelöf space X if and only if A is l-isomorphic to $C(Y)$ for some realcompact weakly Lindelöf space Y. With this in mind we offer the following

Conjecture 7.9. *A non-trivial Φ-algebra A is l-isomorphic to $C(X)$ for some weakly Lindelöf (realcompact) space X if and only if*
(i) A is uniformly closed,
(ii) every z-ideal in A is a Z-ideal, and
(iii) every nsd ideal in A is contained in a strongly divisible maximal l-ideal of A.

We note by way of evidence to support the validity of this statement, Theorem 3.4, which under the additional hypothesis of realcompactness of X equivalently reads:
X is weakly Lindelöf if and only if every nsd ideal of $C(X)$ is contained in a strongly divisible maximal ideal of $C(X)$.
Hence the conditions (i), (ii), and (iii) of (7.10) are certainly necessary. We also note that arguing as in the proof of [P, 3.1], it follows from (i), (ii) and (iii) that A is a Φ-algebra of real-valued functions; unfortunately it seems that a proof of (7.10) would at this point require an alternative to Plank's methods.

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E-mail address: `stokke@math.ualberta.ca`