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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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A GENERALIZATION OF SIEKLUCKI'S THEOREM

J.S. CHOI AND G. KOZLOWSKI

ABSTRACT. We weaken the conditions of Sieklucki's Theorem by using cohomological local connectivity and cohomological dimension based on Alexander-Spanier cohomology with compact supports in a countable principal ideal domain L. We could also use Čech cohomology. The theorem states that if X is a clc^n locally compact separable metric space with $\dim_L X = n$ and $\{X_\lambda\}_{\lambda\in\Lambda}$ is an uncountable collection of closed subsets of X with $\dim_L X_\lambda = n$ for all λ, then there are two distinct indices μ, $\lambda \in \Lambda$ such that $\dim_L(X_\mu \cap X_\lambda) = n$. The proof combines the fact the family of submodules of a finitely generated L-module is countable with a Mayer-Vietoris argument.

1. INTRODUCTION

In 1961 Borsuk [2] proved a theorem about a family of compact finite-dimensional ANR's in a compact ANR of the same dimension, which was subsequently generalized by Sieklucki ([10] and [11]) by removing the hypothesis that members of the family be ANR's. We generalize Sieklucki's result replacing the

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concepts of finite-dimensionality and the ANR property with finite cohomological dimension and cohomological local connectivity in the Theorem below and extend his formulation to locally compact ANR's in Corollary 1.

In this paper we consider only locally compact separable metric spaces. The various metrics and distances will be designated by the letter d. The diameter of a subset A of a metric space will be denoted $diam(A)$. If X is a space and $A \subset X$, then $\overline{A}$ will denote the closure of the set A and $int(A)$ its interior. By a *map* or *mapping* we mean a continuous function.

For cohomology we will use the Alexander-Spanier cohomology modules with compact supports with coefficients in a countable principal ideal domain L. Definitions and notation are exposited in Part I of Massey's book [7], although the basic material may also be found in [12] and in [4]. The q-dimensional cohomology module with compact supports of a locally compact Hausdorff space X with coefficients in L will be denoted by $H^q(X)$, and if f is a proper map from X to Y, $f^*\colon H^q(Y) \to H^q(X)$ will denote the induced homomorphism of f.

Definitions. *A space X is clc^n(cohomologically locally n-connected) if for each $q \leq n$, $x \in X$ and each closed neighborhood N of x, there is a closed neighborhood $M \subset N$ of x such that $0 = i^*\colon H^q(N) \to H^q(M)$, and the cohomological dimension $\dim_L X$ of a space X with respect to the module L is defined to be the least integer n (or ∞) such that $H^q(U) = 0$ for every open subset U of X and $q > n$.*

Theorem. *If X is a clc^n locally compact separable metric space with $\dim_L X = n$ and $\{X_\lambda\}_{\lambda\in\Lambda}$ is an uncountable collection of closed subsets of X with $\dim_L X_\lambda = n$ for all λ, then there are two distinct indices μ, $\lambda \in \Lambda$ such that $\dim_L(X_\mu \cap X_\lambda) = n$.*

Cohomological dimension and the property clc^n are discussed in [1] and extensively treated in [4]. All of the results concerning cohomological dimension which we use in the proof of the theorem can be found in section 16 of Chapter II of [4]. The property clc^n plays an important role through the following lemma which is a part of Theorem 17.4 in [4].

Lemma. *If X is a locally compact Hausdorff clc^n space, then for all compact subsets K, M of X with $K \subset int(M)$ the image of the homomorphism $i^*\colon H^q(M) \to H^q(K)$ induced by inclusion is finitely generated for each $q \leq n$.*

For a compact Hausdorff space Alexander-Spanier cohomology with compact supports is the same as Alexander-Spanier cohomology, and for all spaces the latter is naturally isomorphic to Čech cohomology [12]. Furthermore, for any locally compact Hausdorff space X and for any integer $q > 0$, $H^q(X) \cong H^q(X^+)$, where X^+ is the one point compactification of X. These statements allow one to translate assertions involving cohomology with compact supports into Čech cohomology. On the other hand because the Čech cohomology of a compact pair (X, A) is isomorphic to $H^q(X \setminus A)$ [12], we may use the discussion of large and small cohomological dimension found in the Appendix of [8] to interpret the whole paper in terms of Čech (= Alexander-Spanier) cohomology. Lastly, using standard theorems from [4] and [8] we note that the three definitions of cohomological dimension coincide for locally compact separable metric spaces and agree with covering dimension in case the latter is finite.

We thank the referee for inspiring improvements in our original formulation.

2. Proof of the Theorem

We will use the following notation. If A, B and C are modules and $f\colon A \to C$ and $g\colon B \to C$ are homomorphisms, then $Im\ f$ is the image of f and $\langle f - g\rangle$ denotes the homomorphism from $A \oplus B$ into C defined by $\langle f - g\rangle(a, b) = f(a) - g(b)$.

Proof: Case I. X is compact. Suppose that there is an uncountable collection $\{X_\lambda\}_{\lambda \in \Lambda}$ of closed subsets of X with

$\dim_L X_\lambda = n$ such that for any two distinct indices λ, $\mu \in \Lambda$, $\dim_L(X_\lambda \cap X_\mu) \leq n-1$.

Since X is clc^n, there is a positive ε such that for any closed subset A of X with $diam(A) < \varepsilon$, $i^* = 0$, where $i^*\colon H^n(X) \to H^n(A)$ is induced by inclusion.

Let λ be an arbitrary element of Λ. Since $\dim_L X_\lambda = n$, by the Sum Theorem we can choose a closed subset B_λ of X_λ such that $\dim_L B_\lambda = n$ and $diam(B_\lambda) < \frac{\varepsilon}{3}$. By a standard property of cohomological dimension there exist a closed subset A_λ of B_λ and $u_\lambda \in H^{n-1}(A_\lambda)$ with $u_\lambda \notin Im\ i_\lambda^*$, where $i_\lambda^*\colon H^{n-1}(B_\lambda) \to H^{n-1}(A_\lambda)$ is induced by inclusion $i_\lambda\colon A_\lambda \hookrightarrow B_\lambda$. By the Weak Continuity of Alexander-Spanier cohomology [12], there is a closed neighborhood N_λ (in X) of A_λ such that u_λ has an extension over N_λ; call it v_λ. We may assume that $diam(N_\lambda) < \frac{\varepsilon}{3}$.

Let $\{V_k \mid k \in I\}$ be a countable base for X. For each λ there are finite collections $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ of basic open sets V_k such that for

$$N_\lambda^1 = \bigcup_{V_i \in \mathcal{B}_\lambda} \overline{V_i} \quad \text{and} \quad N_\lambda^2 = \bigcup_{V_i \in \mathcal{A}_\lambda} \overline{V_i}$$

we have

$$A_\lambda \subset N_\lambda^1 \subset int(N_\lambda^2) \subset N_\lambda^2 \subset N_\lambda.$$

Consider $\mathcal{C} = \{(N_\lambda^1, N_\lambda^2) \mid \lambda \in \Lambda\}$. Since we have at most countably many pairs in $\mathcal{C}$ and Λ is uncountable, there exist subsets N_1, N_2 of X and an uncountable subset Λ' of Λ such that for every $\lambda \in \Lambda'$, $(N_\lambda^1, N_\lambda^2) = (N_1, N_2)$.

We now use the fact that a finitely generated L-module has at most countably many distinct submodules, which follows directly from the facts that all its submodules are finitely generated and that L is countable. Consider the inclusions $j\colon N_1 \hookrightarrow N_2$ and $j_\lambda\colon N_2 \hookrightarrow N_2 \cup B_\lambda$ for every $\lambda \in \Lambda'$ and the induced homomorphisms $j^*\colon H^{n-1}(N_2) \to H^{n-1}(N_1)$ and $j_\lambda^*\colon H^{n-1}(N_2 \cup B_\lambda) \to H^{n-1}(N_2)$. Then $Im\ j^* j_\lambda^*$ for $\lambda \in \Lambda'$ form an uncountable family of submodules of $Im\ j^*$, which is finitely generated by the Lemma. Consequently, there exist λ

and $\mu \in \Lambda'$ such that the submodules $Im\ j^*j_\lambda^*$ and $Im\ j^*j_\mu^*$ are the same. We fix λ and μ for the remainder of the proof.

Since $u_\lambda \in H^{n-1}(A_\lambda)$ has an extension v_λ over N_λ and $N_2 \subset N_\lambda$, u_λ has an extension $w_\lambda \in H^{n-1}(N_2)$. Since $u_\lambda \notin Im\ i_\lambda^*$, $w_\lambda \notin Im\ j_\lambda^*$ and $j^*w_\lambda \notin Im\ j^*j_\lambda^*$. Most importantly, $w_\lambda \notin Im\ \langle j_\lambda^* - j_\mu^*\rangle$, for if $w_\lambda \in Im\ \langle j_\lambda^* - j_\mu^*\rangle$, then $j^*w_\lambda \in Im\ j^*\langle j_\lambda^* - j_\mu^*\rangle = Im\ \langle j^*j_\lambda^* - j^*j_\mu^*\rangle$ which is a submodule of $Im\ j^*j_\lambda^*$, because $Im\ j^*j_\lambda^* = Im\ j^*j_\mu^*$. (Actually, one can verify that $Im\ \langle j^*j_\lambda^* - j^*j_\mu^*\rangle = Im\ j^*j_\lambda^*$.)

We now analyze the induced homomorphisms on cohomology in dimension $n-1$ of the inclusions $l\colon N_2 \hookrightarrow N_2 \cup (B_\lambda \cap B_\mu)$ and $l_t\colon N_2 \cup (B_\lambda \cap B_\mu) \hookrightarrow N_2 \cup B_t$ for $t = \lambda, \mu$. By hypothesis, $\dim_L(B_\lambda \cap B_\mu) \leq n-1$. Hence $H^n((B_\lambda \cap B_\mu) \setminus N_2) = 0$, and by the long exact sequence of cohomology, $l^*\colon H^{n-1}(N_2 \cup (B_\lambda \cap B_\mu)) \to H^{n-1}(N_2)$ is an epimorphism. Let w be an element in $H^{n-1}(N_2 \cup (B_\lambda \cap B_\mu))$ such that $l^*w = w_\lambda$. Since $w_\lambda \notin Im\ \langle j_\lambda^* - j_\mu^*\rangle$ and $\langle j_\lambda^* - j_\mu^*\rangle = l^*\langle l_\lambda^* - l_\mu^*\rangle$, w is not in the image of $\langle l_\lambda^* - l_\mu^*\rangle$.

Consider the following Mayor-Vietoris sequence

$$H^{n-1}(B_\lambda \cup N_2) \oplus H^{n-1}(B_\mu \cup N_2) \xrightarrow{\langle l_\lambda^* - l_\mu^*\rangle} H^{n-1}(N_2 \cup (B_\lambda \cap B_\mu))$$

$$\xrightarrow{\Delta} H^n(B_\lambda \cup B_\mu \cup N_2).$$

Since w is not in the image of $\langle l_\lambda^* - l_\mu^*\rangle$, $\Delta(w) \neq 0$. But $diam(B_\lambda \cup B_\mu \cup N_2) \leq diam(B_\lambda) + diam(B_\mu) + diam(N_2) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Therefore, for the inclusion $i\colon B_\lambda \cup B_\mu \cup N_2 \hookrightarrow X$ we have $i^* = 0$ while $i^*\colon H^n(X) \to H^n(B_\lambda \cup B_\mu \cup N_2)$ is an epimorphism. Thus $H^n(B_\lambda \cup B_\mu \cup N_2) = 0$, a contradiction.

Case II. X is not compact. Suppose that there is an uncountable collection $\{X_\lambda\}_{\lambda \in \Lambda}$ of closed subsets of X with $\dim_L X_\lambda = n$ such that for any two distinct indices λ, $\mu \in \Lambda$, $\dim_L(X_\lambda \cap X_\mu) \leq n-1$.

Using a countable base for X consisting of open sets with compact closures we see that there is an ascending sequence

$\{Y_k\}$ of compact subsets of X such that

$$X = \bigcup_{k=1}^{\infty} Y_k \text{ and } Y_k \subset int(Y_{k+1})$$

for all $k \geq 1$. By the Sum Theorem for each $\lambda \in \Lambda$ there is a k such that $\dim_L(X_\lambda \cap Y_k) = n$. It follows that there exists a positive integer m such that $\dim_L(X_\lambda \cap Y_m) = n$ for all λ in an uncountable subset of Λ. Replacing the sets X_λ with the sets $X_\lambda \cap Y_m$ and working in the compact subset Y_m of $int(Y_{m+1})$ we obtain a contradiction as in Case I. □

3. Consequences

As a corollary to the theorem we deduce Sieklucki's Theorem (extended to locally compact spaces). Throughout this section we will use the integers **Z** as coefficients for cohomology. In the statement $\dim X$ denotes the covering dimension of X. For definitions and further discussion see [2], [3], [10], and [11]. The reader interested in a proof using the geometric hypotheses of this corollary may recast the proof of the Theorem as indicated in the proof below proving the Lemma for locally compact ANR's using a technique of Deleanu [6].

Corollary 1. *If X is a locally compact LC^n metric space with $\dim X = n$ and $\{X_\lambda\}_{\lambda\in\Lambda}$ is an uncountable collection of $n-$ dimensional closed subsets of X, then there are two distinct indices μ, $\lambda \in \Lambda$ such that* $\dim X_\mu \cap X_\lambda) = n$.

Proof: Since $\dim X = n$, $\dim_{\mathbf{Z}} Y = \dim Y$ for all spaces Y in the corollary, and because X is LC^n, it is an ANR [3]. Since an ANR is clearly clc^n, the corollary follows from the Theorem. □

The generalization of Sieklucki's Theorem makes it possible to provide a modest improvement in a theorem on cell-like maps by weakening the hypothesis that the domain of the map be an ANR (See [9] or [5]).

Corollary 2. *If $f\colon X \to Y$ is a proper monotone map of a clc^2 locally compact separable metric space X with* $\dim X =$

2 *onto a locally compact separable metric space* Y *such that* $H^1(f^{-1}(y)) = 0$ *for all* $y \in Y$, *then* $\dim Y \leq 2$.

Proof: The proof follows well known lines using the Vietoris-Begle Mapping Theorem and the fact that cohomological dimension ≤ 1 and covering dimension ≤ 1 are the same ([9] or [5]). For $y \in Y$ and for each $\epsilon > 0$ let $B(y,\epsilon) = \{y' \in Y \mid d(y,y') = \epsilon\}$. It follows from the Theorem that $\dim f^{-1}B(y,\epsilon) \leq 1$ and therefore $\dim B(y,\epsilon) \leq 1$ for all but countably many values of ϵ. Hence $\dim Y \leq 2$. □

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Pusan National University, Pusan, Korea 609-735
E-mail address: choijon@hyowon.cc.pusan.ac.kr

Auburn University, Auburn, AL 36849
E-mail address: kozloga@mail.auburn.edu