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CLEAVABILITY OF MANIFOLDS

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ABSTRACT. The aim of this paper is to investigate Arhangelskii's idea of cleavability in the context of manifolds. A key role will be played by "perfectness" type properties.

1. INTRODUCTION

The aim of this note is to investigate the preservation of perfectness (and related properties) and normality (and related properties) by cleavability, and the influence of perfectness (and its relatives) on the topology of manifolds. As defined by Arhangelskii in [1], a space X is cleavable over a class of spaces $\mathcal{P}$ if for every subset A of X there is a mapping f of X into Y, where $Y \in \mathcal{P}$, such that $f(A) \cap f(X - A) = \emptyset$. A space X is absolutely cleavable over a class $\mathcal{P}$ if there exists a one-to-one continuous mapping of X into Y, for some Y in $\mathcal{P}$. Plainly, absolute cleavability implies cleavability. When $\mathcal{P}$ consists of a single space, Y, then X is said to be cleavable over Y.

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Few properties are preserved by absolute cleavability, so, for positive results, we need to place additional restrictions on X. The most effective such restriction, is to assume X to be compact. Two weakenings of compactness then suggest themselves. The first is to suppose that X is not only cleavable over a class $\mathcal{P}$, but also that there is a perfect map of X onto a member of $\mathcal{P}$. The second weakening is to suppose that X is locally compact. This condition, is, in general, too weak, but we will see that good results follow if we additionally assume local connectedness.

Positive results of the first type: 'X cleavable over $\mathcal{P}$, and there is a perfect map of X onto $Y \in \mathcal{P}$, implies $X \in \mathcal{P}$', hold for many classes $\mathcal{P}$ related to the class of perfect spaces. (Among others, spaces with G_δ–diagonal, the class of σ–spaces, and the class of all stratifiable spaces.) Our first new result will be to weaken the perfect map to a quasi–perfect map (in other words, a continuous closed function, with countably compact fibres). We also give examples demonstrating how this fails for $\mathcal{P}$ = perfect or perfectly normal; and fails for strong separation axioms related to perfect normality (hereditarily normal and monotonically normal).

Next, we investigate problems of the type 'X cleavable over $\mathcal{P}$, X locally compact and locally connected, implies X in $\mathcal{P}$'. This is applied to the special case of manifolds.

Finally, we explore some issues related to cleavability over metrisable spaces. In particular, it is shown how 'normal' in two well known results, '(MA(ω_1)) normal, perfect manifolds are metrisable' and 'normal Moore manifolds are metrisable', can be weakened to 'weakly normal', a very weak separation axiom clearly possessed by spaces cleavable over separable metrisable spaces.

Notation and definitions. Most concepts will be described as and when they are used. However it is convenient to make a few technical definitions at the outset.

Let X be a space, and (A, B) a pair of subsets of X. A collection $\mathcal{U}$ of open (closed) subsets of X is T_2–separating open (closed) for (A, B) [respectively, separating] [strongly separating] if, given distinct points $x_0 \in A$ and $x_1 \in B$, there are $U_0, U_1 \in \mathcal{U}$ such that $x_0 \in U_0, x_1 \in U_1$ and $U_0 \cap U_1 = \emptyset$ [respectively, there is $U \in \mathcal{U}$ such that $x_0 \in U$ but $x_1 \notin U$] [respectively, there is $U \in \mathcal{U}$ so that $x_0 \in U$ while $x_1 \notin \overline{U}$]. A T_2–separating family for the pair (X, X) is called a T_2–separating cover of X. (And similarly for separating covers, and strongly separating covers.) A closed subspace, C, of a space X, will be called an S_δ subset (respectively, R_δ subset) if there is a countable open separating (respectively, strongly separating) family for the pair $(C, X - C)$. By hereditary, we mean closed hereditary.

2. Perfect Pre–Images

In this section we examine problems of the first type mentioned in the introduction.

Lemma 2.1. *Let $\mathcal{P}$ be a hereditary class of spaces, whose countably compact members are compact and metrisable. Then for countably compact X, cleavable over $\mathcal{P}$, with countable pseudocharacter, we have $|X| \leq 2^\omega$.*

Proof: We show that X, as in the statement of the lemma, is perfect. For then X has countable spread [Lemma 6.7. [23]], and countable pseudocharacter, from which it follows [Theorem 4.9. [17]], that $|X| \leq 2^\omega$. Take any closed $A \subseteq X$. As X is cleavable over $\mathcal{P}$, there is a continuous surjective mapping $f: X \to Y$, $Y \in \mathcal{P}$, such that $f(A) \cap f(X - A) = \emptyset$. Now Y is countably compact and in $\mathcal{P}$, so, by hypothesis, is compact and metrisable. Similarly, $f(A)$ is compact and metrisable, and in particular is a closed subspace. Since $f^{-1}f(A) = A$, and $f(A)$ is a G_δ in Y, we see that A is a G_δ in X. □

Theorem 2.2. *Let $\mathcal{P}$ be a class of spaces, such that $\mathcal{P}$ is hereditary, countably productive, every members of $\mathcal{P}$ has countable pseudocharacter, and countably compact elements of $\mathcal{P}$ are compact and metrisable. If X is a space cleavable over $\mathcal{P}$, and* *if there is a continuous mapping $f\colon X \to Y$, where $Y \in \mathcal{P}$ such that all preimages of points in Y are countably compact, then X is absolutely cleavable over $\mathcal{P}$.*

Proof: Since the pseudocharacter of space Y is countable, from Proposition 2.1 [2] it follows that the pseudocharacter of space X is countable. By the preceeding Lemma all the fibres $F_y = f^{-1}(y)$, $y \in Y$ have cardinality no greater then 2^ω. Now we can apply Proposition 2.1 [2] to the disjoint family $\{F_y : y \in Y\}$. It follows that there is a continuous mapping g of X into a space Z with a countable pseudocharacter satisfying the following condition: if x_1 and x_2 are in X and $f(x_1) = f(x_2)$, then $g(x_1) \neq g(x_2)$. Then $f \Delta g$ is a one-to-one continuous mapping of X into a space belonging to $\mathcal{P}$. Hence X is absolutely cleavable over $\mathcal{P}$. □

Theorem 2.3. *Let X be a space cleavable over a class $\mathcal{P}$, and suppose there is a quasi-perfect map of X onto a space in $\mathcal{P}$.*

The space X is in $\mathcal{P}$, for the following classes of spaces: spaces with a regular G_δ-diagonal, spaces with a $G^*{}_\delta$-diagonal, spaces with a G_δ-diagonal, spaces with a R_δ-diagonal, spaces with a S_δ-diagonal, spaces with a quasi-$G^*{}_\delta$-diagonal, spaces with a quasi-G_δ-diagonal; semi-stratifiable spaces, σ-spaces, Moore spaces, stratifiable spaces, spaces with point-countable base, metrisable spaces.

Definitions of the standard classes of spaces mentioned above can be found in [15]. The 'quasi' versions are obtained by weakening 'open cover' in the standard property to 'open family'. (So a space has a G_δ–diagonal if there is a countable sequence of open *covers* $\{\mathcal{G}_n\}_{n\in\omega}$ such that, for all points x, $\bigcap_{n\in\omega} \mathrm{st}(x, \mathcal{G}_n) = \{x\}$ $(*)$; while a space has a quasi–G_δ–diagonal, if it has a sequence of open *families* $\{\mathcal{G}_n\}_{n\in\omega}$ satisfying $(*)$.) A space, X, has an S_δ diagonal (respectively, an R_δ diagonal) if the diagonal in X^2 is an S_δ subset (respectively, an R_δ subset).

Proof: This result is known to be true for 'perfect map' in place of 'quasi-perfect map' [5] (or is true by minor variations of the proof given there). However the fibres of the quasi-perfect map are in fact compact. To see this observe that by [Theorem 2.2], all the fibres are countably compact and absolutely cleavable over $\mathcal{P}$. Now recall that a countably compact space belonging to any of the classes listed is compact (and metrisable). □

Perhaps the most interesting example concerning cleavability is Balogh's example of a normal space, whose every subset is a G_δ, but which does not have a G_δ diagonal [9]. As observed by Arhangelskii [2], Balogh's space is cleavable over the space of rational numbers. This demonstrates why our additional assumption (about quasi–perfect pre–images) is a necessary hypothesis in Theorem 2.3.

We now give some examples of classes of spaces closely related to those mentioned in Theorem 2.3, for which the corresponding conclusion is false.

Example 2.4. *There is a space X which is not hereditarily normal (hence, not monotonically normal) which is absolutely cleavable over a metrisable space and which admits a perfect mapping onto a monotonically normal (hence, hereditarily normal) space.*

For. Let $X = (\omega + 1) \times (\mathbf{R}, \mathcal{M})$, $Y = (\mathbf{R}, \mathcal{M})$ and $Z = (\omega + 1) \times (\mathbf{R}, \mathcal{E})$, where $\mathcal{M}$ is the Michael line topology, and $\mathcal{E}$ is the usual Euclidean topology. Then X is not hereditarily normal , Y is monotonically normal, and Z is metrisable. Further, $f = \pi_Y \colon X \to Y$ is a perfect mapping and $i \colon X \to Z$, the identity map, witnesses that X is absolutely cleavable over a metrisable space.

Example 2.5. *There is a space X which is neither perfect nor hereditarily normal, but which is absolutely cleavable over a perfectly normal space and admits a perfect mapping onto a perfectly normal space.*

For. Write DA for the double arrow space, and let $X = DA \times (I, \mathcal{S})$, $Y = (I, \mathcal{S})$, $Z = DA \times (I, \mathcal{E})$; where $\mathcal{S}$ is the Sorgenfrey topology, and $\mathcal{E}$ is the standard Euclidean topology. Note X is not hereditarily normal, Y and Z are perfectly normal (recall that the product of a perfect space and a second countable space is perfect), $f = \pi_Y : X \to Y$ is perfect and $i : X \to Z$, the identity, demonstrates absolute cleavability.

It remains to show X is not perfect. Recall that $DA = I \times \{0,1\}$, with basic open neighbourhoods of $(x,0) \in I \times \{0\}$ having the form $\{(x-\epsilon, x+\epsilon] \times \{0\}\} \cup \{(x-\epsilon, x+\epsilon) \times \{1\}\}$ and basic open neighbourhoods of $(x,1) \in I \times \{1\}$ are $\{(x-\epsilon, x+\epsilon) \times \{0\}\} \cup \{(x-\epsilon, x+\epsilon] \times \{1\}\}$. Let $\Delta_1 = \{((x,0),x) : x \in I\}$, $\Delta_2 = \{((x,1),x) : x \in I\}$.

Then Δ_1 is closed in $DA \times (I, \mathcal{S})$, we show that Δ_1 is not a G_δ. Let U be an open set such that $U \supseteq \Delta_1$. For all $x \in I$ pick a basic neighbourhood $\{(a_x, x] \times \{0\}\} \cup \{(a_x, x) \times \{1\}\} \subseteq U$. As Δ_1 is homeomorphic to the Sorgenfrey line, which is Lindelof, there is a countable set $S_U \subseteq I$ such that $\Delta_1 \subseteq \bigcup_{s \in S_U} \{(a_s, s] \times \{0\}\} \cup \{(a_s, s) \times \{1\}\} \subseteq U$. Note that $\Delta_2 - (S_U \times \{1\}) \subseteq \bigcup_{s \in S_U} ((a_s, s] \times \{0\}) \cup ((a_s, s) \times \{1\}) \subseteq U$. Thus, if $\{U_n\}_{n \in \mathbf{N}}$ is any sequence of open set containing Δ_1, then $\Delta_2 - \bigcup_{n \in \mathbf{N}} (S_{U_n} \times \{1\}) \subseteq \bigcap_{n \in \mathbf{N}} U_n$, and so $\Delta_2 - \bigcup_{n \in \mathbf{N}} (S_{U_n} \times \{1\})$ is uncountable (hence non-empty).

Example 2.6. *There is a compact space X which is absolutely cleavable over a compact space Y with a dense G_δ–Δ, such that X does not have a dense G_δ–Δ. (The map identifying X to a point is a perfect map onto a space with a dense G_δ diagonal.)*

For. Let $X = 2^{\omega_1}$. Then X is a compact space without a dense G_δ–Δ. Let $Y = D(X) = X \times \{0,1\}$, the Alexandrov duplicate of X. Note that $X \times \{0\}$ is homeomorphic to X. Then Y is compact, and X is absolutely cleavable over Y. Further, $X \times \{1\}$ an open dense metrisable subset of Y. So by Arhangelskii's lemma [4], Y has a dense G_δ–Δ.

3. Cleavability over Manifolds

The starting point for our investigation is the following elegant theorem of Arhangelskii, and a companion result which easily follows from a theorem of Balogh & Bennett and (independently) Rudin.

Theorem 3.1. (Arhangel'skii) [2] *Let M be a separable n-dimensional manifold cleavable over $\mathbf{R}^n$, where n is a positive integer. Then X is homeomorphic to an open subspace of $\mathbf{R}^n$.*

Theorem 3.2. *If X is locally compact, locally connected and absolutely cleavable over the class of metrisable spaces. Then X is metrisable.*

Proof of Theorem 3.2 By absolute cleavability over metrisable spaces, there is a coarser metrisable topology on the space X. This metrisable topology has a σ-discrete base, hence X has a σ–discrete open T_2–separating family. The claim follows from Theorem 1.1 [10]. □

Theorem 3.3. *Let X be a locally compact, locally connected space. If X is cleavable over the class $\mathcal{P}$, such that $\mathcal{P}$ is countably productive, hereditary, additive and every compact space cleavable over $\mathcal{P}$ is metrisable, then X is absolutely cleavable over $\mathcal{P}$.*

Proof: Since X is locally connected, it is the disjoint sum of clopen connected subspaces (its connected components). Hence, by additivity of $\mathcal{P}$, it is absolutely cleavable over $\mathcal{P}$ provided each component is absolutely cleavable over $\mathcal{P}$. As each connected component of X is connected, locally connected, locally compact and cleavable over $\mathcal{P}$, we may assume, without loss of generality, that X is connected.

By hypothesis, compact spaces cleavable over $\mathcal{P}$ are metrisable. The proof of Theorem 2.9 of [18] (manifolds have cardinality the continuum) demonstrates that T_3, locally second countable, connected spaces have cardinality no more than the continuum. Thus our particular X has cardinality no more than the continuum.

Consider X as a subset of $\mathbf{R}$. Then by second countability of $\mathbf{R}$, there is a countable family $\mathcal{C}$ of subsets of X which are T_2-separating.

For each $C \in \mathcal{C}$, pick $f_C : X \longrightarrow Y_C$ a continuous map of X into a space $Y_C \in \mathcal{P}$, such that $f_C(C) \cap f_C(X - C) = \emptyset$. Define $\phi : X \longrightarrow \prod_{C \in \mathcal{C}} Y_C$ to be the diagonal map of the family $\{f_C\}_{C \in \mathcal{C}}$. Then ϕ is a one–to–one continuous map of X into a space which belong to class $\mathcal{P}$. In other words, X is absolutely cleavable over class $\mathcal{P}$. □

Corollary 3.4. *Let M be a manifold. If M is cleavable over the class $\mathcal{P}$, such that $\mathcal{P}$ is countably productive, hereditary and additive, then M is absolutely cleavable over $\mathcal{P}$.*

Corollary 3.5. *Let X be a locally compact, locally connected space.*

1. *If M is cleavable over class of spaces with G_δ-diagonal, then M has G_δ-diagonal.*
2. *If M is cleavable over class of spaces with G_δ^*-diagonal, then M is developable.*
3. *If M is cleavable over class of spaces with regular G_δ-diagonal, then M is metrisable.*

Proof: Suppose, first, M is cleavable over class of spaces with G_δ-diagonal. Then, by Theorem 3.3, M is absolutely cleavable over a space with G_δ-diagonal, and hence M has a G_δ-diagonal.

Similarly, if M is cleavable over class of spaces with G_δ^*-diagonal, or regular G_δ-diagonal, then M has the same property. By well known results [15], the claim follows. □

In contrast to Corollary 3.5, in [16] there is an example of a p-adic analytic manifold, absolutely cleavable over the class of metrisable spaces, which is quasi-developable but not perfect (hence not developable). A p-adic analytic manifold, is a space locally homeomorphic to the Cantor set, with these Cantor sets 'sewn together' by 'smooth' functions.

4. Related topics

The main aim of this section is to prove the following two results which are generalisations (respectively) of famous theorems of Rudin [21] and Reed–Zenor [19]. A space X is said to be weakly normal [3] if, for every pair of disjoint closed subsets A and B of X, there is a continuous map f of X into a separable metrisable space such that $f(A) \cap f(B) = \emptyset$. Clearly normal spaces are weakly normal, as is any space cleavable over the class of separable metrisable spaces. That such a weak separation property can replace normality in the theorems of Reed–Zenor and Rudin, was a surprise to the authors.

Theorem 4.1. $(MA(\omega_1))$ *Locally compact, locally connected, perfect and weakly normal spaces, are paracompact.*

Hence, perfect, weakly normal manifolds are metrisable.

Theorem 4.2. *Every weakly normal, locally compact, locally connected Moore space is metrisable.*

We start by elucidating the connection between weak normality, cleavability over the class of separable metrisable spaces, and two properties related to perfectness introduced by Balogh & Bennett.

A space X is called S–perfect (respectively, R–perfect) if every closed subset A of X is an S_δ subset (respectively, a R_δ subset) [11]. It is convenient to introduce two related notions. We shall say that a space X has property (A) (respectively, $(A)_c$) if for every subset A (respectively, closed subset A) of X there is a countable closed family $\mathcal{C}$ which T_2-separates the pair $(A, X - A)$.

Arhangelskii [3] has shown that a space is cleavable over the class of separable metrisable spaces if and only if it is weakly normal and has property (A). It is plain that perfectly normal spaces are R–perfect, spaces with property (A) have property $(A)_c$, and a space is S–perfect if it is either perfect or has property (A). We also remark that the Prufer manifold has a countable closed T_2–separating family, hence it has property (A), but, as it is not metrisable, cannot be weakly normal.

There follows a series of results in which weak normality 'lifts' the weaker perfectness type conditions to R–perfectness.

Lemma 4.3. *A weakly normal space X with property $(A)_c$ is R-perfect.*

Proof: Take any closed $A \subseteq X$. Then there is $\mathcal{C}$, a countable family of closed sets T_2–separating points of A from points of $X - A$. For each pair (C_1, C_2) such that $C_1, C_2 \in \mathcal{C}$ and $C_1 \cap C_2 = \emptyset$, There is, by weak normality, a continuous map $f_{(C_1,C_2)}: X \to \mathbf{R}^\omega$ such that $f(C_1) \cap f(C_2) = \emptyset$.

Fix a countable base, $\mathcal{B}$, for $\mathbf{R}^\omega$, and define $\mathcal{G} = \{f^{-1}_{(C_1,C_2)}(B) : B \in \mathcal{B}, C_1, C_2 \in \mathcal{C}\}$. One can easily show that $\mathcal{G}$ is an open family, strongly separating points of A, from points of $X - A$. □

Proposition 4.4. *Every hereditarily weakly normal space X with a countable point separating open cover has a countable strongly separating open cover.*

Proof: Taking complements it follows that X has a countable point separating closed cover $\mathcal{C}$. By hereditary weak normality, for each pair (C_1, C_2) such that $C_1, C_2 \in \mathcal{C}$, there is a continuous map $f_{(C_1,C_2)}: Y = X - \overline{C_1 - C_2} \cap \overline{C_2 - C_1} \to \mathbf{R}^\omega$, such that $f(C_2 - C_1) \cap f(C_1 - C_2) = \emptyset$. Note that Y is an open subspace of X.

Fix a countable base, $\mathcal{B}$, for $\mathbf{R}^\omega$. Define $\mathcal{G} = \{f^{-1}_{(C_1,C_2)}(B) : B \in \mathcal{B}, C_1, C_2 \in \mathcal{C}\}$.

We claim that $\mathcal{G}$ is an open family, strongly separating points of A, from points of $X - A$. To see this, take any $x \in A, y \notin A$. There are $C_1, C_2 \in \mathcal{C}$ such that $x \in C_1 - C_2, y \in C_2 - C_1$. But $x' = f_{(C_1,C_2)}(x) \neq f_{(C_1,C_2)}(y) = y'$ and there is a $B \in \mathcal{B}$ such that $x' \in B, y' \notin \overline{B}$.

Let $U = f^{-1}_{(C_1,C_2)}(B)$, then $U \in \mathcal{G}$ and $x \in U$ and $y \notin \overline{U}$. □

The same method of proof as Proposition 4.4 yields the following:

Proposition 4.5. *Every hereditarily weakly normal S–perfect space X is R–perfect.*

Proposition 4.6. *Every perfect and weakly normal space X is R–perfect*

Proof: Perfect spaces are S- perfect. From the preceeding proposition, if we can show X is a hereditarily weakly normal, then X is R- perfect.

Actually, we only neėd to show every open subspace of X is weakly normal. Let Y be open subspace of X. As X is perfect, there is a countable family $\mathcal{C} = \{C_n\}_{n\in\mathbf{N}}$ of closed subsets of X such that $Y = \bigcup_{n\in\omega} C_n$.

Let A, B be closed disjoint sets in Y. Define, for each $b \in \omega$, $A_n = A \cap C_n, B_n = B \cap C_n$. Note that the A_ns and B_ns are closed in X. So can find, for each $n, m \in \omega$ a continuous map $f_{(n,m)}\colon X \to \mathbf{R}^\omega$ such that $f_{(n,m)}(A_n) \cap f_{(n,m)}(B_n) = \emptyset$.

Define $f\colon Y \to (\mathbf{R}^\omega)^{\omega\times\omega}$ by $f = \Delta_{(n,m)\in\omega\times\omega} f_{(n,m)}$, in other words, $f(x) = (f_{(n,m)}(x))_{(n,m)\in\omega\times\omega}$. The map f is continuous. We claim $f(A) \cap f(B) = \emptyset$. If not, let $\underline{y} = (y_{(n,m)})_{(n,m)\in\omega\times\omega}$ be in $f(A) \cap f(B)$. Then there is $a \in A$, and there is $b \in B$ such that $f(a) = \underline{y} = f(b)$. As $a \in A$, and $A = \bigcup_{n\in\omega} A_n$, there is $n_0 \in \omega$ such that $a \in A_{n_0}$. Similarly, there is $m_0 \in \omega$ such that $b \in B_{m_0}$. Now $f(a)_{(n_0,m_0)} = f_{(n_0,m_0)}(a) \in f_{(n_0,m_0)}(A_{n_0})$ and $f(b)_{(n_0,m_0)} = f_{(n_0,m_0)}(b) \in f_{(n_0,m_0)}(B_{m_0})$, but $f_{(n_0,m_0)}(A_{n_0}) \cap f_{(n_0,m_0)}(B_{m_0}) = \emptyset$ and this is a contradiction. □

Theorem 4.1 follows from Proposition 4.6 and Balogh & Bennett's [10], '$(MA(\omega_1))$ Every locally compact, locally connected, R-perfect space is paracompact'. Similarly, Theorem 4.2 follows from Proposition 4.6 and Balogh & Bennett's [10], 'Every locally compact, locally connected, R-perfect, quasi–developable space is metrisable'.

Additionally, we observe that from Proposition 4.4 and Theorem 1.1 of [10] follows:

Theorem 4.7. *Every locally compact, locally connected, hereditarily weakly normal space with a countable point-separating open cover is metrisable.*

The existence of a normal manifold with a countable point separating open cover which is not metrisable is independent of set theory. Using $\diamondsuit$, Rudin [20] constructed a hereditarily separable Dowker manifold which has a countable point-separating open cover.

We finish with some remarks on spaces with S_δ or R_δ diagonal, and their relationship to quasi–developable spaces.

Lemma 4.8. *A space X has a quasi–G_δ–diagonal if and only if there is a countable sequence $\{U_n\}_{n\in\mathbf{N}}$ of open subsets in X^2, such that for all $(x,y) \notin \Delta$, there is $n \in \mathbf{N}$ such that $(x,x) \in U_n$ but $(x,y) \notin U_n$.*

Proof: Let $\{\mathcal{G}_n\}_{n\in\mathbf{N}}$ be a quasi–G_δ–diagonal sequence for X.

Define $U_n = \bigcup\{G \times G : G \in \mathcal{G}_n\}$. Then the U_ns are open in X^2. Further, if $(x,y) \in X^2$ such that $x \neq y$, then there is $n \in \mathbf{N}$, such that $x \in st(x,\mathcal{G}_n)$ and $y \notin st(x,\mathcal{G}_n)$. Then $(x,x) \in U_n$ but $(x,y) \notin U_n$.

Conversely suppose we have a sequence $\{U_n\}_{n\in\mathbf{N}}$ as in the statement of the Lemma. Define $\mathcal{G}_n = \{G : G$ is open and $G \times G \subseteq U_n\}$. Suppose distinct x and y are in X. Pick $n \in \mathbf{N}$ so that $(x,x) \in U_n$ but $(x,y) \notin U_n$. Then $x \in st(x,\mathcal{G}_n)$ while $y \notin st(x,\mathcal{G}_n)$. $\square$

Corollary 4.9. *Every space with a S_δ diagonal has a quasi–G_δ diagonal.*

Prompted by the above corollary, and by analogy with 'regular G_δ diagonal', we make the following definition. A space X has a quasi–regular G_δ diagonal if and only if there is a countable sequence $\{U_n\}_{n\in\mathbf{N}}$ of open subsets in X^2, such that for all $(x,y) \notin \Delta$, there is $n \in \mathbf{N}$ such that $(x,x) \in U_n$ but $(x,y) \notin \overline{U_n}$.

Lemma 4.10. *Every space with a R_δ diagonal has a quasi–regular G_δ diagonal.*

Lemma 4.11. *Every space with a quasi–regular G_δ diagonal has a quasi–G_δ^* diagonal.*

Proof: Suppose we have a sequence $\{U_n\}_{n\in\mathbf{N}}$ as in the definition of quasi–regular G_δ diagonal. Define $\mathcal{G}_n = \{G : G$ is open and $G \times G \subseteq U_n\}$. Suppose distinct x and y are in X. Pick $n \in \mathbf{N}$ so that $(x,x) \in U_n$ but $(x,y) \notin \overline{U_n}$. Then $x \in st(x, \mathcal{G}_n)$ while $y \notin \overline{st(x, \mathcal{G}_n)}$. □

Theorem 4.12. *Let X be locally compact and locally connected. If X has a quasi–G_δ^* diagonal, then X is quasi–developable.*

Proof: Let $\{\mathcal{G}_n\}_{n\in\mathbf{N}}$ be a quasi–G_δ^* diagonal sequence for X. Let $c_{\mathcal{G}}(x) = \{n : st(x, \mathcal{G}_n) \neq \emptyset\}$. ¿From the definition of quasi–G_δ^* diagonal, $\bigcap_{n\in c(x)} \overline{st(x, \mathcal{G}_n)} = \{x\}$. Define by induction a sequence of open families $\mathcal{H}_n$ as follows: let $\mathcal{H}_1 = \mathcal{G}_1$, and $\mathcal{H}_{n+1} = \{G \cap H_i$: for some $x \in X, x \in G \in \mathcal{G}_{n+1}$ and i maximal, $i \leq n$ such that $x \in \bigcup \mathcal{H}_i$ and $x \in H_i \in \mathcal{H}_i\}$.

It is easy to check that $\{\mathcal{H}_n\}_{n\in\mathbf{N}}$ satisfies the following:

1. for each $n \in \mathbf{N}$, $\mathcal{H}_n$ is open.
2. $c_{\mathcal{G}} = c_{\mathcal{H}}$.
3. if $x \in H \in \mathcal{H}_n$, then $H \subseteq G$ for some $G \in \mathcal{G}_n$.
4. $st(x, \mathcal{H}_m) \subseteq st(x, \mathcal{H}_n)$, if $m > n, m, n \in c(x)$. In other words, for $m > n, \mathcal{H}_m$ refines $\mathcal{H}_n$, where both are defined.

So, the sequence $\{\mathcal{H}_n\}_{n\in\mathbf{N}}$ is a quasi–G_δ^* diagonal for X, such that $st(x, \mathcal{H}_m) \subseteq st(x, \mathcal{H}_n)$, if $m > n, m, n \in c(x)$. Passing to component (using local connectedness), we may assume that each member of $\mathcal{H}_n$ is connected. We show that $\{\mathcal{H}_n\}_{n\in\mathbf{N}}$ is quasi–development for X. Suppose $x \in X$, U is a compact neighborhood of x, but $\overline{st(x, \mathcal{H}_n)}$ is not a subset of U, for every $n \in c(x)$. Then since $\overline{st(x, \mathcal{H}_n)}$ is connected, $\overline{st(x, \mathcal{H}_n)} \cap \partial U \neq \emptyset$, for each $n \in c(x)$. Since the boundary of U is compact, $\partial U \cap \bigcap_{n\in c(x)} \overline{st(x, \mathcal{H}_n)} \neq \emptyset$, a contradiction. □

The proof of our next results relies on a metrisation theorem of Collins & Roscoe [13].

Theorem 4.13. (Collins & Roscoe) *A space X is metrisable if and only if for each $x \in X$, there is a decreasing local base $\{U(n,x)\}_{n\in\mathbf{N}}$ at x such that if $x \in U$ open then there is an open set V, $x \in V$, and n such that if $y \in V$ then $x \in U(n,y) \subseteq U$.*

Theorem 4.14. *Let X be a space with a sequence $\{\mathcal{G}_n\}_{n\in\mathbf{N}}$ of open families such that , for each $x \in X, \{st^2(x, \mathcal{G}_n)\}_{n\in\mathbf{N}} - \{\emptyset\}$ is a local base at x. Then X is metrisable.*

Proof: By the same procedure as that used in the proof of Theorem 4.12, there is no lose of generality if we assume that:

1. $\mathcal{G}_m$ refines $\mathcal{G}_n$ if $m < n$ where both are defined.
2. $c(x) = \{n \in \mathbf{N} : x \in \bigcup \mathcal{G}_n\}$

Define $U(n, x) = st(x, \mathcal{G}_m)$, where m is minimal, $m \geq n, m \in c(x)$. It is easy to see that $U(n, x)$ is a decreasing local base at x, for all $x \in X$. By hypothesis, there is $n \in \mathbf{N}$, such that $x \in st^2(x, \mathcal{G}_n) \subseteq U$. Let $V = st(x, \mathcal{G}_n)$, take $y \in V$, then $x, y \in G$, for some $G \in \mathcal{G}_n$. So, $x \in st(y, \mathcal{G}_n) \subseteq st^2(x, \mathcal{G}_n) \subseteq U$. Then, $U(n, y) = st(x, \mathcal{G}_n), m$ is minimal, $m \geq n$, and $m \in c(y), (y \in \bigcup \mathcal{G}_n)$. From the definition, $m = n$ and so $st(y, \mathcal{G}_n) = st(x, \mathcal{G}_m)$. Hence, by Theorem 4.13, the claim is done. □

Theorem 4.15. *Let X be a locally compact, locally connected space. If X has a quasi–regular G_δ-diagonal, then X is metrisable.*

Proof: By Theorem 4.14, we only need to show that X has a quasi–development $\{\mathcal{G}_n\}_{n\in\mathbf{N}}$ such that, for each $x \in X$, $\{st^2(x, \mathcal{G}_n)\}_{n\in\mathbf{N}} - \{\emptyset\}$ is a local base at x.

We know already that X has a quasi–development $\{\mathcal{G}_n\}_{n\in\mathbf{N}}$ (Theorem 4.12). Let $\{U_n\}_{n\in\mathbf{N}}$ be as in the definition of quasi–regular G_δ diagonal. So, the U_ns are open in X^2 and for all $(x, y) \notin \Delta$, there is $n \in \mathbf{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$. We may suppose the following about $\{\mathcal{G}_n\}_{n\in\mathbf{N}}$:

1. $st(x, \mathcal{G}_m) \subseteq st(x, \mathcal{G}_n)$, if $m \geq n$ and $m, n \in c(x)$.
2. members of $\mathcal{G}_n$ are connected (for all n).
3. if $x \in G \in \mathcal{G}_m, m \geq n$ and $(x, x) \in U_n$, then $G \times G \subseteq U_n$ (for all $m, n \in c(x)$).

We prove the claim. Take any $x \in X$ and suppose $x \in V$ is open and $\overline{V}$ is compact. Suppose, for a contradiction, for all

$n \in c(x), \overline{st^2(x, \mathcal{G}_n)} \subset X - V$. Since $st^2(x, \mathcal{G}_n)$ is connected when $n \in c(x), \overline{st^2(x, \mathcal{G}_n)} \cap \partial V \neq \emptyset$ (for all $n \in c(x)$). By (1), the family $\{\overline{st^2(x, \mathcal{G}_n)}\}_{n \in \mathbf{N}}$ of closed subsets of a compact space ∂V has the finite intersection property (in fact are decreasing), hence there is $y \in \bigcap_{n \in \mathbf{N}} \overline{st^2(x, \mathcal{G}_n)} \cap \partial V$. Of course $x \neq y$. So by the definition of quasi–regular G_δ diagonal, there is n such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$. We know that, $st(x, \mathcal{G}_r) \subseteq st(x, \mathcal{G}_{m_x})$ if $r \geq m_x$ and $r \in c(x)$. Hence (because $\{\mathcal{G}_n\}_{n \in \mathbf{N}}$ is a quasi–development of X), there is $m_x, m_y, m_x \geq n$ (note that $c(x)$ is infinite), and

$$st(x, \mathcal{G}_{m_x}) \times st(y, \mathcal{G}_{m_y}) \cap U_n = \emptyset.$$

Pick, $G_3 \in \mathcal{G}_{m_y}$ such that $y \in G_3$. As $y \in \overline{st^2(x, \mathcal{G}_n)}$, there exists, $G_1, G_2 \in \mathcal{G}_{m_x}$ such that $x \in G_1, G_1 \cap G_2 \neq \emptyset$ and $G_2 \cap G_3 \neq \emptyset$. Let $z_1 \in G_1 \cap G_2$ and $z_2 \in G_2 \cap G_3$. Then $(z_1, z_2) \in (G_1 \times G_3) \cap (G_2 \times G_2)$. Now, $G_1 \in \mathcal{G}_{m_x}, G_3 \in \mathcal{G}_{m_y}$, so $G_1 \times G_3 \in st(x, \mathcal{G}_{m_x}) \times st(y, \mathcal{G}_{m_y})$. Also, $G_2 \in \mathcal{G}_{m_x}, m_x \geq n$, so $G_2 \times G_2 \subseteq U_n$. In other words, $(z_1, z_2) \in (st(x, \mathcal{G}_{m_x}) \times st(y, \mathcal{G}_{m_y})) \cap U_n$, and this is a contradiction. □

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