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GENERATING DENSE SUBGROUPS OF TOPOLOGICAL GROUPS

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Abstract

If a discrete subset S of a topological group G with the identity 1 generates a dense subgroup of G and $S \cup \{1\}$ is closed in G, then S is called a *suitable set* for G. It turns out that all "good" topological groups have a suitable set, and it takes some work to recognize that there are groups with no suitable set. We present a survey of recent results on the existence of suitable sets in topological groups and discuss several open problems.

0. Introduction

Our aim is to present a survey of recent results in the new and fashionable area: *Suitable sets for topological groups.* It is not an exaggeration to say that 95% of all the material accumulated in the area has been obtained during the last three years, while the notion of a *suitable set* in the present form had

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been introduced by Hofmann and Morris [HM1] in 1990. An intensive study of topological groups with suitable sets started after the lecture delivered by Sydney Morris at the Conference on Set-Theoretic Topology and its Applications in Matsuyama, Japan in December of 1994. The subject has been significantly developed since then, so we systematize our knowledge and discuss some open problems.

Following Hofmann and Morris [HM1], we call a subset S of a topological group G *suitable* for G if S is discrete in itself, generates a dense subgroup of G, and $S \cup \{1_G\}$ is closed in G, where 1_G is the identity of G. Therefore, every finite subset of a topological group G generating a dense subgroup of G is suitable for G. It is well-known that a compact connected Abelian group of weight less that or equal to $\mathfrak{c}$ contains a dense cyclic subgroup, that is, has a one-point suitable set [HR, Theorem 25.14]. A non-Abelian topological group cannot have dense cyclic subgroups simply because the closure of an Abelian subgroup is Abelian too. However, by a theorem in [HM1], every compact connected group G of weight $\leq \mathfrak{c}$ contains a dense subgroup generated by two elements. In other words, such a group has a two-element suitable set (the minimal possible one if G is not Abelian).

Clearly, neither finite nor countable subsets of a topological group G with $w(G) > \mathfrak{c}$ can generate a dense subgroup of G. This makes the following theorem of Hofmann and Morris [HM1] even more intriguing.

Theorem 0.1. *Every locally compact topological group has a suitable set.*

It is worth mentioning that some special cases of this deep result were known long before 1990. In particular, the existence of a suitable set for every compact totally disconnected group was mentioned in 1966 by Douady [Do] (attributed to Tate). A detailed proof of the latter fact appears in Chapter 12 of

[HM2].

From 1995, the study of topological groups with a suitable set focuses on the non-locally compact case. A significant contribution to the subject is the following theorem proved in [CMRST]:

Theorem 0.2. *Every metrizable topological group has a suitable set.*

Theorems 0.1 and 0.2 mean that every "basic" topological group (i.e., any group we find at the initial period of the study of topological groups) does have a suitable set. To emphasize how wide the class $\mathcal{S}$ of topological groups with a suitable set is, we just mention here that the cartesian product of arbitrarily many groups in $\mathcal{S}$ is again in $\mathcal{S}$. In Section I we will present several results generalizing both Theorem 0.1 and Theorem 0.2. In particular, every almost metrizable topological group has a suitable set (see Theorem 1.16).

All this makes the problem of constructing a group with no suitable set fairly difficult. The first example of a topological group without a suitable set was an infinite countably compact group G having no convergent sequences constructed by van Douwen under Martin's Axiom MA (see [Dou1]). Additionally, van Douwen's group G is separable and boolean, that is, $x^2 = 1_G$ for each $x \in G$. The fact that $G \notin \mathcal{S}$ was found out in [CMRST], and its proof is really easy. Indeed, if $S \subseteq G$ is a suitable set for G then S is infinite because a finite subset of a boolean group generates a finite subgroup. Since G is countably compact, S must have an accumulation point in G. The unique accumulation point of S can only be the identity 1_G of G. Therefore, $S \cup \{1_G\}$ is a closed (hence compact) subset of G, so that S converges to 1_G. This contradicts the fact that G does not have non-trivial convergent sequences. We have thus proved the following result (see Theorem 3.15 of [CMRST]):

Theorem 0.3. (MA) *There exists a separable countably compact topological group without a suitable set.*

We conclude, in particular, that under MA, not every topological group has a suitable set. In other words, $\mathcal{S}$ does not coincide with the class of all topological groups. Fortunately, the latter conclusion does not require Martin's Axiom. In Section II we will present a few examples of topological groups without a suitable set, which are based on the use of free topological groups and the spaces of continuous functions with the pointwise convergence topology. One of them will be a boolean group G in which the closure of every countable subset is compact, so that G is countably compact but $G \notin \mathcal{S}$.

In Section III we discuss the problem whether the class $\mathcal{S}$ is stable under cartesian products, taking subgroups, quotient groups, extensions, etc.

The existence of a suitable set for free topological groups is considered in Section IV. Denote by $\mathcal{C}_f$ the class of all spaces X such that the free topological group $F(X)$ on X has a suitable set. We will see that this class is very wide: it contains all metrizable spaces and all *polyadic* compact spaces (that is, continuous images of products of compact spaces with at most one non-isolated point). In particular, all compact dyadic spaces are in $\mathcal{C}_f$. In other words, the free topological group $F(K)$ on a compact dyadic space K has a suitable set.

The minimal and totally minimal topological groups are considered in Section V. It turns out that minimality is a very useful concept when looking for a suitable set. Section V also contains several results on suitable sets in topological groups endowed with the Bohr topology.

Notation and Terminology

We denote respectively by $\mathbb{R}, \mathbb{I}, \mathbb{Z}$ and $\mathbb{N}$ the reals, the unit interval [0,1], the integers and natural numbers. The circle group $\mathbb{R}/\mathbb{Z}$ is denoted by $\mathbb{T}$. The groups $\mathbb{R}$ and $\mathbb{T}$ are assumed to carry their usual additive group operations and topology.

Let G be a group. The neutral element of G is denoted by 1 or 1_G, or respectively by 0 or 0_G if G is Abelian. The minimal subgroup of G containing a subset $A \subseteq G$ is $\langle A \rangle$.

Topological groups are assumed to be Hausdorff. We denote by $\widehat{G}$ the two-sided (Raĭkov) completion of G and by $c(G)$ the connected component of 1 in G. A group G is *precompact* (or *totally bounded*) if $\widehat{G}$ is compact, *pseudocompact* if every continuous real-valued function on G is bounded, and *countably compact* if each open countable cover of G contains a finite subcover.

Let $\mathcal{S}$ (resp., $\mathcal{S}_c$) be the class of groups G having a suitable (resp., closed suitable) set. It turns out that very often a suitable set S for the group G has the stronger property to *generate* G, instead of generating just a dense subgroup of G. We denote by $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ the corresponding subclasses of $\mathcal{S}$ and $\mathcal{S}_c$.

The closure of a subset $Y \subseteq X$ in X is denoted by $cl_X Y$ or simply $cl\, Y$ if there is no ambiguity. When convenient, we also use $\overline{Y}^X$ or $\overline{Y}$ for the same purpose.

A space X is called a *P-space* if an intersection of countably many open sets in X is open. The base of the *ω-modification* of the topology of X is defined as the collection of all G_δ-sets in X. A space X with the ω-modified topology is obviously a P-space. A space X is called *ω-bounded* if the closure of every countable subset of X is compact. An ω-bounded space is countably compact, but not vice versa. A topological group G is ω-bounded iff every countable subset of G is contained in a compact subgroup of G.

We will frequently use the notions of σ- and *Σ-products* both applied to spaces and topological groups. Let us describe them briefly. Choose a point $a \in \Pi = \prod_{\alpha \in A} X_\alpha$ in the cartesian product Π of spaces X_α and for each $x \in \Pi$, define $\mathrm{supp}\,(x) = \{\alpha \in A : x_\alpha \neq a_\alpha\}$. Now the *$\sigma$-product* and *$\Sigma$-product* of the spaces X_α with the *center* a are defined respectively as follows:

$$\sigma(A, a) = \{x \in \Pi : |\mathrm{supp}\,(x)| < \aleph_0\}$$

and

$$\Sigma(A, a) = \{x \in \Pi : |\mathrm{supp}\,(x)| \leq \aleph_0\}.$$

The spaces $\sigma(A, a)$ and $\Sigma(A, a)$ are considered with the topology inherited from Π. If each X_α is a topological group, we will always fix the neutral element 1 of Π as the central point, so the denotations $\sigma(A, 1)$ and $\Sigma(A, 1)$ can be abbreviated respectively to $\sigma(A)$ and $\Sigma(A)$. In the latter case, both $\sigma(A)$ and $\Sigma(A)$ are dense subgroups of Π, and $\sigma(A)$ is also called the *direct sum* of the groups X_α, $\alpha \in A$.

Let X be a space, $x_0 \in X$ and $S \subseteq X \setminus \{x_0\}$. We call S a *supersequence converging to* x_0 if S is an infinite discrete subset of X and $S \cup \{x_0\}$ is a compact subset of X. In other words, $S \cup \{x_0\}$ is the one-point compactification of S and we also call it a *compact supersequence.*

The cardinality of continuum 2^ω will be denoted by $\mathfrak{c}$. The notation for cardinal functions is standard: $w(X)$, $nw(X)$, $iw(X)$, $d(X)$, $\chi(X)$, $\psi(X)$, $L(X)$ and $t(X)$ stand for the weight, network weight, i-weight, density, character, pseudocharacter, Lindelöf number and tightness of X respectively.

The abbreviations CH and MA are used for the Continuum Hypothesis and Martin's Axiom respectively. As usual, $\mathrm{V} = \mathrm{L}$ stands for the axiom of constructibility. The symbol $\diamondsuit$ refers to a special set-theoretic axiom concerning subsets of ω_1. It is well known that $\mathrm{V} = \mathrm{L} \Longrightarrow \diamondsuit \Longrightarrow \mathrm{CH} \Longrightarrow \mathrm{MA}$ (see [Ku]).

1. Topological Groups With a Suitable Set

A non-compact countable space has many infinite closed discrete subsets. It seems natural, therefore, to ask whether every countable topological group has a suitable set. The answer is in the positive (see Theorem 1.2 of [CMRST]).

Theorem 1.1. *Every countable topological group G has a closed discrete subset S such that $\langle S \rangle = G$. In particular, S is suitable for G and $G \in \mathcal{S}_{cg}$.*

A key to the proof of this theorem is the following simple observation:

Lemma 1.2. *Let G be a non-discrete Hausdorff topological group and U a non-empty open subset which generates G. Then every point $x \in U$ has an open neighborhood $V_x \subseteq U$ such that $\langle U \setminus \overline{V_x} \rangle = G$.*

To prove Theorem 1.1, it suffices to consider the case when G is infinite and non-discrete. Being countable, G is zero-dimensional, that is, G has a base of clopen sets. We enumerate the elements of G, say $G = \{g_n : n \in \omega\}$ and construct by induction the sequence $\{V_n : n \in \omega\}$ of clopen subsets of G and the sequence $\{S_n : n \in \omega\}$ of finite subsets of G satisfying the following conditions for each $n \in \omega$:
(i) $g_n \in V_0 \cup V_1 \cup \cdots \cup V_n$;
(ii) $G = \langle G \setminus (V_0 \cup V_1 \cup \cdots \cup V_n) \rangle$;
(iii) $V_i \cap V_n = \emptyset$ if $i < n$;
(iv) $V_i \cap S_n = \emptyset$ if $i < n$, and
(v) $g_n \in \langle S_0 \cup S_1 \cup \cdots \cup S_n \rangle$.

All this is possible because of Lemma 1.2. One easily verifies then that the set $S = \bigcup_{n \in \omega} S_n$ is closed discrete in G and $\langle S \rangle = G$. We conclude, therefore, that $G \in \mathcal{S}_{cg}$.

Suppose that G is a separable topological group and $A \subseteq G$ is a countable dense subset of G. Then $H = \langle A \rangle$ is a countable dense subgroup of G, and by Theorem 1.1, there exists a suitable set S for H. However, S can have many accumulation points in G, so the following problem arises (see Open Question 1 of [CMRST]):

Problem 1.1. *Does there exist in ZFC a separable topological group with no suitable set?*

Note that Theorem 0.3 of Introduction gives us such a group, but under the assumption of Martin's Axiom.

More generally, suppose that H is a dense subgroup of a topological group G and H has a suitable set. Will then G have a suitable set? Combining Theorems 0.3 and 1.1 we immediately conclude that the answer is negative, but again this argument requires MA. We will see in Section II, however, that there exists a relatively simple way to construct such groups G and H without the use of additional set-theoretic assumptions.

The following problem arises if one tries to extend Theorem 1.1 to all topological groups of cardinality $< \mathfrak{c}$.

Problem 1.2. *Is it true that every topological group of cardinality less than $\mathfrak{c}$ has a suitable set?*

As we know, separable topological groups need not have a suitable set (at least under MA). What kind of additional conditions can we impose on a separable group G in order to have $G \in \mathcal{S}$? It turns out that it suffices to assume that either G is not totally bounded or G has countable pseudocharacter (see Corollary 5.8 and Theorem 5.13 of [CMRST]).

Theorem 1.3. *Let G be a separable topological group. Then: (a) if G is not totally bounded, then it has a closed suitable set; (b) if G is of countable pseudocharacter, then it has a suitable set.*

Let us give a sketch of the proof of Theorem 1.3. We will use the following simple but important observation.

Lemma 1.4. *Let H be an open subgroup of a topological group*

G. If H has a (closed) suitable set, then G also has a (closed) suitable set.

Suppose now that G is a separable topological group which is not totally bounded. There exists a neighborhood U of the identity in G such that $G \neq F \cdot U$ for each finite subset F of G. Therefore, we can construct a sequence $\{x_n : n \in \omega\} \subseteq G$ such that $x_n \notin x_k \cdot U$ whenever $k < n$. Choose an open symmetric neighborhood V of the identity in G such that $V^4 \subseteq U$. It is easy to see that for each $x \in G$, $x \cdot V$ intersects at most one element of the family $\gamma = \{x_n \cdot V : n \in \omega\}$. This means that the family γ is discrete. Since G is separable and V is open in G, there exists a countable set $\{y_n : n \in \omega\} \subseteq V$ which is dense in V. Define

$$S = \{x_n : n \in \omega\} \cup \{x_n \cdot y_n : n \in \omega\}.$$

Then S is closed and discrete in G. Denote by H the subgroup of G generated by $V \cup \{x_n : n \in \omega\}$. It is clear that H is open in G and $S \subseteq H$. In addition, $\{y_n : n \in \omega\} \subseteq \langle S \rangle$, so that $\langle S \rangle$ is dense in H. We conclude, therefore, that S is a closed suitable set for H. By Lemma 1.4, G has a closed suitable set.

The case where G is a separable group of countable pseudocharacter requires another strategy. We can assume without loss of generality that G is totally bounded. Let L be a countable dense subgroup of G, say $L = \{x_n : n \in \omega\}$. Choose a decreasing sequence $\{U_n : n \in \omega\}$ of open neighborhoods of the identity 1_G in G satisfying the conditions:

(1) $U_{n+1}^2 \subseteq U_n$ for each $n \in \omega$;
(2) $\{1_G\} = \bigcap_{n\in\omega} U_n$.

Since G is totally bounded, one can construct by induction a sequence $\{S_n : n \in \omega\}$ of finite subsets of L to satisfy the following conditions for each $k \in \omega$:

(i) $x_k \in \langle S_k \rangle$;
(ii) $S_{k+1} \setminus S_k \subseteq U_k$;

(iii) $G = \langle S_k \rangle \cdot U_k$.

Put $S = \bigcup_{k\in\omega} S_k$. From (ii) it follows that $S \setminus U_k \subseteq S_k$ is finite for each $k \in \omega$. The latter, together with (1) and (2), implies that S is a closed discrete subset of $G \setminus \{1_G\}$. Finally, we apply (i) to conclude that $L = \langle S \rangle$, so S is suitable for G. This finishes the proof of Theorem 1.3.

As an immediate corollary to Theorem 1.3 we have the following.

Corollary 1.5. *Every topological group with a countable network has a suitable set.*

Indeed, a topological group with a countable network is separable and has countable pseudocharacter, so Theorem 1.3 (b) applies.

Theorem 1.3 suggests the question whether every topological group of countable pseudocharacter is in $\mathcal{S}$. Theorem 2.2.4 (a) of Section II answers the question in the negative: there exists a Lindelöf topological group L of countable pseudocharacter with $L \notin \mathcal{S}$. However, every σ-compact group of countable pseudocharacter has a suitable set, because such a group has a countable network (note that every compact subspace of a topological group of countable pseudocharacter has a diagonal of type G_δ, and hence is metrizable). On the other hand, not every σ-compact topological group is in $\mathcal{S}$ (see Corollary 2.1.5). However, the things change in the separable case.

Proposition 1.6. *A separable σ-compact topological group has a suitable set.*

In fact, we prove here a more general result which appears as Lemma 3.5 of [DTT1].

Theorem 1.7. *If a separable topological group G is not pseu-*

docompact, then G has a closed suitable set.

Proof. Let $P = \{x_n : n \in \omega\}$ be a countable dense subgroup of G. Since G is not pseudocompact, there exists a sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of G such that $\overline{U}_{n+1} \subseteq U_n$ for each $n \in \omega$ and $\emptyset = \bigcap_{n\in\omega} U_n$. We can assume without loss of generality that G is totally bounded — otherwise the conclusion follows from Theorem 1.3 (a). The construction that follows is close to that in the proof of Theorem 1.3 (b), but we give more details.

We will construct an increasing sequence $\{S_k : k \in \omega\}$ of finite subsets of P satisfying the following conditions for each $k \in \omega$:

(1) $x_k \in \langle S_k \rangle$;
(2) $S_{k+1} \setminus S_k \subseteq U_k$;
(3) $G = \langle S_k \rangle \cdot U_k$.

Since P is dense in G, we have $G = P \cdot U_0$. By our assumption, G is totally bounded, so there exists a finite subset K_0 of P such that $K_0 \cdot U_0 = G$. In particular, there are $a_0 \in K_0$ and $u_0 \in U_0$ such that $x_0 = a_0 \cdot u_0$. Then $u_0 = a_0^{-1} \cdot x_0$ and we put $S_0 = K_0 \cup \{u_0\}$.

Suppose that for some $n \in \omega$ we have defined an increasing sequence $S_0, \ldots, S_n$ of finite subsets of P which satisfies (1) and (3) for each $k \leq n$ and (2) for every $k < n$. Since P is dense in G, the set $U_n \cap P$ generates a dense subgroup of the group $G_n = \langle U_n \rangle$. Therefore, we have the equality $\langle U_n \cap P \rangle \cdot U_{n+1} = G_n$. Using total boundedness of G_n, we find a finite subset F_{n+1} of $\langle U_n \cap P \rangle$ such that $F_{n+1} \cdot U_{n+1} = G_n$. Choose a finite subset K_{n+1} of $U_n \cap P$ with $F_{n+1} \subseteq \langle K_{n+1} \rangle$ and note that $\langle K_{n+1} \rangle \cdot U_{n+1} = G_n$. Define $S'_{n+1} = S_n \cup K_{n+1}$ and apply (3) (with $k = n$) to obtain

$$(*) \qquad \begin{aligned} \langle S'_{n+1} \rangle \cdot U_{n+1} &= \langle S'_{n+1} \rangle \cdot \langle K_{n+1} \rangle \cdot U_{n+1} \\ &= \langle S'_{n+1} \rangle \cdot G_n \supseteq \langle S_n \rangle \cdot U_n = G. \end{aligned}$$

By $(*)$, there are $a_{n+1} \in \langle S'_{n+1} \rangle$ and $u_{n+1} \in U_{n+1}$ such that $x_{n+1} = a_{n+1} \cdot u_{n+1}$. Since $a_{n+1} \in \langle S'_{n+1} \rangle \subseteq P$ and $x_{n+1} \in P$, we conclude that $u_{n+1} \in P$ and put $S_{n+1} = S'_{n+1} \cup \{u_{n+1}\}$. Clearly, S_{n+1} is a finite subset of P and $S_n \subseteq L_{n+1}$. From $(*)$ it follows that $\langle S_{n+1} \rangle \cdot U_{n+1} = G$. This gives (1) and (3) for $k = n + 1$. Since $S_{n+1} \setminus S_n \subseteq K_{n+1} \cup \{u_{n+1}\} \subseteq U_n$, the condition (2) holds for $k = n$ as well.

Put $S = \cup\{S_n : n \in \omega\}$. It follows from (2) that $S \backslash \overline{U}_k \subseteq S_k$ is a finite set for each $k \in \omega$. The latter, along with $\bigcap_{n\in\omega} \overline{U}_n = \emptyset$ implies that S is a closed discrete subset of G. It remains to apply (1) in order to conclude that $\langle S \rangle = P$. □

As we have mentioned before Proposition 1.6, neither separability nor σ-compactness, nor countable pseudocharacter imply the existence of a suitable set. However, by Theorem 1.3 and Proposition 1.6, any two of the three properties do imply this.

An important corollary of Theorem 1.7 concerns Problem 1.1: if a separable topological group has no suitable set, then it has to be *pseudocompact*. This observation makes Problem 1.1 even more intriguing. We can also note that if a topological group G with $|G| < \mathfrak{c}$ is separable, then $G \in \mathcal{S}_c$. Indeed, this is trivial if G is finite. Otherwise G is not pseudocompact, because infinite pseudocompact groups have cardinality greater than or equal to $\mathfrak{c}$. It remains to apply Theorem 1.7 to conclude that G has a closed suitable set. Thus, we have obtained the positive answer to Problem 1.2 in the special case when G is separable. We do not know, however, if one can replace "separable" by "σ-compact" (or "of countable pseudocharacter"):

Problem 1.3. *Let G be a σ-compact topological group with $|G| < \mathfrak{c}$. Is it true that $G \in \mathcal{S}$?*

Let us turn to metrizable topological groups. We can refine Theorem 0.2 of Introduction as it appears in [CMRST]:

Theorem 1.8. *Every metrizable topological group G has a suitable set. Further, if G is not compact, it has a closed suitable set.*

We present here the proof of the first part of Theorem 1.8 which depends on a couple of simple lemmas.

Lemma 1.9. *Let X be a subset of a topological group G and U an open neighborhood of the identity in G. Then there exist an ordinal γ and a subset $Y = \{x_\alpha : \alpha < \gamma\}$ of X such that $x_\beta \notin x_\alpha \cdot U$ whenever $\alpha < \beta < \gamma$ and $X \subseteq Y \cdot U$. Further, if V is a symmetric neighborhood of 1_G in G with $V^4 \subseteq U$, then the set Y is uniformly V-discrete in G (i. e., the set $x \cdot V$ contains at most one point of Y for each $x \in G$).*

The proof of Lemma 1.9 is straightforward: if the sequence $A_\beta = \{x_\alpha : \alpha < \beta\}$ has been defined for some ordinal β and $X \setminus A_\beta \cdot U \neq \emptyset$, we simply choose a point $x_\beta \in X \setminus A_\beta \cdot U$.

Lemma 1.10. *If $\{O_n : n \in \omega\}$ is a symmetric open basis at the identity of a topological group G, and for each $n \in \omega$ the set $F_n \subseteq G$ satisfies $F_n \cdot O_n = G$, then $F = \bigcup_{n\in\omega} F_n$ is dense in G.*

Indeed, if W is a non-empty open subset of G, then there are $x \in G$ and $n \in \omega$ such that $x \cdot O_n \subseteq W$. Then $x \in y \cdot O_n$ for some $y \in F_n$, whence $y \in x \cdot O_n^{-1} = x \cdot O_n \subseteq W$. This implies that $y \in W \cap F_n \subseteq W \cap F \neq \emptyset$.

Proof of Theorem 1.8. Let $\{V_n : n \in \omega\}$ be a base at the identity in G satisfying $V_0 = G$, $V_{n+1}^4 \subseteq V_n$ and $V_n^{-1} = V_n$ for each $n \in \omega$. By Lemma 1.9, for every $n \in \omega$ we can define a subset $F_n = \{x_{n,\alpha} : \alpha < \gamma_n\}$ of V_n so that

(i) $V_n \subseteq F_n \cdot V_{n+1}$;

(ii) $x_{n,\beta} \notin x_{n,\alpha} \cdot V_{n+1}$ whenever $\alpha < \beta < \gamma_n$.

Put $S = \bigcup_{n\in\omega} F_n$. We claim that the set S is suitable for G. First, we prove that

(1) $\langle S\rangle \cdot V_n = G$ for each $n \in \omega$.

Indeed, it suffices to show that

(2) $F_0 \cdot F_1 \cdot \dots \cdot F_n \cdot V_{n+1} = G$ for each $n \in \omega$.

We prove (2) by induction on n. The equality $F_0 \cdot V_1 = G$ follows from (i). If the equality (2) is valid for some $n \in \omega$, then (i) implies that

$$F_0 \cdot F_1 \cdot \dots \cdot F_n \cdot F_{n+1} \cdot V_{n+2} \supseteq F_0 \cdot F_1 \cdot \dots \cdot F_n \cdot V_{n+1} = G.$$

This proves (2), and hence (1). Since the sets V_n form a base at the identity of G, the definition of S, (1) and Lemma 1.10 together imply that $\langle S\rangle$ is dense in G. It remains to show that S is closed and discrete in $G \setminus \{1\}$. Let $x \in G$ be arbitrary, $x \neq 1$. There exists $n \in \omega$ such that $x \notin \overline{V_n}$. Since $F_k \subseteq V_k$ for each $k \in \omega$, we have $F_k \cap (G \setminus \overline{V_n}) = \emptyset$ whenever $k > n$. Therefore, $G \setminus \overline{V_n}$ can intersect only the sets $F_0, F_1, \dots, F_n$. From (ii) and $V_{n+2}^4 \subseteq V_{n+1}$ it follows that the set F_n is uniformly V_{n+2}-discrete in G, and hence is closed discrete in G, $n \in \omega$. Therefore, the union $F = F_0 \cup F_1 \cup \dots \cup F_n$ is a closed discrete subset of G, and there exists an open neighborhood W of x in G whose intersection with F contains at most one point. Clearly, the neighborhood $O = W \cap (G \setminus \overline{V}_n)$ of x has the property $|O \cap S| \leq 1$. This proves that S is closed and discrete in $G \setminus \{1\}$. □

Theorem 1.8 admits several generalizations obtained in [DTT1] and [OTk]. Let us call a subset Y of a space X *strictly*

σ-discrete (in X) if Y can be represented as a countable union of closed discrete subsets of X. It is clear that every separable space contains a dense strictly σ-discrete subset. Since a metrizable space has a σ-discrete base, we conclude that such a space also contains a dense strictly σ-discrete subset. It is well known that a pseudocompact topological group of countable pseudocharacter is compact [CS, lemma 3.1], and hence it has a suitable set by a result of Hofmann and Morris [HM]. So, the following result generalizes both Theorem 1.7 and Theorem 1.8.

Theorem 1.11. *Let G be a non-pseudocompact topological group. If G has a dense strictly σ-discrete subset then it has a closed suitable set, i. e., $G \in \mathcal{S}_c$.*

In fact, the proof of the latter result leans on Theorem 1.7 (see [DTT1, Theorem 3.6]), but much work is required in this case.

Theorem 1.11 implies several corollaries (see [DTT1]).

Corollary 1.12. *Any locally separable non-pseudocompact topological group has a closed suitable set.*

Corollary 1.13. *Every topological group with a σ-discrete network has a suitable set.*

To make the last assertion clearer, note that a group G with a σ-discrete network has countable pseudocharacter. If G is not pseudocompact, the conclusion follows from Theorem 1.11. Otherwise G has countable character by Lemma 3.1 of [CS] and the Birkhoff–Kakutani metrization theorem implies that G is compact metrizable, so that $G \in \mathcal{S}$ by Theorem 0.1.

Corollary 1.14. *If a topological group is a union of countably many closed metrizable subspaces, then it has a suitable set.*

Note that such a group has a σ-discrete network. Therefore, the above assertion follows from Corollary 1.13. It is not known if one can omit "closed" in Corollary 1.13. We thus have the following open problem posed in [DTT1]:

Problem 1.4. *Suppose that a topological group G is a union of countably many metrizable subspaces. Does G have a suitable set?*

Metrizable spaces form a proper subclass of a (significantly wider) class of *stratifiable* spaces. Stratifiable spaces were introduced by Ceder [Ce] under a different name, and their present name is due to Borges [Bo]. Since every stratifiable space has a σ-discrete network [Gr, Theorem 5.9], Corollary 1.13 implies the following.

Corollary 1.15. *Every stratifiable topological group has a suitable set.*

The ideas used in [HM] and [CMRST] to prove respectively that locally compact and metrizable groups are in $\mathcal{S}$ were completely different. It appears interesting, therefore, to unify these results in a single assertion. An idea of such a unification can be the use of the notion of an *almost metrizable* topological group introduced in [Pa]. A topological group G is said to be *almost metrizable* if it contains a non-empty compact set K of countable character in G. It is easy to verify that G is almost metrizable iff it contains a compact subgroup N such that the quotient space G/N is metrizable [Pa]. Clearly, all locally compact and all metrizable groups are almost metrizable. Therefore, the following result (see [OTk, Corollary 1.2]) generalizes Theorems 0.1 and 0.2.

Theorem 1.16. *Every almost metrizable topological group has a suitable set.*

Linearly ordered topological groups form a very special subclass of topological groups (see [NR] for basic information on this subject). It is very likely that these groups are in $\mathcal{S}$.

Problem 1.5. *Does every linearly orderable topological group have a (closed) suitable set?*

Let us now turn to topological groups which have no suitable set.

2. Topological Groups Without a Suitable Set

The reader has probably seen that the technique used in Section I to prove "positive" results does not take too much from algebra, but has a topological origin. The explanation of this phenomenon is quite easy: the notion of a suitable set is "almost" topological. The results of this section go in the same topological direction, but they are more sophisticated from both topological and algebraic sides.

Theorems 1.3, 1.7, 1.8 and their generalizations presented in Section I make it clear that any topological group with no suitable set has to be complicated in some sense. The tools we use for construction of counterexamples are *free topological groups* and *spaces of continuous functions.*

2.1. Application of Free Topological Groups. Let X be a space. Recall that a topological group G with a fixed topological embedding $i : X \to G$ is called the *free topological group* on X if G satisfies the following conditions:

(1) $i(X)$ algebraically generates G;

(2) given a continuous mapping $f\colon X \to H$ of X to an arbitrary topological group H, there exists a continuous homomorphism $\hat{f}\colon G \to H$ such that $\hat{f} \circ i = f$.

The free topological group on X is usually denoted by $F(X)$.

It is a common practice to identify X with its image $i(X) \subseteq F(X)$ and we shall follow that. Then (2) can be reformulated by saying that every continuous mapping $f\colon X \to H$ extends to a continuous homomorphism $\hat{f}\colon F(X) \to H$.

If the groups G and H in the above definition are Abelian, we obtain the definition of the *free Abelian topological group* on X which is denoted by $A(X)$.

The elements of $F(X)$ are the "irreducible words" $g = x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n}$, where $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. The number n is the *length* of g. It is known that for each $n \in \omega$, the subset $F_n(X)$ of $F(X)$ consisting of all words of length $\leq n$ is closed in $F(X)$ (see [Gra]).

Let $\widehat{X}$ be the free sum of X, its copy X^{-1} and the identity e of $F(X)$. Then for every $n \in \omega$, the subspace $F_n(X)$ of $F(X)$ is a continuous image of $\widehat{X}^n$ under the multiplication mapping $(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) \mapsto x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n}$. Therefore, if X is a compact space, then $F_n(X)$ is also compact for each $n \in \omega$.

By a theorem of Graev [Gra], the free topological group $F(X)$ on a compact space X has the following remarkable property: a set $K \subseteq F(X)$ is closed in $F(X)$ iff $K \cap F_n(X)$ is compact for each $n \in \omega$. This result plays a crucial role in our considerations.

The following fact about suitable sets in free topological groups was established in [CMRST].

Theorem 2.1.1. *The free topological group $F(X)$ on a separable space X has a closed suitable set. The same is true for the free Abelian topological group $A(X)$.*

The idea of the proof is rather easy. Let $D = \{x_n : n \in \omega\}$ be a dense subset of X. For every $n \geq 1$, define the element $g_n = x_1 \cdot \cdots \cdot x_n$ of $F(X)$. Then the set $S = \{g_n : n \in \omega\}$ generates the subgroup $\langle S \rangle$ of $F(X)$ which contains D, so $\langle S \rangle$ is dense in $F(X)$. To see that S is closed and discrete in $F(X)$, consider the free topological group $F(\beta X)$ on the Čech–Stone compactification βX of X. The identity mapping $i_X : X \to X$ extends to a continuous monomorphism $\hat{\imath}_X : F(X) \to F(\beta X)$. Clearly, for each $n \in \omega$ the intersection $\hat{\imath}_X(S) \cap F_n(\beta X)$ consists exactly of n elements, and hence is compact. By Graev's theorem, $\hat{\imath}_X(S)$ is closed in $F(\beta X)$. Since $\hat{\imath}_X$ is a continuous one-to-one mapping, S is closed in $F(X)$. The same argument applied to an arbitrary subset T of S shows that T is closed in $F(X)$. We conclude, therefore, that S is closed and discrete in $F(X)$. In particular, S is suitable for $F(X)$. One easily verifies that this reasoning also works for the free Abelian group $A(X)$.

Recall that X is called an *F-space* if every F_σ-set in X is C^*-embedded in X. It is well known that for every infinite discrete space D, βD and $\beta D \setminus D$ are F-spaces (see pp. 210 and 215 of [GJ]).

The following theorem provides a large number of σ-compact topological groups without suitable sets (see Theorem 3.8 of [CMRST]).

Theorem 2.1.2. *Let X be a non-separable compact F-space. Then the free Abelian topological group $A(X)$ does not have a suitable set.*

Note that "non-separable" appears here because of Theorem 2.1.1. The proof of Theorem 2.1.2 is based on the following auxiliary topological result (see Lemma 3.5 of [CMRST]):

Lemma 2.1.3. *Let X be a compact F-space, and let n be any positive integer. Denote by Y_n the subspace of X^n consisting of all points in general position or, in other words, a point $y = (x_1, \ldots, x_n) \in X^n$ belongs to Y_n if all coordinates of y are pairwise distinct. Then Y_n is countably compact.*

An analysis of the proof of Lemma 2.1.3 given in [CMRST] shows that the result remains valid for a compact space X in which countable discrete subsets are C^*-embedded. Therefore, the conclusion of Theorem 2.1.2 also holds for such a space X (see Remark 3.9 of [CMRST]). In fact, the argument in the proof of Theorem 3.8 of [CMRST] gives even more:

Theorem 2.1.4. *Let X be a compact space in which every countable discrete subset is C^*-embedded. Then the free Abelian topological group $A(X)$ does not have non-trivial convergent sequences.*

A relation between convergent sequences and suitable sets is not evident, but one can show that a non-separable σ-compact topological group with a suitable set has many convergent sequences. A partial result in this direction is Lemma 4.6 of Section IV. On the other hand, the problem of the existence of non-trivial convergent sequences in free topological groups on compact spaces is well-studied (see [EOY]).

All this leaves, however, the following open problem.

Problem 2.1. *Let X be a non-separable compact space without non-trivial convergent sequences. Is it true that $A(X) \notin \mathcal{S}$?*

Note that the conclusions of Lemma 2.1.3 and Theorem 2.1.4 are false for the spaces as in Problem 2.1: the Alexandroff duplicate Y of $\beta\mathbb{N} \setminus \mathbb{N}$ is a counterexample (nevertheless, the group $A(Y)$ does not have a suitable set by Corollary 4.11 of Section IV).

Since $\beta\mathbb{N} \setminus \mathbb{N}$ is a non-separable compact F-space, Theorem 2.1.2 immediately implies the following.

Corollary 2.1.5. *The free Abelian topological group $A(\beta\mathbb{N}\setminus\mathbb{N})$ does not have a suitable set.*

A generalization of Theorem 2.1.2 has recently been obtained by Tomita and Trigos-Arrieta [TT]. They show that if X is a non-separable compact F-space and H is a sequentially compact topological group, then the product $A(X)^n \times H$ does not have a suitable set. In particular, no finite power of $A(X)$ has a suitable set.

The free Abelian topological group $A(\beta\mathbb{N})$ on the separable compact space $\beta\mathbb{N}$ has a closed suitable set by Theorem 2.1.1. Note that the group $A(\beta\mathbb{N}\setminus\mathbb{N})$ is topologically isomorphic to a closed subgroup of $A(\beta\mathbb{N})$ (extend the embedding $i\colon \beta\mathbb{N}\setminus\mathbb{N} \to \beta\mathbb{N}$ to a continuous monomorphism $\hat{\imath}\colon A(\beta\mathbb{N}\setminus\mathbb{N}) \to \mathbb{A}(\beta\mathbb{N})$ and apply the Graev's description of closed subsets of the free topological group on a compact space to conclude that $\hat{\imath}$ is a topological embedding). We have obtained, therefore, the following result.

Corollary 2.1.6. *A closed subgroup of a topological group with a closed suitable set does not necessarily have a suitable set.*

Let us mention that a dense subgroup of a topological group with a closed suitable set can also fail to have a suitable set. We will see later more convincing examples of this kind, but this is a good place to mention here a tricky one which makes use of free topological groups (see Remark 3.13 of [CMRST]).

Recall that a *variety of topological groups* is a class of topological groups closed under formation of subgroups, quotient groups, and arbitrary cartesian products. An analysis of the proof of Theorem 2.1.2 given in [CMRST] shows that the result remains valid if we replace free Abelian topological group by free topological group in the variety $\mathcal{B}(\mathbb{T})$ of topological groups generated by the circle group, $\mathbb{T}$. In particular, if X is one of the two connected components of $\beta\mathbb{R}\setminus\mathbb{R}$ (which is an a compact F-space), then the group $A(X,\mathcal{B}(\mathbb{T}))$ has no suitable set. This group, however, is a σ-compact connected dense subgroup of its closure in $\mathbb{T}^{\mathfrak{c}}$, and this closure, like every compact connected Abelian group of weight $\leq \mathfrak{c}$, is monothetic by Theorem

25.14 of [HR] and hence has a one-element suitable set. This argument proves the following result.

Corollary 2.1.7. *A dense subgroup of a monothetic group need not have a suitable set.*

A similar problem posed in [CMRST] as Open Question 2 was to find out whether there exists (in ZFC) an example of a Hausdorff topological group G which has a dense subgroup H with a suitable set, but G itself does not have a suitable set. The answer again is negative. A simple example based on Theorem 2.1.2 was suggested by A. Tomita. Let D be an uncountable discrete space and let $G = A(\beta D)$ be the free Abelian topological group on βD. Denote by H the subgroup of $A(\beta D)$ algebraically generated by D. Then H is dense in G, the discrete set D generates H algebraically, and $D = H \cap \beta D$ is closed in H. Therefore, D is a closed suitable set for H, while βD is a compact non-separable F-space, and according to Theorem 2.1.2, the group G has no suitable set. Our conclusion is the following:

Corollary 2.1.8. *There exist a σ-compact topological group G and a dense subgroup H of G such that $H \in \mathcal{S}_c$, but $G \notin \mathcal{S}$.*

The latter result motivates the problem that follows.

Problem 2.2. *Is it true that every σ-compact topological group has a dense subgroup with a suitable set?*

We will show below that under $\diamondsuit$ there exists a hereditarily Lindelöf non-separable topological group G such that no dense subgroup of G has a suitable set (see Theorem 2.2.4 (b)). We do not know, however, if it possible to construct in ZFC a topological group with no dense subgroups in $\mathcal{S}$. This might indicate that the answer to Problem 2.1 is positive.

2.2. Application of C_p-techniques. Let X be a space. Denote by $C_p(X)$ the space of all real-valued continuous functions on X equipped with the topology of pointwise convergence. This means that $C_p(X)$ is considered as the subspace of $\mathbb{R}^X$ and the latter carries the product topology. The space $C_p(X)$ with the usual sum of functions and multiplication by reals becomes a locally convex topological linear space [Ar1], and hence a topological group.

Given another space Y, one can define in a similar way the space $C_p(X, Y)$ of continuous functions from X to Y identifying it with the corresponding subspace of the product Y^X. If Y is a topological group, then $C_p(X, Y)$ is also a topological group (being a subgroup of the product group Y^X).

A continuous bijection $f: X \to Y$ is called a *condensation* of X onto Y. We will also say that f *condenses* X onto Y. If $g: X \to Y$ is a continuous map and $g: X \to g(X)$ is one-to-one, we say that g *condenses* X *into* Y.

We present here a short list of the main facts of C_p-theory which will be used in the sequel (the proofs and discussions can be found in [Ar1]).

Theorem 2.2.1. (References to C_p-theory.) *Let Y be $\mathbb{I}$, $\mathbb{T}$ or $\mathbb{R}$. Then:*

(a) $nw(C_p(X, Y)) = nw(X)$;

(b) X is separable iff $C_p(X, Y)$ condenses onto a second countable space; X condenses onto a second countable space iff $C_p(X, Y)$ is separable;

(c) $C_p(X, Y)$ has countable tightness iff all finite powers of X are Lindelöf; all finite powers of $C_p(X, Y)$ are hereditarily Lindelöf iff all finite powers of X are hereditarily separable;

(d) If X is the one-point compactification of a discrete space, then $C_p(X)$ topologically embeds into a Σ-product of unit segments.

Topological groups with a suitable set satisfy certain cardinal constraints as the following result from [DTT1] shows. We will use it to construct a special topological group without a suitable set.

Lemma 2.2.2. *A topological group $G \in \mathcal{S}$ satisfies $d(G) \leq L(G) \cdot \psi(G)$. In particular, a non-separable Lindelöf topological group of countable pseudocharacter does not have a suitable set.*

Proof. Suppose that A is a suitable set for G. If U is an open neighborhood of the identity in G, then $A \setminus U$ is closed and discrete in G, and hence $|A \setminus U| \leq L(G)$. Fix a family γ of open sets in G such that $\cap\gamma = \{1\}$ and $|\gamma| = \psi(G)$. It follows from

$$A \setminus \{1\} \subseteq \bigcup\{A \setminus U) : U \in \gamma\}$$

that $|A| \leq L(G) \cdot \psi(G)$. The subgroup $H = \langle A \rangle$ of G obviously satisfies $|H| \leq |A| \cdot \aleph_0$. Since A is a suitable set, the group H is dense in G which implies that

$$d(G) \leq |H| \leq |A| \cdot \aleph_0 \leq L(G) \cdot \psi(G).$$

The rest of the lemma is immediate. □

By a result of Okunev and Tamano [OTa], there exists a σ-compact separable space X such that $nw(X) > \aleph_0$ and $C_p(X)$ is Lindelöf. We use this space to prove the following.

Theorem 2.2.4. *(a) There exists a Lindelöf non-separable linear topological space L of countable tightness and countable i-weight (and hence of countable pseudocharacter). Thus, L considered as a topological group does not have a suitable set.*
(b) Under $\diamondsuit$ there exists a hereditarily Lindelöf non-separable linear topological space L of countable tightness. Therefore, L considered as a topological group does not have a suitable set. In addition, no dense additive subgroup of L has a suitable set.

Proof. (a) Let X be the space constructed by Okunev and Tamano [Ota] and $L = C_p(X)$. Then L is Lindelöf. Since X is σ-compact (and hence all finite powers of X are Lindelöf), Theorem 2.2.1 (b) implies that $t(L) \leq \aleph_0$. In addition, $iw(L) \leq \aleph_0$ because X is separable. However, L can not be separable for otherwise the i-weight of X would be countable by Theorem 2.2.1 (d1) and X would have countable network — a contradiction with the choice of the space X.
(b) Under $\diamondsuit$, Ivanov [Iv] constructed a compact non-metrizable space X such that X^n is hereditarily separable for each $n \in \mathbb{N}$. If we put $L = C_p(X)$, then L is hereditarily Lindelöf by Theorem 2.2.1 (c), and hence L has countable pseudocharacter. The space $C_p(X)$ is not separable, because otherwise the compact space X would be metrizable by Theorem 2.2.1 (b). If H is a dense additive subgroup of L, then H is a Lindelöf non-separable group of countable pseudocharacter, so Lemma 2.2.2 implies that H does not have a suitable set. □

By Theorem 0.3, there exists under MA a countably compact topological group with no suitable set. It was natural, therefore, to ask whether one can construct in ZFC a pseudocompact topological group which has no suitable set (see Open Question 3 of [CMRST]). This question was recently solved in the positive in [DTT1]. The group constructed in [DTT1] is not only pseudocompact, it is *ω-bounded* (hence countably compact). We present here a brief description of the construction given in [DTT1].

Theorem 2.2.5. *There exists a connected, locally connected Abelian topological group G with the following properties:*
(1) G is ω-bounded;
(2) G is a dense subgroup of $\mathbb{T}^{2^{\mathfrak{c}}}$;
(3) G^{κ} does not have a suitable set for each $\kappa \geq 1$; in particular, $G \notin \mathcal{S}$.

It is quite easy to define the group G as in Theorem 2.2.5, but the proof of the fact that it satisfies (1)–(3) requires a lot of work. Let X be the Tikhonov cube $\mathbb{T}^{\mathfrak{c}}$ with the ω-modified topology and $G = C_p(X, \mathbb{T})$. Then G is a dense subgroup of $\mathbb{T}^{2^{\mathfrak{c}}}$. Note that X is a P-space, so one can apply the following assertion proved in [DTT1] to conclude that G is ω-bounded (and hence countably compact):

Proposition 2.2.6. *Let Y be a compact metrizable space. If X is a P-space, then $C_p(X, Y)$ is ω-bounded.*

Since G is dense in $\mathbb{T}^X$ and countably compact, the projections of G cover all countable faces of $\mathbb{T}^X$. Therefore, G is connected by a result of [Tk]. A similar argument gives local connectedness of G.

To show that G does not have a suitable set, we first note that G is not separable — otherwise X would condense onto a second countable space by Theorem 2.2.1 (b), and we would have $|X| \leq \mathfrak{c}$, a contradiction. Thus, a suitable set S for G must be uncountable. Since the group G is countably compact, $Y = S \cup \{0_G\}$ is the one-point compactification of S.

The second step of the proof is to note that Y *separates points* of X, that is, for any distinct $x, y \in X$ there exists $f \in Y$ such that $f(x) \neq f(y)$. If this were not so, the dense subgroup $H = \langle Y \rangle$ of G generated by Y would not separate points of X either, which is obviously impossible. It is a standard fact of C_p-theory (see Proposition 0.5.4 (a) of [Ar1]) that if a family $E \subsetneq C_p(X, Z)$ separates the points of X, then the diagonal product φ of the mappings of E is a one-to-one continuous mapping of X to $C_p(E, Z)$. But $C_p(Y, \mathbb{T})$ embeds into a Σ-product $\Sigma(Y)$ of circles by Theorem 2.2.1 (d). Thus, φ condenses X into $\Sigma(Y)$, and this contradicts the following result of [DTT1]:

Lemma 2.2.7. *The space $\mathbb{T}^{\mathfrak{c}}$ with the ω-modified topology does* *not admit a condensation into a Σ-product of second countable spaces.*

This proves that G does not have a suitable set. To show that G^κ does not have a suitable set for each $\kappa \geq 1$, we use a simple but important fact below (see Proposition 2.7 of [DTT1]):

Lemma 2.2.8. *Let $f\colon K \to L$ be a continuous epimorphism of countably compact groups. Then $K \in \mathcal{S}$ implies $L \in \mathcal{S}$.*

So, let $p\colon G^\kappa \to G$ be a projection where $\kappa \geq 1$. Then G^κ is ω-bounded and hence countably compact. If G^κ had a suitable set, Lemma 2.2.8 would imply that G has a suitable set, a contradiction. This proves Theorem 2.2.5. $\square$

The group G in Theorem 2.2.5 has the cardinality $2^{\mathfrak{c}}$, so it is natural to ask whether there exists a topological group having similar properties to G, but of cardinality $\mathfrak{c}$. The answer is "yes". By Corollary 2.9 of [DTT1], the group G contains a dense, connected, locally connected subgroup H with $|H| = \mathfrak{c}$ such that H^κ does not have a suitable set for each κ satisfying $1 \leq \kappa \leq \omega$. This leaves, however, the following problems.

Problem 2.3. *Does there exist a topological group H of size $\mathfrak{c}$ such that H^κ has no suitable set for each $\kappa \geq 1$?*

Problem 2.4. *Does there exist an ω-bounded topological group of size $\mathfrak{c}$ without a suitable set?*

Let us mention here that a group H as in Problem 2.2 has to be *countably compact.* In fact, the following theorem proved in [DTT1] implies even more.

Recall that a cardinal $\tau \geq \aleph_0$ is called *measurable* if there exists an $\aleph_0$-complete free ultrafilter on the set τ. It is consistent with ZFC that there are no measurable cardinals: V = L implies that all cardinals are non-measurable. All small cardinals such as $\aleph_0$, $\mathfrak{c} = 2^{\aleph_0}$, $2^{\mathfrak{c}}$, etc., are non-measurable.

Theorem 2.2.9. *Let G be a topological group of non-measurable cardinality. If no positive power of G has a suitable set, then G^κ is countably compact for each $\kappa \geq 1$.*

A key to the proof of Theorem 2.2.9 is the following simple lemma.

Lemma 2.2.10. *Let G be a topological group.*
(a) If G contains a closed discrete subset A such that $|A| \geq d(G)$, then $G \times G$ has a closed suitable set.
(b) If G contains a closed discrete subset A of size $|G|$, then $G \times G$ has a closed generating set, that is, $G \times G \in \mathcal{S}_{cg}$.

Proof. Let us prove the first part of the lemma. Suppose that A is a closed discrete subset of G with $|A| \geq d(G)$. Choose a dense subset D of G with $1 \notin D$ satisfying $|D| = d(G)$ and denote by φ any mapping of A onto D. We define a subset S of $G \times G$ by

$$S = (A \times \{1\}) \cup (\{1\} \times A) \cup \{(x, \varphi(x)) : x \in A\} \cup \{(\varphi(x), x) : x \in A\}.$$

It is clear that S is closed discrete in $G \times G$ and the subgroup $\langle S \rangle$ of $G \times G$ generated by S is dense in $G \times G$. The second assertion of the lemma can be proved in a similar way. $\square$

Let us turn to Theorem 2.2.9. Suppose that G^τ is not countably compact for some cardinal τ. Then, by a theorem of Ginsburg and Saks [GS], $G^{2^{\mathfrak{c}}}$ is not countably compact either. Therefore, $G^{2^{\mathfrak{c}}}$ contains an infinite closed discrete subset, say, a copy of $\mathbb{N}$. Denote $\kappa = |G| \cdot \mathfrak{c}$. Since $(G^{2^{\mathfrak{c}}})^{2^\kappa} \cong G^{2^\kappa}$, the group G^{2^κ} contains a closed homeomorph of $\mathbb{N}^{2^\kappa}$. By a result of Juhász [Ju], $\mathbb{N}^{2^\kappa}$ contains a closed discrete subset of size 2^κ. We conclude, therefore, that G^{2^κ} contains a closed discrete subset of cardinality 2^κ. It remains to note that $d(G^{2^\kappa}) \le 2^\kappa$, so Lemma 2.2.10 implies that the group $G^{2^\kappa} \times G^{2^\kappa} \cong G^{2^\kappa}$ has a closed suitable set. This proves Theorem 2.2.9.

Was connectedness (or local connectedness) of the group G in Theorem 2.2.5 important to imply that $G \notin \mathcal{S}$? The answer is negative as the following theorem of [DTT1] shows. In the sequel we denote the discrete two-element group $\{0, 1\}$ by 2.

Theorem 2.2.11. *There exists a zero-dimensional topological group G with the following properties:*
(1) G is ω-bounded;
(2) G is a dense subgroup of $2^{\mathfrak{c}}$;
(3) G^κ does not have a suitable set for each $\kappa \ge 1$.

The construction of such a group G is very much like to that in Theorem 2.2.5. Let $\Sigma(\mathfrak{c})$ be the Σ-product of $\mathfrak{c}$ many copies of the discrete doubleton 2 with an arbitrary center, $\Sigma \subseteq 2^{\mathfrak{c}}$. Consider the ω-modification X of the space $\Sigma(\mathfrak{c})$. One can verify (see Theorem 2.11 of [DTT1]) that the group $G = C_p(X, 2)$ has all required properties.

It is worth mentioning that Lemma 2.2.10 has many other applications to suitable sets. One can slightly generalize it as follows. Let G and H be topological groups such that G contains a closed discrete subset A with $|A| \ge d(H)$ and H contains a closed discrete subset B with $|B| \ge d(G)$. Then $G \times H$ contains a closed suitable set. In particular, the following result is valid (see [TT]).

Corollary 2.2.12. *Let G and H be two separable topological groups that are not countably compact. Then $G \times H$ has a* *closed suitable set.*

We will see in Section V how Lemma 2.2.10 works for the study of Bohr topologies on locally compact Abelian groups.

3. Categorical Properties of The Classes $\mathcal{S}$ and $\mathcal{S}_c$

We study here the problem whether the classes $\mathcal{S}$, $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ are closed under cartesian and direct products, taking quotients and (open or closed) subgroups, etc.

The following result is proved in [CMRST].

Theorem 3.1. *Let $\{G_i : i \in I\}$ be a family of topological groups in $\mathcal{S}$. Then the direct sum $\sigma(I)$, the Σ-product $\Sigma(I)$ and the cartesian product $\Pi_I = \prod_{i\in I} G_i$ of this family have a suitable set.*

Note that both $\sigma(I)$ and $\Sigma(I)$ are dense subgroups of Π_I, so it suffices to show that Π_I has a suitable set S contained in $\sigma(I)$. For every $i \in I$, let 1_i be the identity of G_i and $1_{I\setminus\{i\}}$ the identity of $\prod_{j\in I\setminus\{i\}} G_j$. Denote by S_i a suitable set for G_i, $i \in I$. One can easily verify that the set

$$(*) \qquad S = \bigcup_{i\in I} S_i \times \{1_{I\setminus\{i\}}\}$$

is as required.

Note that if S_i is a generating suitable set for G_i for each $i \in I$, then the set S defined above is a generating suitable set for $\sigma(I)$. This implies Corollary 3.2.4 of [DTT2]:

Corollary 3.2. *The classes $\mathcal{S}$ and $\mathcal{S}_g$ are closed with respect to arbitrary direct sums.*

The previous corollary is not valid for the classes $\mathcal{S}_c$ and $\mathcal{S}_{cg}$ as we will see below. In the countable case, however, the conclusion still holds by Theorem 3.1.2 of [DTT2]:

Assertion 3.3. *The classes $\mathcal{S}_c$ and $\mathcal{S}_{cg}$ are closed with respect to countable direct sums.*

The proof of Theorem 3.1 does not work in this case, so we have to change the construction. Let $\{G_i : i \in \omega\}$ be a family of topological groups in $\mathcal{S}_c$, and let $G = \sigma(\omega)$ be the direct sum of these groups, $\sigma(\omega) \subseteq \prod_{i\in\omega} G_i$. For each $i \in \omega$, choose a closed suitable set S_i for G_i and define

$$A_i = S_0 \times \cdots \times S_i \times \{\overline{1}_i\},$$

where $\overline{1}_i$ is the identity of the group $\prod_{j>i} G_j$. A direct verification shows that $S = \bigcup_{i\in\omega} A_i$ is a closed suitable set for $\sigma(\omega)$. In addition, if each S_i generates the group G_i, then S generates $\sigma(\omega)$. Therefore, $G \in \mathcal{S}_{cg}$ in the latter case.

If the index set I in Theorem 3.1 is infinite, the set S in $(*)$ has the identity 1 of Π_I as a cluster point, so S chosen in that way cannot be closed. It turns out that no closed suitable set can exist for a product group in some cases (see Proposition 3.2.5 of [DTT2]):

Theorem 3.4. *The classes $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ are closed with respect to finite products but fail to be closed with respect to infinite products.*

In fact, the construction of the set S given after Theorem 3.1 still works to prove the first assertion of Theorem 3.4. As for the second one, we present a single counterexample which serves for the three classes. The key is the following simple fact, the proof of which is left to the reader: if a compact group G is in some of these three classes, then G has a dense finitely generated subgroup. Now let C be a finite cyclic group with $|C| > 1$ (so that $C \in \mathcal{S}_{cg}$), and put $G = C^\omega$. Then every finitely generated subgroup of G is finite, whence $G \notin \mathcal{S}_c$ and $G \notin \mathcal{S}_g$.

It is worth mentioning that the product operation can improve the properties of topological groups. For example, by Theorem 2.2.9, if a G is a topological group of non-measurable cardinality and G is not countably compact, then G^κ has a suitable set for some cardinal κ (while G is not necessarily in $\mathcal{S}$). The following result is another example of this kind.

Theorem 3.5. *Let $\{G_i : i \in I\}$ be an infinite family of non-trivial topological groups each of which has a dense strictly σ-discrete subset. Then the direct sum $\sigma(I)$ of these groups has a suitable set.*

Proof. Suppose first that $|I| = \aleph_0$. One easily verifies then that $\sigma(I)$ has a strictly σ-discrete subset and is not pseudocompact. By Theorem 1.11, $\sigma(I)$ has a suitable set. If $|I| > \aleph_0$, we can partition the index set I into countably infinite subsets and then apply Corollary 3.2. □

Corollary 3.6. *The direct sum of infinitely many non-trivial separable topological groups has a suitable set.*

If the family $\{G_i : i \in I\}$ in Theorem 3.5 (or in Corollary 3.6) is countably infinite, then the direct sum $\sigma(I)$ has a *closed* suitable set by Theorem 1.11. If, however, the index set I is uncountable, the latter conclusion can be false. Indeed, let $G = \sigma(\omega_1)$ be the direct sum of ω_1 copies of the two-element group $\{0, 1\}$. The group G is σ-compact and non-separable, so that a suitable set S for G has to be uncountable. Let $G = \bigcup_{i \in \omega} K_i$ be a union of compact sets K_i. If S were closed, every intersection $S \cap K_i$ would be finite, and hence S would be countable, a contradiction.

By Corollaries 2.1.6 and 2.1.7, neither a closed nor dense subgroup of a group in $\mathcal{S}_c$ has to be in $\mathcal{S}$. It turns out that even open subgroups of the groups in $\mathcal{S}_{cg}$ can be arbitrarily bad (see Theorem 4.7 of [CMRST]).

Theorem 3.7. *For every topological group G, the group $H = G \times G_d$ belongs to $\mathcal{S}_{cg}$, where G_d denotes the group G equipped with the discrete topology. So, if $G \notin \mathcal{S}$, then the projection $p : H \to G$ is an open homomorphism which does not preserve the classes $\mathcal{S}$, $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$.*

The proof of Theorem 3.7 is straightforward. Indeed, define

$$S = \{(x, x) : x \in G\} \cup \{(1, x) : x \in G_d\}.$$

Then S generates $G \times G_d$ algebraically. It is easy to see that S is closed and discrete in $G \times G_d$. Note that $G \times \{1\}$ is an open subgroup of the group $G \times G_d$, so we have the following.

Corollary 3.8. *Every topological group can be embedded as an open subgroup into a group in $\mathcal{S}_{cg}$.*

Theorem 3.7 shows that a quotient group of a group in $\mathcal{S}_{cg}$ need not even belong to $\mathcal{S}$. Closed homomorphisms are much better, as the following result of [DTT2] shows.

Theorem 3.9. *The classes $\mathcal{S}$, $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ are invariant with respect to closed homomorphic images.*

Indeed, let $f : G \to H$ be a closed continuous surjective homomorphism, and S a suitable set for G. Define $S_1 = f(S) \setminus \{1_H\}$. Since $\overline{S} \subseteq S \cup \{1_G\}$ and f is closed, we have $S_1 \cup \{1_H\} = f(S \cup \{1_G\}$, and the latter set is closed in H. A similar argument applied to an arbitrary subset S' of S shows that S_1 is discrete, and the identity 1_H of H can be the only accumulation point of S_1, i. e., S_1 is suitable for H. It remains to note that if S is closed (generating), then S_1 has the same property.

Corollary 3.10. *If $G \in \mathcal{S}$ and K is a compact normal subgroup of G, then $G/K \in \mathcal{S}$. The same is true for the classes $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$.*

Recall that every continuous homomorphic image of a group $G \in \mathcal{S}$ belongs to $\mathcal{S}$ if G is *countably compact* (see Lemma 2.2.8).

Another interesting problem posed in [DTT2] is to investigate whether the class $\mathcal{S}$ is invariant under *extensions.*

Problem 3.1. *Let $\mathcal{C}$ be one of the classes $\mathcal{S}$, $\mathcal{S}_c$, $\mathcal{S}_g$ or $\mathcal{S}_{cg}$. Suppose that N is a closed normal subgroup of a topological group G such that $G/N \in \mathcal{C}$ and $N \in \mathcal{C}$. Is it true that $G \in \mathcal{C}$? What is the answer if N is either compact or metrizable?*

By Theorems 3.4 and 3.1, the answer to Problem 3.1 is "yes" if N is a topological direct summand, i. e., $G \cong N \times G/N$. This remains valid for semidirect products as well (see Corollary 3.13 below). Note that Lemma 1.4 answers Problem 3.1 affirmatively for both $\mathcal{S}$ and $\mathcal{S}_c$ when N is open, i. e., when G/N is discrete. This observation suggests us to impose some additional restrictions on the quotient group G/N in order to get $G \in \mathcal{S}$. The first positive general result was obtained for the class $\mathcal{S}_c$ (see Theorem 3.4.5 of [DTT2]).

Theorem 3.11. *Let N be a closed normal subgroup of a topological group G such that $H = G/N \in \mathcal{S}_c$. Then (a) if $N \in \mathcal{S}$, then $G \in \mathcal{S}$; (b) if $N \in \mathcal{S}_c$, then $G \in \mathcal{S}_c$.*

It is also possible to show that if $G/N \in \mathcal{S}_{cg}$ and $N \in \mathcal{S}_g$ (resp., $N \in \mathcal{S}_{cg}$), then $G \in \mathcal{S}_g$ (resp., $G \in \mathcal{S}_{cg}$).

Let us present another condition to ensure the positive answer to Problem 3.1. We say that a homomorphism $f : G \to H$ *has a closed section* if there exists a closed subset X of G such that $f(X) = H$, i. e., there exists a (not necessarily continuous) section $s : H \to G$ such that $X = s(H)$ is closed in G. The following result coincides with Lemma 3.4.7 of [DTT2].

Theorem 3.12. *Let $f : G \to G/N$ be a quotient homomorphism with a closed section, and $N \in \mathcal{S}$. Then $G/N \in \mathcal{S}$ implies $G \in \mathcal{S}$.*

The following corollary to Theorems 3.11 and 3.12 is immediate.

Corollary 3.13. *The classes $\mathcal{S}$ and $\mathcal{S}_c$ are closed under the semidirect product operation.*

In fact, Corollary 3.13 remains valid for the classes $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ as well — this follows from the construction of a suitable set given in the proof of Lemma 3.4.7 of [DTT2].

A quotient group of a pseudocompact topological group $G \in \mathcal{S}$ (or even $G \in \mathcal{S}_c$) does not necessarily have a suitable set (see Corollary 3.4 of [DTT1]). The situation changes if G is connected and the kernel of the corresponding homomorphism is sufficiently big.

Corollary 3.14. *If G is a connected pseudocompact group and $N \in \mathcal{S}$ (resp. $N \in \mathcal{S}_c$) is a closed normal G_δ-subgroup of G, then $G \in \mathcal{S}$ (resp. $G \in \mathcal{S}_c$).*

Indeed, in this case G/N is a compact connected metrizable group by Theorem 3.2 of [CR], so that G/N has a two-element suitable set by a theorem of Hofmann and Morris [HM1]. We conclude, therefore, that $G/N \in \mathcal{S}_c$, and Theorem 3.11 implies that $G \in \mathcal{S}$.

We do not know, however, if Corollary 3.14 remains valid without the assumption that G is connected.

Several interesting results related to Problem 3.1 have recently been obtained by Dikranjan and Trigos-Arrieta [DTr]. They showed that if H is closed normal subgroup of a topological group G and the quotient group G/H has a closed suitable set S such that $d(H) \leq |S|$, then G also has a closed suitable set. It is worth to mention that the subgroup H of G *need not* belong to $\mathcal{S}$.

A topological group G has a closed suitable set if it contains a discrete normal subgroup H such that $G/H \in \mathcal{S}_c$ (no additional restrictions on H are required in this case). Finally, if H is again a discrete normal subgroup of G and $G/H \in \mathcal{S}$, then $G \in \mathcal{S}$ [DTr].

4. Free Topological Groups

By Theorem 2.1.2, the free Abelian topological group $A(X)$ on a non-separable compact F-space X does not necessarily have a suitable set. In fact, the proof of this result presented in [CMRTS] works in the non-Abelian case as well. It seems interesting, therefore, to characterize the spaces X for which the free topological group $F(X)$ (or the free Abelian topological group $A(X)$) has a suitable set. It is not difficult to show that if X is compact and $F(X) \in \mathcal{S}$, then $A(X) \in \mathcal{S}$. The converse is not evident and apparently is an open problem.

We denote by $\mathcal{C}_f$ the class of spaces X such that $F(X)$ has a suitable set. According to Theorem 2.1.2, not every compact space is in $\mathcal{C}_f$: the spaces of the form $\beta D \setminus D$ with D discrete and infinite do not belong to $\mathcal{C}_f$. On the other hand, all separable spaces are in $\mathcal{C}_f$ by Theorem 2.1.1. The class $\mathcal{C}_f$ also includes all metrizable spaces according to Corollary 3.14 of [DTT1]:

Theorem 4.1. *The free topological group $F(X)$ on a metrizable space X has a closed suitable set.*

It is known that for a metrizable space X, the group $F(X)$ is a countable union of its closed metrizable subspaces (see Fact α_1 of [Ar2]). Since $F(X)$ is never pseudocompact, Theorem 1.11 implies that $F(X)$ has a closed suitable set.

We call a X a *σ-space* if it has a σ-discrete network [Oku]. By a result of Arhangel'skiĭ [Ar2], the free topological group $F(X)$ on a paracompact σ-space X is also paracompact σ-space, and hence has a σ-discrete network. Therefore, Theorem 1.11 implies the following result which generalizes Theorem 4.1:

Theorem 4.2. *If X is a paracompact σ-space, then the free topological group $F(X)$ has a closed suitable set.*

Since every stratifiable space is a paracompact σ-space [Gru], we conclude that all stratifiable spaces are in $\mathcal{C}_f$.

It is very easy to show that every compact supersequence (a one-point compactification of a discrete space) is in the class $\mathcal{C}_f$. In fact, a more general result holds (see Theorem 17 of [TT]):

Assertion 4.3. *If a space X has at most one non-isolated point, then $F(X) \in \mathcal{C}_f$.*

Indeed, let $X = Y \cup \{a\}$, where all points of Y are isolated in X. Then $S = \{a\} \cup \{a^{-1} \cdot y : y \in Y\}$ is a suitable set for $F(X)$.

Ordinal spaces form another subclass of $\mathcal{C}_f$. By an ordinal space we mean any ordinal α endowed with the order topology, and the corresponding space is denoted by $T(\alpha)$. The following result has been proved independently in [OTk] and [TT].

Theorem 4.4. *$T(\alpha) \in \mathcal{C}_f$ for every ordinal α.*

It is not surprising that the set $S = \{0\} \cup \{\beta^{-1} \cdot (\beta + 1) :$

$\beta+1<\alpha\}$ is suitable for $F(T(\alpha))$. The proof of this fact given in [OTk] depends on the following property of ordinal spaces:

Lemma 4.5. *If α is an ordinal with $cf(\alpha)\neq\omega$, then the set $R=\{(\beta,\beta+1):\beta+1<\alpha\}$ converges to the diagonal Δ in $T(\alpha)^2$. In other words, every neighborhood of Δ in $T(\alpha)^2$ contains all but finitely many points of R.*

Assertion 4.3 and Theorem 4.4 suggest the hypothesis that all "good" compact spaces are in $\mathcal{C}_f$. For example, we can ask whether Tychonoff cubes, Cantor cubes or their continuous images (i. e., dyadic spaces) belong to $\mathcal{C}_f$. Surprisingly, the answer to this question is in affirmative. The first step in the proof of this fact is the following simple characterization of compact spaces in $\mathcal{C}_f$ (see [OTk, Lemma 2.2]):

Lemma 4.6. *For a compact space X the following conditions are equivalent:*
(i) $X\in\mathcal{C}_f$;
(ii) there is a subset S of $F(X)$ that generates a dense subgroup of $F(X)$ and is a countable union of compact supersequences.

The second (still simple) step is to show that continuous images of compact spaces in $\mathcal{C}_f$ are in $\mathcal{C}_f$ (Lemma 2.3 of [OTk]).

Lemma 4.7. *If X is compact, $X\in\mathcal{C}_f$ and Y is a continuous image of X, then $Y\in\mathcal{C}_f$.*

In fact, Lemma 4.7 follows directly from Lemma 4.6. The third and the most difficult part of the proof that all dyadic compact spaces belong to $\mathcal{C}_f$ is the *productivity* of compact spaces in $\mathcal{C}_f$ (Theorem 2.15 of [OTk]).

Theorem 4.8. *If $\{X_i : i\in I\}$ is a family of compact spaces and $X_i\in\mathcal{C}_f$ for each $i\in I$, then $\prod_{i\in I}X_i\in\mathcal{C}_f$.*

By Theorem 2.1.1, every separable space is in $\mathcal{C}_f$, so Theorem 4.8 and Lemma 4.7 imply the following result (see [OTk, Corollary 2.16]).

Corollary 4.9. *Every continuous image of a product of separable compact spaces belongs to $\mathcal{C}_f$.*

A compact space is called *polyadic* [Mr] if it is a continuous image of a product of compact supersequences. Obviously, the class of polyadic spaces contains the class of all dyadic spaces. Since every compact supersequence is in $\mathcal{C}_f$ (Assertion 4.2), Theorem 4.8 and Lemma 4.7 together imply that all polyadic spaces belong to $\mathcal{C}_f$:

Corollary 4.10. *Every polyadic, in particular, every dyadic space belongs to the class $\mathcal{C}_f$.*

Note that every compact supersequence is a continuous image of an ordinal space of the form $T(\alpha+1)$ (identify all limit ordinals $\leq \alpha$ to a point). Therefore, the class of spaces X as in Corollary 4.11 below contains all polyadic spaces.

Corollary 4.11. *If α is an ordinal and X is a continuous image of an arbitrary power of the space $T(\alpha+1)$, then $X \in \mathcal{C}_f$.*

Again, Corollary 4.11 is immediate if we use Theorems 4.4, 4.8 and Lemma 4.7. Corollaries 4.10 and 4.11 are due to Okunev and Tkačenko [OTk].

Since $\beta\mathbb{N} \setminus \mathbb{N} \notin \mathcal{C}_f$ by Corollary 2.1.5, we conclude that $\beta\mathbb{N} \setminus \mathbb{N}$ is not a continuous image of any power of any ordinal space $T(\alpha+1)$.

Recall that two spaces X and Y are called *M-equivalent* if the free topological groups $F(X)$ and $F(Y)$ are topologically isomorphic [Gra], [Ok]. The proof of Theorem 4.8 given in

[OTk] essentially depends on this useful notion. It is known that the Alexandroff duplicate of an infinite compact space X is M-equivalent to the free sum $X \oplus C$, where C is a compact supersequence of cardinality $|X|$. Since the class $\mathcal{C}_f$ is closed under finite sums, we obtain the following result:

Corollary 4.12. *The Alexandroff duplicate of a compact space from $\mathcal{C}_f$ also belongs to $\mathcal{C}_f$.*

As we have seen, the class $\mathcal{C}_f$ is very wide (even if we restrict ourselves only to considering compact spaces). However, the following problems still remain open (see [OTk]):

Problem 4.1. *Does every scattered compact space belong to $\mathcal{C}_f$? Is every compact space of cardinality less that $\mathfrak{c}$ in $\mathcal{C}_f$?*

Problem 4.2. *Does every Eberlein (Corson) compact space belong to $\mathcal{C}_f$?*

Problem 4.3. *Does $\mathcal{C}_f$ contain the class of linearly ordered compact spaces?*

5. Minimal Groups and Bohr Topologies

By a theorem of Hofmann and Morris [HM], every locally compact group has a suitable set. None of the other compact-like properties considered so far (ω-boundedness, completeness, being Lindelöf, σ-compactness) imply the existence of a suitable set.

The property of a topological group to be *minimal* or *totally minimal* is close in some sense to local compactness: minimal groups often have a suitable set. A topological group (G, τ) is called *minimal* if τ is a minimal element of the partially ordered (with respect to inclusion) set of Hausdorff group topologies on the group G. We say that G is *totally minimal* if every

Hausdorff quotient group of G is minimal. To stress the relation between minimality and compactness, we just mention here a fundamental result of Prodanov and Stoyanov [PS]: every minimal Abelian group is totally bounded, or equivalently, *precompact.*

All results of this Section were proved in [DTT2], so in the sequel we often omit the references to the authority. The following result summarizes several facts on preimages of topological groups taken from different sources (see Lemma 4.1.1 of [DTT2]).

Proposition 5.1. *Let $f : G \to G_1$ be a continuous surjective homomorphism of compact groups and H_1 a dense subgroup of G_1. Then the subgroup $H = f^{-1}(H_1)$ of G is dense in G and:*
(a) $H_1 \cong H/\ker f$;
(b) H is totally minimal if H_1 is totally minimal;
(c) H is minimal if H_1 is minimal;
(d) H is countably compact if H_1 is countably compact;
(e) H is ω-bounded if H_1 is ω-bounded;
(f) $H_1 \in \mathcal{S}$ (resp., $H_1 \in \mathcal{S}_c$, $\mathcal{S}_g$, $\mathcal{S}_{cg}$) if $H \in \mathcal{S}$ (resp., $H \in \mathcal{S}_c$, $\mathcal{S}_g$, $\mathcal{S}_{cg}$).

By Theorems 0.2 and 3.1, a cartesian product of metrizable topological groups has a suitable set. On the other hand, dense subgroups of such a product group can fail to have a suitable set even if all the factors are second countable (see Theorems 2.2.5 and 2.2.11). It turns out that total minimality improves the situation.

Proposition 5.2. *A dense totally minimal subgroup of a product of metrizable groups has a suitable set.*

The proof of Proposition 5.2 is based on the fact that a dense totally minimal subgroup H of a product group contains the direct sum of the factors, so that we can apply the construction given after Theorem 3.1 to obtain a suitable set for H.

By Ivanovskiĭ–Kuz'minov theorem, all compact topological groups are dyadic, i. e., continuous images of 2^κ for a sufficiently big cardinal κ. If, however, a compact group G is not boolean (that is, $x^2 \neq 1_G$ for some $x \in G$), there is no *homomorphism* of 2^κ onto G by the obvious algebraic reason. We can still represent compact groups as continuous homomorphic images of cartesian products of compact second countable groups in two important cases.

Theorem 5.3. *Let G be a compact topological group which is either Abelian or connected. Then G is a quotient group of a cartesian product of compact metrizable groups. In addition, if G is Abelian (connected), then the factors can be chosen Abelian (connected).*

Note that Theorem 5.3 immediately implies that compact Abelian groups and compact connected groups are dyadic. Our aim, however, is different: we apply Theorem 5.3 to deduce the following.

Theorem 5.4. *Every totally minimal Abelian group has a suitable set.*

Proof. Let G be a totally minimal Abelian group. Then the completion $\widehat{G}$ of G is a compact Abelian group by Prodanov–Stoyanov theorem [PS]. By Theorem 5.3, there exists a continuous surjective homomorphism $f\colon K \to \widehat{G}$, where $K = \prod_{i\in I} K_i$ and every K_i is a compact metrizable group. According to Proposition 5.1, $H = f^{-1}(G)$ is a dense totally minimal subgroup of K and $H/N \cong G$, where N is the kernel of f. From Proposition 5.2 it follows that $H \in \mathcal{S}$. Now Proposition 5.1 (f) completes the proof. □

It is not known if "Abelian" is essential in Theorem 5.4:

Problem 5.1. *Does the class $\mathcal{S}$ contain all totally minimal groups? Are all precompact totally minimal groups in $\mathcal{S}$?*

The following result gives a partial positive answer to Problem 5.1.

Theorem 5.5. *Every totally minimal connected precompact group G has a suitable set.*

Proof. Since the completion $\widehat{G}$ of G is a compact connected group, we can find a continuous surjective homomorphism $f\colon \prod_{i\in I} K_i \to \widehat{G}$, where every K_i is a compact metrizable group. It remains to apply the final part of the reasoning in the proof of Theorem 5.4 based essentially on Proposition 5.1 to obtain $G \in \mathcal{S}$. □

An interesting example of a totally minimal topological group can be obtained as follows. Consider the group $S(X)$ of all permutation ofaan infinite set X. This group is endowed with the *topology of pointwise convergence* a base at the identity of which consists of the sets U_K of all functions $f : X \to X$ that do not move the points of a finite set $K \subseteq X$. Total minimality of $S(X)$ was established by Dierolf and Schwanengel [DS]. A short proof of the fact that $S(X)$ has a closed suitable set in given in [DTT2, Example 4.1.7]. Note that the group $S(X)$ is very far from being Abelian or precompact.

The wider class of minimal topological groups contains many groups without a suitable set. The following example is taken from [DTT2].

Example 5.6. *There exists an ω-bounded (and hence countably compact) minimal Abelian group H without suitable sets.*

Let G be the ω-bounded dense subgroup of $\mathbb{Z}(2)^{\mathfrak{c}}$ with $G \notin \mathcal{S}$ constructed in Theorem 2.2.11. The group G is not minimal (since a minimal group of exponent 2 must be compact, see [DPS, Example 2.5.3 (b)]). In order to correct this, we consider a surjective continuous homomorphism $f : K \to \mathbb{Z}(2)^{\mathfrak{c}}$ where K is a totally disconnected compact Abelian group. Then by Proposition 5.1 (e), (f) the subgroup $H = f^{-1}(G)$ of K is ω-bounded and $H \notin \mathcal{S}$. To guarantee the minimality of H one has to choose f in such a way that every closed non-trivial subgroup of K meets $\ker f$ (cf. [DPS, Theorem 2.5.1]). The group $K = \mathbb{Z}(4)^{\mathfrak{c}}$ and the homomorphism $f : \mathbb{Z}(4)^{\mathfrak{c}} \to \mathbb{Z}(2)^{\mathfrak{c}}$ defined by $f(x) = x + x$ satisfy the latter condition, thus finishing our construction.

As in the case of totally minimal groups, connectedness improves the properties of minimal groups.

Theorem 5.7. *Every minimal countably compact connected Abelian group has a suitable set.*

The proof of this result is based on Theorem 5.3 and Corollaries 3.2 and 3.14 (see [DTT2, Theorem 4.2.1]). The non-Abelian case again presents difficulties:

Problem 5.2. *Is it true that every minimal countably compact connected group has a suitable set?*

We finish with a discussion of topological groups equipped with the *Bohr topology*. Given a topological group (G, τ), consider the weakest group topology τ^+ on G which makes all τ-continuous homomorphisms of G to compact groups τ^+-continuous. The new topology τ^+ is called the *Bohr topology* on G. Clearly, τ^+ is weaker that τ and the group $G^+ = (G, \tau^+)$ is totally bounded. However, the topology τ^+ is not necessarily Hausdorff. When it is, the group G is said to be *maximally al-*

most periodic (MAP). It is well known that all discrete Abelian groups as well as Abelian totally bounded groups are MAP.

Pontryagin–van Kampen duality theory is based on the fact that all locally compact Abelian groups are in the class MAP. Furthermore, continuous homomorphisms of a locally compact Abelian group G to the circle group $\mathbb{T}$ separate the elements of G. It is also known that for an Abelian group (G, τ), the Bohr topology τ^+ on G is the weakest one which makes the τ-continuous homomorphisms to the circle group τ^+-continuous.

There are many totally bounded topological groups which have no suitable set; one can even find an ω-bounded minimal Abelian group which is not in $\mathcal{S}$ (see Example 5.6). On the other hand, every locally compact group has a suitable set by Theorem 0.1. The following result (proved independently in [DTT2] and [TT]) shows that the functor $^+$ assigning to a group G its modification G^+ preserves the latter property of locally compact Abelian groups.

Theorem 5.8. *G^+ has a suitable set for every locally compact Abelian group G.*

We do not know, however, if Theorem 5.8 remains valid for maximally almost periodic groups (see [TT]):

Problem 5.3. *Does $H^+ \in \mathcal{S}$ for every locally compact MAP group H?*

Note that the group G in Theorem 2.2.5 is MAP and $G^+ = G$, so there exist Abelian MAP groups H with $H \notin \mathcal{S}$.

If a group G is discrete, we follow van Douwen [Dou2] and write $G^\#$ instead of G^+. Clearly, every discrete group belongs to $\mathcal{S}_{cg}$, i. e., has a closed generating suitable set. By Theorem 5.7 of [DTT2], the functor $^\#$ preserves this property in the Abelian case.

Theorem 5.9. *$G^{\#} \in \mathcal{S}_{cg}$ for every discrete Abelian group G.*

Theorem 5.8 admits one generalization that gives a partial positive answer to Problem 5.3. To present it, we first note that the functor $^+$ preserves arbitrary products of locally compact Abelian group, that is, $(\prod_{i\in I} G_i)^+ \cong \prod_{i\in I} G_i^+$ for every family $\{G_i : i \in I\}$ of locally compact Abelian groups (see Theorem 5.1 (d) of [DTT2]). Combining this fact with Theorems 5.9 and 3.1, we obtain the following result.

Theorem 5.10. *Let G be a cartesian product of locally compact Abelian groups. Then $G^+ \in \mathcal{S}$.*

The study of Bohr topologies on topological groups concerning suitable sets is far from being completed, and we present here only one (of many other) open problems in this area (see [DTT2]):

Problem 5.4. *(a) Let G be a locally compact Abelian group. Does $G \in \mathcal{S}_c$ always imply $G^+ \in \mathcal{S}_c$?*
(b) Does every Abelian topological group that satisfies the Pontryagin–van Kampen duality admit a suitable set?

Very recently Dikranjan and Trigos-Arrieta [DTr] gave a complete solution to Problem 5.4. First, they proved that for every locally compact Abelian group G, the conditions $G \in \mathcal{S}_c$ and $G^+ \in \mathcal{S}_c$ are equivalent, thus answering (a) of Problem 5.4 in the positive. To answer (b) of Problem 5.4, Dikranjan and Trigos-Arrieta applied the following fact established by Pestov in [Pe]: the free Abelian topological group $A(X)$ on a zero-dimensional compact space X satisfies Pontryagin–van Kampen's duality. Therefore, by Corollary 2.1.5, the group $A(\beta\mathbb{N} \setminus \mathbb{N})$ on $\beta\mathbb{N} \setminus \mathbb{N}$ is a counterexample to the conjecture in (b) of the problem.

It is also showen in [DTr] that Theorem 5.8 admits a generalization to the class of locally compact *Moore* groups, that is, $G^+ \in \mathcal{S}$ for every locally compact Moore group G. Recall that a topological group G is called *Moore* if every continuous unitary irreducible representation of G is finite dimensional. The class of Moore groups contains all locally compact Abelian groups and all compact groups, and it is closed under the operations of taking closed subgroups, quotients, finite products and extensions. Since locally compact Moore groups are MAP, the above result of [DTr] gives a partial positive answer to Problem 5.3. In addition, for every locally compact Moore group G, the conditions $G \in \mathcal{S}_c$ and $G^+ \in \mathcal{S}_c$ are equivalent [DTr].

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