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# DEFINITIONS OF $\Sigma$-SPACES

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## Abstract

We point out that the definition of a $\Sigma$-space in Gruenhage's article [G] in Handbook of Set-Theoretic Topology differs from the usual one in two points. By giving a counterexample, we show that at least the first point should be modified. The second point raises an interesting question whether the local finiteness in the usual definition of $\Sigma$-spaces can be replaced by discreteness or not.

## Introduction

All spaces are assumed to be regular $T_1$-spaces.

The class of $\Sigma$-spaces, introduced by K. Nagami [N], plays an important role in the theory of generalized metric spaces. It contains all $M$-spaces (in particular, countably compact spaces) and all $\sigma$-spaces. $\Sigma$-spaces are defined in a quite natural way, but sometimes we fail to recall the full definition. Indeed, it may be interesting to guess which one of the following conditions is used in the usual definition of a $\Sigma$-space:

## Conditions

(a): There are a cover $\mathcal{C}$ of $X$ by closed countably compact sets and a $\sigma$-locally finite family $\mathcal{F}$ of closed sets of $X$ such that for any $C \in \mathcal{C}$ and an open set $U$ with $C \subset U$, there is $F \in \mathcal{F}$ satisfying $C \subset F \subset U$.

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(b): There are a cover $\mathcal{C}$ of $X$ by closed countably compact sets and a $\sigma$-discrete family $\mathcal{F}$ of closed sets of $X$ such that for any $C \in \mathcal{C}$ and an open set $U$ with $C \subset U$, there is $F \in \mathcal{F}$ satisfying $C \subset F \subset U$.

(c): There are a cover $\mathcal{C}$ of $X$ by closed countably compact sets and a $\sigma$-discrete family $\mathcal{F}$ of subsets of $X$ such that for any $C \in \mathcal{C}$ and an open set $U$ with $C \subset U$, there is $F \in \mathcal{F}$ satisfying $C \subset F \subset U$.

A space $X$ is called a $\Sigma$-space if it satisfies (a). This definition is a little different from the original definition due to Nagami [N]. But it is commonly used and it is easy to check the equivalence of both definitions. Gruenhage's article [G] in Handbook of Set-Theoretic Topology is an excellent survey on generalized metric spaces. There a $\Sigma$-space is defined as a space satisfying (c). Hence "a $\sigma$-locally finite, closed family" in Nagami's definition is replaced by "a $\sigma$-discrete (not necessarily closed) family" in Gruenhage's definition. This difference was found in a seminar at University of Tsukuba, after which some graduate students asked me whether the two definitions are equivalent or not.

Note that (b) implies both (a) and (c). Compare our case with the case of $\sigma$-spaces. In the definition of a $\sigma$-space, such a difference can be ignored (see [G]). That is, those three conditions are equivalent if $\mathcal{C} = \{\{x\} : x \in X\}$. It is easy to see that every normal space satisfying ($c$) has property ($b$). But in general, we have the following example:

**Example.** *There is a space satisfying (c) which is not a $\Sigma$-space.*

Hence (c) is equivalent to neither (a) nor (b). I have discussed with Gruenhage whether (a) and (b) are equivalent or not, but we still do not know the answer. Thus we get the following very interesting question raised by a mistake of Gruenhage:

**Problem.** *Does every Σ-space satisfy condition (b)?*

It is natural to consider condition (a') which is obtained from (a) by not requiring members of $\mathcal{F}$ to be closed. We don't know whether (a') is equivalent to (c) or not.

**Proof of Example.** Let $R$, $Q$ and $P$ be the set of real numbers, rational numbers and irrational numbers respectively. $\omega_1 + 1 = [0, \omega_1]$ is the set of ordinals less than or equal to the first uncountable ordinal with the order topology. Define $X = (\omega_1 + 1) \times R - \{\omega_1\} \times P$.

For each $r \in R$, let $l_r = X \cap ((\omega_1 + 1) \times \{r\})$. Note that $l_r = (\omega_1 + 1) \times \{r\}$ if $r \in Q$, and $l_r = \omega_1 \times \{r\}$ if $r \in P$. Let $\pi_1 : X \to \omega_1 + 1$ and $\pi_2 : X \to R$ be the projections.

First we show that $X$ satisfies (c). Define $\hat{\mathcal{C}} = \{l_r : r \in R\}$. Let $\mathcal{B}$ be a countable base of $R$ consisting of open intervals with rational endpoints. Define a countable family $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}} = \{l_r : r \in Q\} \cup \{\omega_1 \times B : B \in \mathcal{B}\}$. Then it is easy to check that $\hat{\mathcal{C}}$ and $\hat{\mathcal{F}}$ satisfy (c).

Next we show that $X$ is not a Σ-space. To see this, suppose the contrary and assume that $\mathcal{C}$ and $\mathcal{F}$ are families satisfying (a).

CLAIM 1. For each $r \in P$, there is some $C \in \mathcal{C}$ such that $\pi_1(l_r \cap C)$ is unbounded in $\omega_1$.

*Proof of Claim 1.* For each $r \in P$, the countably compact set $l_r$ meets only countably many elements of $\mathcal{F}$ because $\mathcal{F}$ is $\sigma$-locally finite. In particular, $\{F \in \mathcal{F}$: $\pi_1(l_r \cap F)$ is bounded in $\omega_1\}$ is a countable family. Hence there is $\alpha \in \omega_1$ such that $\pi_1(l_r \cap F)$ is unbounded for each $F \in \mathcal{F}$ with $[\alpha, \omega_1) \cap \pi_1(l_r \cap F) \neq \emptyset$. Since $\mathcal{C}$ is a cover of

$X$, there is $C \in \mathcal{C}$ containing $\langle \alpha, r \rangle$. Then $\pi_1(l_r \cap C)$ is unbounded in $\omega_1$. Indeed, suppose not. Then there is $\beta \in \omega_1$ such that $\pi_1(l_r \cap C) \subset \beta$. Let $U = X - \{\langle \gamma, r \rangle : \gamma \geq \beta\}$. Then $U$ is an open neighborhood of $C$. But there is no element $F \in \mathcal{F}$ satisfying $C \subset F \subset U$. □

By Claim 1, for each $r \in P$, we can take $C_r \in \mathcal{C}$ such that $\pi_1(l_r \cap C_r)$ is unbounded in $\omega_1$. Since $C_r$ is countably compact, there is some $B_r \in \mathcal{B}$ containing $r$ such that $C_r \cap (\{\omega_1\} \times \mathrm{cl}_R B_r) = \emptyset$. Let $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$, where each $\mathcal{F}_n$ is a locally finite family of closed sets of $X$. By (a), for each $r \in P$, there is $n_r \in \omega$ and $F_r \in \mathcal{F}_{n_r}$ such that $C_r \subset F_r \subset X - (\{\omega_1\} \times \mathrm{cl} B_r)$.

Now by the Baire Category Theorem, there are $\hat{n} \in \omega$, $\hat{B} \in \mathcal{B}$, and a sequence $\{r_n\}_{n \in \omega}$ of irrational numbers in $\hat{B}$ converging to a rational number $q \in Q \cap \mathrm{cl}_R \hat{B}$ such that $n_{r_n} = \hat{n}$, $B_{r_n} = \hat{B}$ for any $n \in \omega$.

Let $n \in \omega$. Since $\langle \omega_1, q \rangle \notin F_{r_n}$, there are $\alpha < \omega_1$ and a neighborhood $U$ of $q$ in $R$ such that $F_{r_n} \cap ((\alpha, \omega_1] \times U) = \emptyset$. Note that $\pi_1(l_r \cap F_r)$ contains a closed unbounded set $\pi_1(l_r \cap C_r)$ for each $r \in P$. So whenever $r \in U \cap P$, we have $F_r \cap ((\alpha, \omega_1] \times U) \neq \emptyset$, which implies that $F_r \neq F_{r_n}$. Hence by taking a subsequence of $\{r_n\}_{n \in \omega}$, we may assume that $F_{r_n}$'s are distinct, which contradicts the local finiteness of $\mathcal{F}_{\hat{n}}$ at $\langle \omega_1, q \rangle$. □

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