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CONNECTIVITY OF STABLY COMPACT SPACES

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Abstract

Kopperman and Wilson, in [7], have recently introduced a definition of approximation of compact metric spaces by inverse sequences of finite T_0-spaces, which introduces a formal notion of approximation to (topological) digital topology. They propose the consideration of stably compact spaces in this context, and give various results concerning connectivity. This work is a development of that theory.

1. Introduction

Freudenthal proved in 1937 that a space is compact metric if and only if it is the limit of an inverse sequence of polyhedra (see Nagata's account in [11]). Apart from its significance in classical topology (and in continuum theory in particular), this result is also of interest in computer science. Kopperman and Wilson (K & W) have very recently formulated a version of the result in terms of the types of structure considered in digital topology: where an inverse sequence of spaces is said to *approximate* the T_2-reflection of its limit (the T_2-reflection of a space X consists of a Hausdorff space Y and a continuous map $f : X \to Y$ such that any continuous map from X to

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any Hausdorff space factors through f in a unique way), they proved in [7] that:

Theorem 1.1. *A space is compact metric if and only if it can be approximated by an inverse sequence of finite T_0-spaces.*

Thus a space may be considered in a mathematically precise way as the limit of its computational models. K & W then show that connectivity of a space may be expressed in terms of connectivity of its models: a compact metric space is connected if and only if it can be approximated by connected finite T_0-spaces (in fact, K & W consider the approximation of compact Hausdorff spaces by inverse spectra of finite T_0-spaces, but, for simplicity, we will only consider inverse sequences here). Moreover, they argue that the *stably compact* (the terminology they use is *skew compact*) spaces are an important class of spaces to consider in this context, since they are closed under inverse limits and include compact metric and finite T_0-spaces. This work is an attempt to develop the theory by expressing further connectivity properties of a space in terms of connectivity properties of its models, and by developing a theory of connectivity on stably compact spaces in general. Such a theory would embrace the standard theory on compact metric spaces and would apply equally well to their finite models in digital topology. A well-known example in this regard is a *connected ordered topological space* (COTS), which is a connected space such that, for any three distinct points, one separates the other two, and which is a definition of a *line* that generalizes the classical definition and also that of the *Khalimsky line* (see [5]). In fact, we shall argue that the definition of a COTS should be adjusted slightly.

2. Stably Compact Spaces

We will follow Lawson's description of stably compact spaces and proper maps in terms of compact ordered spaces and continuous order-preserving maps; see [8] (to which we also refer for basic definitions and results in non-Hausdorff topology) for a more detailed approach and for the proofs of various standard results stated below; for a background to compact ordered spaces, see Nachbin [9].

Notation and Terminology. Where $\leq$ is a partial order on a set X, for $x \in X$: $\uparrow x$ denotes the set $\{y \mid x \leq y\}$ and $\downarrow x$ denotes the set $\{y \mid y \leq x\}$. For $P \subseteq X$: $\uparrow P$ denotes the set $\bigcup\{\uparrow x \mid x \in P\}$ and $\downarrow P$ denotes the set $\bigcup\{\downarrow x \mid x \in P\}$. P is an *upper* set if $P = \uparrow P$, and is a *lower* set if $P = \downarrow P$. In a compact ordered space, this notation and terminology will always be used with respect to the order; in a stably compact space, it will always be used with respect to the *specialization order*. The specialization order on a space X is denoted $\leq_X$.

A binary relation on a space is *closed* if it is a closed subset of the product space. A *compact ordered space* is a compact Hausdorff space together with a closed partial order. Associated are two significant topologies: the collection of all open upper sets is called the *stably compact topology*, and the collection of all open lower sets is called the *dual-stably compact topology*. A space is *stably compact* if it is the stably compact space corresponding to some compact ordered space. For a purely topological characterization: stably compact spaces are compact, locally compact, sober, and such that the intersection of any two compact upper subsets is compact. Every stably compact space is T_0. Examples include compact Hausdorff spaces, which are precisely the stably compact Hausdorff spaces, and finite T_0-spaces, which are precisely the finite stably compact spaces.

Any stably compact space X contains all the information necessary to reconstruct the compact ordered space to which it corresponds. The *dual topology* on X is the collection of complements of compact upper subsets, and the resultant space is called the *dual space*, which is denoted X^D. Any stably compact space is the dual of its dual, i.e. $X = (X^D)^D$. The *patch topology* on X is that which has as subbase the topology on X together with the dual topology on X, and the resultant space is called the *patch space*, which is denoted X^P (open subsets of X^D are called *dual-open*; open subsets of X^P are called *patch-open*). The patch topology is compact Hausdorff, and $\leq_X$ is closed in the patch topology. This gives a compact ordered space $(X^P, \leq_X)$, with respect to which X is the corresponding stably compact space, and X^D is the corresponding dual-stably compact space.

A subset of a space is *stably compact* if its subspace topology is stably compact; K & W showed that:

Proposition 2.1. *If a subset of a stably compact space is patch-closed then it is a stably compact subset of X and of X^D.*

Examples of patch-closed subsets of a stably compact space are the closed sets and the compact upper (= dual-closed) sets. For any compact subset C, $\uparrow C$ is compact upper and so is patch-closed. The result does not have a converse, however: even if a subset of a space is stably compact and stably compact in the dual-space it is not necessarily patch-closed.

A continuous map between stably compact spaces is *proper* if it is continuous with respect to the respective dual topologies. Between compact Hausdorff spaces, or between finite T_0-spaces, the proper maps are precisely the continuous maps. Proper maps are called *de Groot* maps by K & W. The proper maps are precisely those that are continuous and order-preserving as maps between the corresponding compact ordered spaces. A further characterization of proper maps was given by Escardo in [3]:

Notation. For any continuous map $f : X \to Y$ and for any $U \subseteq X$, $\forall_f(U)$ denotes the greatest open subset of Y whose preimage is a subset of U.

Proposition 2.2. *A continuous map $f : X \to Y$ between stably compact spaces is proper if and only if for all open $U \subseteq X$, $\forall_f(U) = \{y \mid f^{-1}(\uparrow y) \subseteq U\}$.*

We now consider inverse sequences of stably compact spaces. To recall some standard terminology, a *thread* of an inverse sequence (X_i, f_i) is a sequence of points (x_i) such that each $x_i = f_i(x_{i+1})$. Where X_ω is the set of all threads, for each j the *jth-projection* is the map $p_j : X_\omega \to X_j$, $(x_i) \mapsto x_j$. The *limit* of the inverse sequence is X_ω together with the *inverse limit topology*, which has as base the collection of preimages of open sets under the projections. Although the limit of an inverse sequence of compact Hausdorff spaces is compact Hausdorff, the limit of an inverse sequence of compact spaces is not necessarily compact (a counterexample is given in [1] Ex.1.9.7). The following result was given (with a different proof) by K & W, but the idea of considering stable compactness in the context of inverse sequences was suggested independently to the author by Mike Smyth.

Proposition 2.3. *The limit of an inverse sequence of stably compact spaces and proper maps is stably compact, and each projection on the limit is proper.*

Proof. Consider an inverse sequence $((X_i, \leq_i), f_i)$ of compact ordered spaces and continuous order-preserving maps. Let X_ω be the limit of the inverse sequence of spaces (X_i, f_i), with (p_i) the projections. Then $\leq_\omega = \bigcap_i (p_i \times p_i)^{-1}(\leq_i)$ is a closed partial order on X_ω, with respect to which each projection is order-preserving. It suffices to show that the stably compact topology corresponding to the compact ordered space $(X_\omega, \leq_\omega)$ is the inverse limit topology of the stably compact topologies corresponding to the compact ordered spaces $(X_i, \leq_i)$. For any open upper $U \subseteq (X_i, \leq_i)$, $p_i^{-1}(U)$ is open upper in $(X_\omega, \leq_\omega)$. Conversely, suppose that $U \subseteq (X_\omega, \leq_\omega)$ is an open upper neighbourhood of the thread (x_i). Then $\uparrow(x_i)$ is a subset of U, and is the intersection of the compact sets $p_i^{-1}(\uparrow x_i)$, so there is some j such that $p_j^{-1}(\uparrow x_j) \subseteq U$. Translating Proposition 2.2 to the context of compact ordered spaces, we have that $(x_i) \in p_j^{-1}(\forall_{p_j}(U)) \subseteq U$ and $\forall_{p_j}(U)$ is open upper in $(X_j, \leq_j)$. □

3. Finite Approximation of Spaces

In this section we give a construction of an inverse sequence of finite T_0-spaces approximating a given compact metric space, which is different to the construction given by K & W, but which is needed in a later proof. We consider a *cover* of a space X as a finite *multiset* $\mathcal{C}$ of non-empty *compact* subsets whose union is X (multisets are needed for the proof of Theorem 4.9). $\mathcal{S} \subseteq \mathcal{C}$ is a *simplex* if $\mathcal{S} \neq \emptyset$ and $\bigcap \mathcal{S}$ $(= \bigcap\{C \mid C \in \mathcal{S}\}) \neq \emptyset$.

Definition 3.1. The finite T_0-space $T(\mathcal{C})$ *corresponding* to a cover $\mathcal{C}$ is the set of simplexes together with the (unique) topology for which $\mathcal{S}_1 \leq_{T(\mathcal{C})} \mathcal{S}_2$ if and only if $\mathcal{S}_1 \supseteq \mathcal{S}_2$.

A mapping $f : \mathcal{C}_2 \to \mathcal{C}_1$ between covers of the same space is an *inclusion map* if it maps each element to an element that contains it, in which case the map $T(f) : T(\mathcal{C}_2) \to T(\mathcal{C}_1)$, $\mathcal{S} \mapsto f[\mathcal{S}]$ is well-defined and continuous. A cover of a metric space an ϵ-*cover* if each of its elements has diameter $\leq \epsilon$. An inverse sequence $(\mathcal{C}_i, f_i)$ of covers and inclusion maps is *approximating* if it contains some ϵ-cover for all $\epsilon > 0$; corresponding is an inverse sequence $(T(\mathcal{C}_i), T(f_i))$ of finite T_0-spaces. Every compact metric admits an approximating sequence of covers.

Proposition 3.2. *The inverse sequence of finite T_0-spaces corresponding to an approximating inverse sequence of covers of a compact metric space X approximates X.*

Proof. Suppose $(\mathcal{C}_i, f_i)$ is an approximating sequence for X; let X_ω be the limit of $(T(\mathcal{C}_i), T(f_i))$, with (p_i) the projections.

For any thread $(\mathcal{S}_i) \in X_\omega$, $(\bigcap \mathcal{S}_i)$ is a decreasing sequence of arbitrarily small non-empty compact subsets of X, so the function $\phi : X_\omega \to X$, $(\mathcal{S}_i) \mapsto x$, where $\{x\} = \bigcap_i(\bigcap \mathcal{S}_i)$, is well-defined.

If U is an open neighbourhood of $\phi((\mathcal{S}_i))$ then there is some j such that $\bigcap \mathcal{S}_j \subseteq U$. Then $p_j^{-1}(\uparrow\mathcal{S}_j)$ is an open neighbourhood of $(\mathcal{S}_i)$ and is a subset of $\phi^{-1}(U)$, so ϕ is continuous.

An inverse sequence of finite multisets can be considered as a finitely branching tree in which the nodes are the elements of the sets, and in which x is a leaf of y if y is the image of x under some bonding map. For any $x \in X$, by König's lemma there exists some sequence (C_i) such that, for all i, $x \in C_i = f_i(C_{i+1})$. Then x is the image under ϕ of the thread $(\{C_i\})$, so ϕ is onto. Any onto continuous map from a stably compact space to a compact Hausdorff space is a quotient map (think of the map as a map from the patch space).

Now suppose $g : X_\omega \to Y$ is continuous and Y is Hausdorff. Any two elements of X_ω that are related by $\leq_{X_\omega}$ have the same image under g. If $(\mathcal{R}_i), (\mathcal{S}_i) \in \phi^{-1}(x)$ then $(\mathcal{R}_i \cup \mathcal{S}_i) \in \phi^{-1}(x)$ and is below both $(\mathcal{R}_i), (\mathcal{S}_i)$ in $\leq_{X_\omega}$. Then any two elements of X_ω that have the same image under ϕ have the same image under g. Together with the fact that ϕ is onto, this means that $\hat{g} : X \to Y$, $x \mapsto g(y)$ for any $y \in \phi^{-1}(x)$, is the unique function such that $g = \hat{g} \circ \phi$, and $\hat{g}$ is continuous because ϕ is a quotient map. □

4. Connectivity of Stably Compact Spaces

Our intention is to develop a theory of connectivity for stably compact spaces. Departures from standard terminology will be stated explicitly, otherwise standard definitions will be assumed thoughout; for example: 'connected', 'locally connected', 'cut-point' mean exactly the same thing here as they do normally. We will consider a *separation* of a subset P of a space X as a pair (U, V) of open subsets of X such that

$P \subseteq U \cup V$, $P \cap U \neq \emptyset$, $P \cap V \neq \emptyset$, $P \cap U \cap V = \emptyset$. This is slightly different to the standard definition: U, V are usually considered as subsets of P that are open in the subspace topology on P and that satisfy the latter four conditions. Both definitions are of course in agreement as to which sets admit a separation.

Terminology. For any connectivity property of a space, a subset or a point, the object in question has the *dual-property* if it has the property in the dual space, has the *bi-property* if it has the property *and* the dual-property, and has the *di-property* if it has the property *or* the dual-property. A connectivity property is *self-dual* if it holds with respect to a topology exactly when it holds with respect to the dual topology.

For example, a subset of stably compact space X is *dual-connected* if it is connected in X^D, is *bi-connected* if it is connected in X and in X^D, and is *di-connected* if it is connected in X or in X^D.

Definition 4.1. A *continuum* is a connected stably compact space.

The definition is due to K & W, although they use the term *skew continuum.* A *subcontinuum* is a connected stably compact subspace. The crucial difference between the above and the standard definition is that it allows for non-trivial finite continua. K & W showed that:

Proposition 4.2. *A patch-closed subset of a stably compact space is connected if and only if it is dual-connected.*

Thus the property of being a patch-closed subcontinuum is self-dual, although the property of being a subcontinuum is not self-dual. It seems broadly the case that the interesting and original part of the theory of connectivity on stably compact spaces arises from the fact that the result does *not* necessarily hold for open sets. There are, however, two very important exceptions to this: duality is not an issue in Hausdorff spaces, and the theory presented here is, for such spaces, exactly the standard theory; secondly, every subset of a finite T_0-space is patch-closed, and so every connectivity property is self-dual on such spaces. A further result of K & W's is that:

Proposition 4.3. *The intersection of any collection of patch-closed subcontinua that is totally ordered by inclusion is a patch-closed subcontinuum.*

The terminology of *continuum* rather than *stably compact continuum* is justifed, we shall argue, because continua have many properties similar to those of Hausdorff continua, the following result being our first example of this.

A patch-closed subcontinuum C is *irreducibly connected* about a set P if $P \subseteq C$ and there is no patch-closed subcontinuum D such that $P \subseteq D \subset C$.

Theorem 4.4. *The following properties of compact Hausdorff spaces also hold for stably compact spaces:*

1. *Any subset of a continuum has a patch-closed subcontinuum irreducibly connected about it;*

2. *The components are the quasi-components.*

Proof. The proof of either result is analogous to the proof of the corresponding classical result (see Hocking & Young [4] Th. 2-11 and Nadler [10] Ex. 5.18). For 1: the (non-empty) collection of patch-closed subcontinua containing some subset P ordered by inclusion contains some maximal totally ordered set $\mathcal{C}$, and $\bigcap \mathcal{C}$ is a patch-closed subcontinuum irreducible about P.

For 2. it suffices to show that each quasi-component is connected. A quasi-component Q is the intersection of clopen sets and so is patch-closed and upper. If Q is disconnected then it is dual-disconnected: when (U', V') is a separation of Q in the dual space then $C = Q - U'$ and $D = Q - V'$ are disjoint non-empty patch-closed upper sets whose union is Q. Then there exist disjoint open sets U, V such that $C \subseteq U$ and $D \subseteq V$ (because there exist disjoint patch-open sets containing C, D respectively and, for any patch-open set U, the set $\{x \mid \uparrow x \subseteq U\}$ is open). Each $x \in X - Q$ is contained in some clopen set that is disjoint from Q (this is true in any space), so the compact set $X - (U \cup V)$ is contained in some clopen set W that is disjoint from Q. But this is impossible because Q then intersects both the disjoint clopen sets $U \cup W$ and $V - W$. □

4.1. Local Connectivity

As an example of the non-trivial role of duality in the theory of connectivity on stably compact spaces, the property of being locally connected is not self-dual:

Example 4.5. Take any point a in the Cantor space, and put $x < a$ for all other points. The reflexive closure of this relation is a closed partial order. The corresponding stably compact space is locally connected because all of its open sets are connected; however, the dual space is only locally connected at the point a.

A standard result proved by Capel in [2] is that the limit of an inverse sequence of locally connected compact Hausdorff spaces and monotone bonding maps is locally connected. We will use the following generalization of monotonicity:

Definition 4.6. A proper map between stably compact spaces is *monotone* if the preimage of every connected upper set is connected.

The preimage of an upper set under a continuous map is upper, so the composition of two monotone maps is monotone. For a map between compact Hausdorff spaces to be monotone, it is sufficent that the preimage of each point be connected, which is generalized by:

Proposition 4.7. *A proper map $f : X \to Y$ between stably compact spaces is monotone if and only if for all $y \in Y$, $f^{-1}(\uparrow y)$ is connected.*

Proof. ($\Rightarrow$): For any $y \in Y$, $\uparrow y$ is a connected upper set. ($\Leftarrow$): Suppose $P \subseteq Y$ is an upper set, and that (U, V) is a separation of $f^{-1}(P)$. For any $y \in P$, if $f^{-1}(\uparrow y)$ is connected then it must be a subset of whichever one of U, V it intersects, otherwise it is separated by these sets. It then follows from Proposition 2.2 that $(\forall_f(U), \forall_f(V))$ is a separation of P. □

Proposition 4.8. *The limit of an inverse sequence of locally connected stably compact spaces and monotone bonding maps is locally connected, and each projection on the limit is monotone.*

Proof. Let (X_i, f_i) be such an inverse sequence, with limit X_ω and projections (p_i). For any $x \in X_i$, $f_i^{-1}(\uparrow x)$, $(f_{i+1})^{-1}(f_i^{-1}(\uparrow x)), \ldots$ is a sequence of patch-closed subcontinua; this sequence together with the relevant restrictions of bonding maps is then an inverse sequence of continua and proper maps whose limit is $p_i^{-1}(\uparrow x)$ (see [2] for the details of this property in the context of Hausdorff spaces). K & W showed that the limit of an inverse sequence of continua and proper maps is a continuum, so each projection p_i is monotone. If U is a neighbourhood of the thread (x_i), there is some j and some open neighbourhood V of x_j such that $p_j^{-1}(V) \subseteq U$. Then there is some connected open set W such that $x_j \in W \subseteq V$. Then $p_j^{-1}(W)$ is a connected open neighbourhood of (x_i) that is contained in U. □

Theorem 4.9. *A compact metric space is locally connected if and only if it can be approximated by an inverse sequence of finite T_0-spaces and monotone bonding maps.*

Proof. ($\Leftarrow$): Every finite space is locally connected, K & W showed that the T_2-reflection of a stably compact space X is a quotient of X, and any quotient of any locally connected space is locally connected.

($\Rightarrow$): Nadler in [10] (Th. 8.9) showed that, for any $\epsilon > 0$, any locally connected compact metric space X admits an ϵ-cover by compact connected and locally connected sets; let $\mathcal{C}_1$ be such a 1-cover. Each $C \in \mathcal{C}_1$ admits a 1/2-cover $\mathcal{C}_2(C)$ by connected and locally connected sets whose union is C. Let $\mathcal{C}_2$ be the cover of X that is the multiset union of the sets $\mathcal{C}_2(C)$, for all $C \in \mathcal{C}_1$, and let $f_1 : \mathcal{C}_2 \to \mathcal{C}_1$, $D \mapsto C$, where $D \in \mathcal{C}_2(C)$. In this way may be obtained an approximating inverse sequence $(\mathcal{C}_i, f_i)$ by covers consisting of connected sets, and such that each map f_i has the property that, for any $C \in \mathcal{C}_i$, $C = \bigcup\{D \mid D \in f_i^{-1}(C)\}$ (multisets are used in order to obtain this property).

It remains to show that each map $T(f_i)$ is monotone. For $\mathcal{S} \in T(\mathcal{C}_i)$, let $\mathcal{R} = (T(f_i))^{-1}(\uparrow\mathcal{S})$ and suppose (U, V) is a separation of $\mathcal{R}$. Because the elements of $\mathcal{C}_i$ are connected and because $\mathcal{S}$ is a simplex, $\bigcup\mathcal{S}$ $(= \bigcup\{C \mid C \in \mathcal{S}\})$ is a connected subset of X. From the above-mentioned property of the maps f_i it follows that $\bigcup\mathcal{S} = \bigcup\{C \in \mathcal{C}_{i+1} \mid \{C\} \in \mathcal{R}\}$. Then there exist $\{C\}, \{D\} \in \mathcal{R}$ such that $\{C\} \in U, \{D\} \in V$, and such that $C \cap D \neq \emptyset$. But then $\{C, D\} \in \mathcal{R}$, and $\{C, D\}$ is below both $\{C\}, \{D\}$ in $\leq_{T(\mathcal{C}_i)}$, so whichever of U, V contains $\{C, D\}$ must contain both $\{C\}$ and $\{D\}$. □

Theorem 4.10. *A compact metric space is a Peano continuum if and only if it can be approximated by an inverse sequence of finite continua and monotone bonding maps.*

Proof. ($\Leftarrow$): Is a consequence of the previous result and K & W's result that an inverse sequence of finite continua approximates a connected space. ($\Rightarrow$): In the construction used in the previous proof, if X is connected then each space $T(\mathcal{C}_i)$ is connected. □

4.2. Cut-points

A *cut-point* of a continuum is any point whose complement is disconnected; points that are not cut-points are called *endpoints.* The property of being a cut-point is not self-dual: the point a in Example 4.5 is an endpoint in the stably compact topology, but is a cut-point in the dual topology. This necessitates the use of some fairly unusual terminology:

Terminology. For any subset P of a stably compact space X, (U, V) is a *dual-separation* of P if it is a separation of P in X^D, and is a *di-separation* of P if it is a separation or a dual-separation of P. A point x of a continuum X is a *dual-cut-point* if it is a cut-point of X^D, and is a *di-cut-point* if it is a cut-point or a dual-cut-point. Points that are not di-cut-points are *bi-endpoints.* The point x *separates* the points a, b if there exists a separation (U, V) of $X - \{x\}$ such that $a \in U - \{x\}$ and $b \in V - \{x\}$ (in which case x separates a, b *via* (U, V)); x *dual-separates* a, b if it separates a, b in X^D; x *di-separates* a, b if it separates or dual-separates a, b.

Lemma 4.11. *Let x be a di-cut-point of a continuum X, with (U, V) a di-separation of $X - \{x\}$. Then*

1. *x is maximal or minimal in $\leq_X$;*
2. *If x is minimal in $\leq_X$ then $U \cup \{x\}$ and $V \cup \{x\}$ are closed;*
3. *If x is maximal in $\leq_X$ then $U \cup \{x\}$ and $V \cup \{x\}$ are dual-closed;*
4. *$U \cup \{x\}$ and $V \cup \{x\}$ are subcontinua.*

Proof. Suppose first that x is a cut-point of X and that (U, V) is a separation of $X - \{x\}$. Either $U \cap V = \{x\}$ (in which case x is maximal) or $U \cap V = \emptyset$ (in which case x is minimal), otherwise (U, V) would be a separation of X. $U \cup \{x\}$ is the union of the patch-closed sets $\{x\}$ and $X - V$, and so is patch-closed. If x is minimal then $U \cup \{x\}$ is lower and so is closed; if x is maximal then $U \cup \{x\}$ is upper and so is dual-closed. That $U \cup \{x\}$ is connected is given by a standard result which holds for any space (see [10] Prop. 6.3).

Now suppose that x is a dual-cut-point of X and that (U, V) is a dual-separation of $X - \{x\}$. In other words, x is a cut-point of X^D and (U, V) is a separation of $X^D - \{x\}$. For any stably compact space X it is the case that $\leq_{X^D} = (\leq_X)^{-1}$. The proof that each of the four properties holds may then be obtained from the proof of the first part of this result. □

Lemma 4.12. *If (U, V) is a di-separation of the complement of a di-cut-point x and if C is a patch-closed subcontinuum such that $x \notin C$ then $C \subseteq U$ or $C \subseteq V$.*

Proof. Patch-closed subcontinuua are bi-connected, and so admit no di-separation. If C intersects both U and V then (U, V) is a di-separation of C. □

Lemma 4.13. *If x, y are* distinct *di-cut-points of a continuum, with $(U_x, V_x), (U_y, V_y)$ di-separations of their respective complements, then*

1. *If $x \in U_y$ and $y \in V_x$ then $U_x \subseteq U_y$ and $V_y \subseteq V_x$;*
2. *If $x \in V_y$ and $y \in V_x$ then $U_x \subseteq V_y$ and $U_y \subseteq V_x$.*

Proof. 1: $U_x \cup \{x\}$ is a patch-closed subcontinuum that intersects U_y and does not contain y, and similarly for $V_y \cup \{y\}$. 2. is simply 1. in a different form, stated for convenience. □

Theorem 4.14. *The following properties of Hausdorff continua also hold for stably compact continua:*

1. *Both sets in a di-separation of the complement of a di-cut-point contain a bi-endpoint;*

2. *Every continuum with ≥ 2 points has ≥ 2 bi-endpoints;*

3. *Every continuum is irreducibly connected about its set of bi-endpoints.*

Proof. The proofs are similar to the proofs of the corresponding classical results (see [10] Ch. 6). 1: Suppose c is a di-cut-point of a continuum, with (U, V) a di-separation of its complement. Suppose that, for each $x \in U - \{c\}$, there exists a di-separation (U_x, V_x) of the complement of $\{x\}$ such that $c \in V_x$. From the previous result it follows that $U_x \subseteq U$ for all $x \in U - \{c\}$. Also, if $y \in U_x - \{x\}$ then $U_y \subseteq U_x$ (if $x \in U_y$ then $V_x \subseteq U_y$, which is impossible because then $c \in U_y$: therefore $x \in V_y$, in which case $U_y \subseteq U_x$). The collection of patch-closed subcontinua $\{U_x \cup \{x\} \mid x \in U - \{c\}\}$ ordered by inclusion contains some maximal totally ordered set $\mathcal{C}$, and $\bigcap \mathcal{C} \neq \emptyset$. If $p \in \bigcap \mathcal{C}$ then $U_p \subseteq \bigcap \mathcal{C}$, in which case $\mathcal{C}$ is not maximal because it does not contain $U_q \cup \{q\}$ for $q \in U_p - \{p\}$.

2. Is an immediate corollary of 1. 3: By Theorem 4.4 there is some patch-closed subcontinuum C irreducibly connected about the set of bi-endpoints. If $x \notin C$ then there exists some di-separation (U, V) of its complement: C is a subset of one of U, V, which is impossible because the other contains a bi-endpoint. □

4.3. The Separation Order

For any two points a, b of a continuum, the set of all points that di-separate a, b is denoted (a, b), and $[a, b]$ denotes the union of this set with $\{a, b\}$ (it is then obvious what sets are denoted by $[a, b)$ and by $(a, b]$).

Definition 4.15. For any two points a, b of a continuum, the *separation order* on $[a, b]$, defined on *distinct* points, is the relation $x \prec y$ if $x = a$ or x di-separates a, y.

The definition differs from the standard definition only in its use of di-separation rather than separation. As with the standard definition, there are in fact two separation orders, one with bottom element a (as defined above), and one with bottom element b. It is not usually important which order is considered, since one is the inverse of the other. A total order is a transitive relation $\prec$ such that, for all x, $\neg(x \prec x)$, and such that, for all $x \neq y$, $x \prec y$ or $y \prec x$.

Proposition 4.16. *The separation order is a total order.*

Proof. The proof is very similar to the proof of the corresponding classical result (see [10] Prop. 6.15). Let a, b be points of a continuum, and let $\prec$ be the separation order on $[a, b]$. That $\prec$ is transitive: suppose $x \prec y$ and $y \prec z$. If $x = a$ then $x \prec z$; otherwise, suppose that x di-separates a, y via (U_x, V_x), and that y di-separates a, z via (U_y, V_y). If $x \in V_y$ then, by Lemma 4.13(2), $U_y \subseteq V_x$. But then $a \in V_x$, which is impossible because $a \in U_x$, therefore $x \in U_y$. Then, by Lemma 4.13(1), $V_y \subseteq V_x$, in which case $x \prec z$.

Suppose x, y are distinct points of $[a, b]$: if $x = a$ or $y = b$ then $x \prec y$, and if $x = b$ or $y = a$ then $y \prec x$. Otherwise, suppose that x, y di-separate a, b via (U_x, V_x), (U_y, V_y) respectively. If $y \in V_x$ then $x \prec y$; if $x \in V_y$ then $y \prec x$; if $x \in U_y$ and $y \in U_x$ then $V_y \subseteq U_x$, in which case $b \in U_x$, which is impossible because $b \in V_x$. □

With respect to a total order $\prec$, a point y is *between* the points x, z if $x \prec y \prec z$ or $z \prec y \prec x$.

Proposition 4.17. *For any two points a, b of a continuum X and for all $x, y, z \in [a, b]$, y di-separates x, z in X if and only if y is between x, z in the separation order on $[a, b]$.*

Proof. ($\Leftarrow$): Suppose $x \prec y \prec z$ in the separation order on $[a, b]$. Then y di-separates a, z via some pair (U, V) and $x \in U$ (because if $x \in V$ then $y \prec x$).

($\Rightarrow$): Suppose y di-separates x, z via (U_y, V_y), and assume that $x \prec z$. If $y \prec x$ then there is some pair (U_x, V_x) via which

x di-separates y, z. But then $V_y \subseteq U_x$, which is impossible because then $z \in U_x$. If $z \prec y$ then there is some pair (U_z, V_z) via which z di-separates x, y. But then $U_y \subseteq V_z$, which is impossible because then $x \in V_z$. □

5. Arcs

Definition 5.1. An *arc* is a continuum such that, for any three distinct points, one di-separates the other two.

The definition is clearly based on that of a COTS, the difference being in the use of di-separation rather than separation. The two definitions are, however, equivalent on finite T_0-spaces and on compact Hausdorff spaces.

We will now show that arcs can be described in terms of a total order (the separation order) together with a partial order (the specialization order). Let $\prec$ be a total order on a stably compact space X. For any point x, Lx denotes the set $\{y \mid y \prec x\}$, and Rx denotes the set $\{y \mid x \prec y\}$. The points x, y are *adjacent* if there is no point between them (thus adjacency is a reflexive and symmetric relation). x is *left-adjacent* to y if $x \prec y$ and x, y are adjacent. An *open ray* is any set of the form Lx or Rx, when x is *minimal* in $\leq_X$; a *dual-open ray* is any such set when x is *maximal* in $\leq_X$. The *order topology* on X is that which has subbase the collection of open rays, and the *order dual-topology* is that which has subbase the collection of dual-open rays. (Our definition of 'order topology' is non-standard: the order topology is usually the topology having as subbase *all* open rays, regardless of whether or not points are maximal or minimal.)

A stably compact space X is *orderable* if it admits a total order $\prec$ for which the order topology is the topology on X, and the order dual-topology is the topology on X^D (in which case $\prec$ is said to *order* X).

Theorem 5.2. *The following properties of a continuum with*

≥ 2 points are equivalent:

1. *It is an arc;*
2. *It contains exactly two bi-endpoints;*
3. *It is orderable.*

Proof. $(3 \Rightarrow 1)$: Let X be a continuum ordered by $\prec$; we claim that every point is maximal or minimal in $\leq_X$. If $x <_X y$ then, because X is T_0, there is some open set U containing y but not x. Assuming that $x \prec y$, we may also assume that U is a subbasic open set Rz for some minimal z. But then $z = x$, otherwise $x \in Lz$ and $\neg(x <_X y)$. So every point that is below some other point in $\leq_X$ is minimal, and our claim follows. Then when $x \prec y \prec z$, y di-separates x, z via (Ly, Ry).

$(1 \Rightarrow 2)$: Every arc has ≤ 2 bi-endpoints and every continuum has ≥ 2 bi-endpoints (by Theorem 4.14).

$(2 \Rightarrow 3)$: If a, b are the two bi-endpoints of the continuum X then, by Theorem 4.14 (1), $[a, b]$ is the underlying set in X. We claim that X is ordered by the separation order $\prec$ on $[a, b]$.

Take $x \in (a, b)$: x di-separates a, b via some pair (U, V). For any $y \in U - \{x\}$, x di-separates y, b, so $y \prec x \prec b$ by Proposition 4.17. Conversely, for any $y \prec x$, if $y \in V$ then $x \prec y$, so $y \in U$. Therefore $U - \{x\} = Lx$, and therefore $V - \{x\} = Rx$. x is maximal or minimal in $\leq_X$: if x is minimal then $Lx \cup \{x\}$ and $Rx \cup \{x\}$ are closed, in which case Lx and Rx are open; similarly, if x is maximal then Lx and Rx are dual-open.

The point a is maximal or minimal in $\leq_X$: if $x <_X a <_X y$ then x cannot di-separate a, y and y cannot separate a, x. If a is minimal then Ra is open, and if a is maximal then Ra is dual-open. Similar reasoning applies to b, and therefore the order topology is refined by the topology on X, and the order-dual topology is refined by the dual topology on X.

We have shown that every point of X is maximal or minimal in $\leq_X$ and that, for all x, Lx, Rx are disjoint sets which are either both open (if x is minimal) or both dual-open (if x is maximal). It follows that if $x \leq_X y$ then x, y are adjacent with respect to $\prec$. Conversely, if $x \prec y$ are adjacent and unrelated by $\leq_X$, then both the disjoint non-empty patch-open sets Rx, Ly are upper, and therefore form a separation of X. So x, y are adjacent with respect to $\prec$ if and only if x, y are related by $\leq_X$.

For all $x \prec y$, by Proposition 4.17 the set (x, y) is the set $(Rx \cap Ly)$. Then $[x, y]$ is the complement of $Lx \cup Ry$: Lx and Ry are patch-open, so $[x, y]$ is patch-closed.

Now suppose that U is an open neighbourhood of x in X. We claim that if $x \neq a$ then either $(a, x] \subseteq U$ or there is some $p \prec x$ such that p is minimal in $\leq_X$ and $(p, x] \subseteq U$. If a, x are adjacent then $(a, x] \subseteq U$, otherwise the collection $\{[p, x] \mid p \prec x,\ p, x \text{ not adjacent}\}$ is non-empty. This collection is totally ordered by inclusion, and its intersection is the set C containing x together with, if it exists, the point y that is left-adjacent to x, together with, if it exists, the point z that is left-adjacent to y. If y exists and $y \notin U$ then y is minimal (otherwise $x \leq_X y$ and $y \in U$) and $(y, x] \subseteq U$. If y, z exist and $y \in U$ and $z \notin U$ then, similarly, z is minimal and $(z, x] \subseteq U$. If $C \subseteq U$ then there is some $p \prec x$ such that p, x are not adjacent and $[p, x] \subseteq U$. If p is not minimal then there is some minimal p' adjacent to p, in which case $p' \neq x$ and $(p', x] \subseteq U$.

Similarly, if $x \neq b$ then either $[x, b) \subseteq U$ or there is some minimal $q \succ x$ such that $[x, q) \subseteq U$. This suffices to show that the order topology refines the topology on X, and the proof that the order dual-topology refines the topology on X^D is similar. □

Proposition 5.3. *In any arc: (1) Every point is maximal or minimal in the specialization order; (2) Two points are adjacent in the separation order if and only if they are related by the specialization order; (3) Every patch-closed subset has a top and a bottom element with respect to the separation order.*

Proof. (1) and (2) were proved in the proof of the previous result. (3): If the patch-closed subset C has no bottom element then $\{Rx \mid x \in C\}$ is a cover of C by patch-open sets. There is some finite $D \subseteq C$ such that $\{Rx \mid x \in D\}$ is a cover of C. Then, where d is the bottom element of D, $C \subseteq Rd$, which is impossible because $d \in C - Rd$. The proof that C has a top element is similar. □

Various definitions of an ordered topological space, or of an ordered bi-topological space, which is a space (or bi-topological space) together with a total order have been proposed in [6]. Our definition of an 'arc' is essentially that of a bi-topological space which has a meaningful *intrinsic* total order (the separation order), and upon which an extraneous order does not need to be added. Moreover, the definition of a continuous order-preserving map between ordered spaces has a purely topological characterization with regard to our definition, as will be shown in the following section.

The final result of this section concerns the internal structure of arcs. An *interval* of an arc is any subset P such that $x, y \in P$ implies $[x, y] \subseteq P$.

Proposition 5.4. *A subset of an arc is an interval if and only if it is bi-connected.*

Proof. ($\Leftarrow$): If P is not an interval then there is some $y \notin P$ and some $x, z \in P$ such that y is between x and z in the separation order, in which case (Ly, Ry) is a di-separation of P.

($\Rightarrow$): If (U, V) is a di-separation of an interval P of an arc X, then it is a di-separation of some $[x, y] \subseteq P$. Then $[x, y]$ is the disjoint union of the non-empty patch-closed sets $C = [x, y] - U$ and $D = [x, y] - V$. Moreover, no element of C is related to any element of D by $\leq_X$. Assume without loss of generality that $x \in C$. If $y \in D$ then X is the disjoint union of the non-empty patch-closed sets $C \cup (Lx \cup \{x\})$ and $D \cup (Ry \cup \{y\})$, both of which are lower by Proposition 5.3. If $y \in C$ then X is the disjoint union of the non-empty patch-closed lower sets $C \cup (Lx \cup \{x\}) \cup (Ry \cup \{y\})$ and D. □

5.1. Inverse Sequences of Arcs

A proper map is *dual-monotone* if the preimage of every connected lower set is connected, and is *bi-monotone* if it is both monotone and dual-monotone. A function $f : X \to Y$ between arcs is *order-preserving* if there exist separation orders $\prec_X, \prec_Y$ on X, Y respectively such that f is order-preserving with respect to the respective reflexive closures of $\prec_X, \prec_Y$. The definition is stated as such to take into account of the fact that there are *two* separation orders on an arc, one of which is the inverse of the other.

Proposition 5.5. *A proper map $f : X \to Y$ between arcs is bi-monotone if and only if it is order-preserving.*

Proof. ($\Leftarrow$): For $y \in Y$, the set $\uparrow y$ is an interval by Proposition 5.3. Then $f^{-1}(\uparrow y)$ is an interval (and therefore connected) because f is order preserving, so f is monotone. The proof that the dual of Proposition 4.7 holds is straightforward, and then the proof that f is bi-monotone is similar to the proof that f is monotone.

($\Rightarrow$): Choose a separation order $\prec_X$ on X, and let a, b be its respective bottom and top elements. Let $\prec_Y$ be the separation order on Y such that $f(a) \prec_Y f(b)$ or $f(a) = f(b)$, and let c be its top element. For any $x \in X$, if $f(x) \prec_Y f(a)$, then (Lx, Rx) is a di-separation of $f^{-1}([f(a), c])$. But $[f(a), c]$ is a patch-closed subcontinuum, and is upper (if $f(a)$ is maximal) or lower (if $f(a)$ is minimal), in which case $f^{-1}([f(a), c])$ is a patch-closed subcontinuum. So either $f(a) = f(x)$ or $f(a) \prec_Y f(x)$ and, similarly, either $f(x) = f(b)$ or $f(x) \prec_Y f(b)$. It follows that if $x \prec_X y$ and $f(x) \succ_Y f(y)$ then (Ly, Ry) is a di-separation of the patch-closed subcontinuum $f^{-1}([f(x), c])$.
□

The limit of an inverse sequence of COTS is not necessarily a COTS, even when the spaces in the sequence are finite; a counterexample is given by Smyth in [12]. Moreover, the bonding maps in Smyth's counterexample are bi-monotone. However, our adjustment to the definition of COTS allows:

Proposition 5.6. *The limit of an inverse sequence of arcs and bi-monotone bonding maps is an arc.*

Proof. Suppose (X_i, f_i) is such a sequence, with limit X_ω and projections (p_i). There exists a sequence $(\prec_i)$ such that each $\prec_i$ is a separation order on X_i and such that each f_i is order-preserving. For any two threads, put $(x_i) \prec_\omega (y_i)$ if there is some j such that $x_k \prec_k y_k$ for all $k \geq j$. Then $\prec_\omega$ is a total order on X_ω.

It is the case that $\leq_{X_\omega} = \bigcap_i (p_i \times p_i)^{-1}(\leq_{X_i})$ (this is true for any inverse sequence). From Proposition 5.3 it follows that every thread is maximal or minimal in $\leq_{X_\omega}$, and that if two threads are not adjacent in $\prec_\omega$ then they are not related by $\leq_{X_\omega}$. Then, for any thread (x_i), $L(x_i)$ and $R(x_i)$ are both upper or both lower; it now suffices to show that both are patch-open. If $(y_i) \in L(x_i)$ then there is some j such that $y_j \in Lx_j$. From the fact that each bonding map is order-preserving, $p_j^{-1}(Lx_j) \subseteq L(x_i)$. The set Lx_j is patch-open, and the preimage of a patch-open set under a proper map is patch-open. □

Say that an inverse sequence of finite T_0-spaces is *non-degenerate* if the space it approximates contains ≥ 2 points:

Theorem 5.7. *Any non-degenerate inverse sequence of finite COTS and bi-monotone bonding maps approximates the unit interval.*

Proof. Let X be the limit of such a sequence: K & W showed that the T_2-reflection of (any stably compact space) X is its quotient by the smallest equivalence relation that contains $\leq_X$ and is closed in X^P; let R denote this relation, and let $\prec$ denote the separation order on X. We claim that, for all $x \in X$, the set $R(x) = \{y \mid xRy\}$ is an interval.

The relation $S = \{(x,z) \mid xRz \wedge \forall y.\ y$ between $x, z \Rightarrow xRy\}$ is a subset of R, is clearly reflexive and symmetric, and contains $\leq_X$ by Proposition 5.3. If xSy and ySz then xRz and (assuming that $x \prec z$) if $x \prec p \prec z$ and $\neg xRp$ then either $\neg xSy$ (if $p \prec y$) or $\neg ySz$ (if $y \prec p$), so S is an equivalence relation.

We claim that S is closed in X^P, in which case it is equal to R and our original claim follows. Suppose $\neg xSz$: if $\neg xRz$ then there is an open neighbourhood of (x, z) in $X^P \times X^P$ that is disjoint from R and therefore from S. If xRz then (assuming that $x \prec z$) there is some y such that $x \prec y \prec z$ and $\neg xRy$. The set $R(y)$ is closed (this is true for any closed equivalence relation on any space), so the set $R(y) \cap [x, z]$ is patch-closed and non-empty, and so has a bottom element p and a top element q with respect to $\prec$. Then x, z are elements of the patch-open sets Lp, Rq respectively; we claim that $(Lp \times Rq) \cap S = \emptyset$. Suppose $u \in Lp$ and $v \in Rq$ and uSv: then $u \prec y \prec v$ so uRy. Then $u \prec x$ because $u \prec p$ and $u \in R(y)$. Then uRx because $u \prec x \prec v$, which is impossible because then xRy.

Where $\phi : X \to X/R$ is the canonical quotient map then, for all $x \in X/R$, $\phi^{-1}(x)$ is a patch-closed (because ϕ is proper) interval $[p, q]$. If x is not the image of an endpoint of X then both Lp, Rq are non-empty. Then by Proposition 2.2 and its dual, $(\forall_\phi(Lp), \forall_\phi(Rq))$ is a separation of $X/R - \{x\}$. Then X/R is a connected metric space that has exactly 2 endpoints. □

6. Related & Further Work/Questions

Many of the results given here have been obtained by adapting similar results in the context of the approximation of spaces by inverse sequences of finite *graphs*, a construction that was introduced by Smyth in [13]. Given a cover $\mathcal{C}$ of a space, one considers its *intersection graph* $G(\mathcal{C})$, in which the vertices represent the elements of $\mathcal{C}$, and in which the relation represents non-empty intersection. Corresponding to an approximating inverse sequence of covers $(\mathcal{C}_i, f_i)$ of a compact metric space X is an inverse sequence $(G(\mathcal{C}_i), G(f_i))$ of finite graphs and relation-preserving maps (where the vertex representing $C \in \mathcal{C}_{i+1}$ is mapped by $G(f_i)$ to the vertex representing $f_i(C)$). This inverse sequence is considered in the category of *topological graphs* (also introduced by Smyth: the objects are spaces together with closed relations; the morphisms are continuous relation-preserving maps), where the set of vertices in each $G(\mathcal{C}_i)$ is considered with the discrete topology. The limit of this inverse sequence is a topological graph (X_ω, R) (X_ω is typically Cantor space) in which R is an equivalence relation. Then X is (homeomorphic to) the quotient space X_ω/R, and the inverse sequence of graphs is said to *approximate* X. The quotienting operation is analogous to taking the T_2-reflection in K & W's work, although we will not state this analogy formally here.

In [15] we showed that many basic and some more esoteric properties of Hausdorff continua - such as dendrites; simple closed curves; arc-like, circle-like and tree-like continua; weakly chainable continua; indecomposable continua - could all be expressed in terms of properties of finite graphs and bonding maps in approximating inverse sequences. It seems reasonable to conjecture that most, if not all, of these results can be translated to the context of approximation by finite T_0-spaces.

Smyth in [14] has also given an analysis of *lines* in the context of topological graphs; such structures are closed under inverse limits and may be characterized using total orders.

We finish with two questions which we have not been able to answer one way or the other, but which express results we feel would be essential (in some form) to any well-developed theory of connectivity for stably compact spaces. Where a *bi-Peano continuum* is a 2nd-countable connected and bi-locally connected continuum:

Question 1. Is every bi-Peano continuum arcwise connected?

Question 2. Is every bi-Peano continuum the continuous (or proper) image of an arc? Is there an arc that is universal in this respect?

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