Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY
PROCEEDINGS
Volume 23, 1998

LOCAL PROPERTIES OF HYPERSPACES

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ABSTRACT. Let X be a space and 2^X ($C(X)$, $\mathcal{K}(X)$, $C_K(X)$) denote the hyperspace of nonempty closed subsets (connected closed subsets, compact subsets, subcontinua) of X with the Vietoris topology. We investigate the relationships between the space X and its hyperspaces concerning the properties of local compactness and local connectedness, viewed as pointwise properties in Section 2 and as global properties in Section 3. The following results are obtained: (1) If X is a Hausdorff space and $x \in X$, then each of 2^X, $\mathcal{K}(X)$, $C_K(X)$, and $C(X)$ is locally compact at $\{x\}$ if and only if X is locally compact at x; (2) If X is a Hausdorff space, then each of $\mathcal{K}(X)$ and $C_K(X)$ is locally compact if and only if X is locally compact; (3) If X is a locally compact Hausdorff space, then each of $C(X)$ and $C_K(X)$ is locally connected if and only if X is locally connected.

INTRODUCTION

Let X be a space and 2^X $(C(X), \mathcal{K}(X), C_K(X))$ denote the hyperspace of nonempty closed subsets (closed and connected subsets, compact subsets, subcontinua) of X, each with the Vietoris topology.

1991 *Mathematics Subject Classification.* Primary 54B20, 54B15.
Key words and phrases. hyperspace, continuum, local compactness, local connectedness, connected im kleinen.

A continuum is a compact connected Hausdorff space. One of the earliest results about local properties of hyperspaces of metric continua, due to Wojdyslawski [19], was that each of 2^X and $C(X)$ is locally connected if and only if X is locally connected. Michael [15, Theorem 4.12] proved that if X is a Hausdorff space, then $\mathcal{K}(X)$ is locally connected if and only if X is locally connected. Xie [20] proved that 2^X is locally compact if and only if X is compact. Goodykoontz [6, 7, 8] investigated local connectedness as a pointwise property in the hyperspaces 2^X and $C(X)$ of metric continua. Dorsett [2,3,4] extended many of Goodykoontz's results to more general spaces. Tsahmetov [18] investigated local connectivity in hyperspaces when X is a complete metric space. For additional information about hyperspaces, an excellent reference is the text by Nadler [17].

The purpose of this paper is to investigate the relationships between the space X and its hyperspaces 2^X, $C(X)$, $\mathcal{K}(X)$, and $C_K(X)$ concerning the properties of local compactness and local connectedness. In Section 1 we collect some known results and establish some new results which will be used in the proofs in Sections 2 and 3. Section 2 deals with connectedness im kleinen, local connectedness, and local compactness as pointwise properties. Section 3 deals with compactness and connectedness and with local compactness and local connectedness as global properties.

For notational purposes, small letters will denote elements of X, capital letters will denote subsets of X and elements of 2^X, and script letters will denote subsets of 2^X. If $A \subset X$, then $\overline{A}$ ($Int(A)$) will denote the closure (interior) of A in X. If $A \subset Y \subset X$, then $Cl(A, Y)$ denotes the closure of A in the relative topology on Y.

1. Preliminaries

Let X be a Hausdorff space. If $A_1, ..., A_n$ are subsets of X, then $< A_1, ..., A_n > = \{E \in 2^X :$ for each i=1,...,n, $E \cap A_i \neq \emptyset$ and $E \subset \cup_{i=1}^n A_i\}$. The collection of all sets of the form

$< U_1, ..., U_n >$, with $U_1, ..., U_n$ open in X, is a basis for the Vietoris topology T_v for 2^X. When we restrict T_v to each of $C(X) = \{E \in 2^X : E \text{ is connected}\}$, $\mathcal{K}(X) = \{E \in 2^X : E \text{ is compact}\}$, $C_K(X) = C(X) \cap \mathcal{K}(X)$, $\mathcal{F}(X) = \{E \in 2^X : E \text{ is finite}\}$, and $\mathcal{F}_n(X) = \{E \in 2^X : E$ has at most n elements$\}$, then these spaces are also called hyperspaces of X. We note that if X is Hausdorff, then $\mathcal{F}_n(X)$ is closed in 2^X, and X and $\mathcal{F}_1(X)$ are homeomorphic. If X is connected Hausdorff, then $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are connected.

Let X be a space and $x \in X$. Then X is locally compact at x provided that there exists a neighborhood U of x with compact closure. The space X is locally connected at $x \in X$ provided that for each neighborhood U of x there is a connected neighborhood V of x such that $V \subset U$. The space X is connected im kleinen at x provided for each neighborhood U of x there is a component of U which contains x in its interior. The space X is locally compact (locally connected) provided that X is locally compact (locally connected) at each of its points. If a space X is connected im kleinen at each of its points, then X is locally connected.

We will use the next theorem and the next lemma in several proofs.

Theorem 1.1. **[14, Theorem, p.1209]** *If X is a compact connected Hausdorff space, then 2^X and $C(X)$ are (arcwise) connected compact spaces.*

In proving Theorem 1.1, McWater observed that, for each $E \in C(X)$, the set $\mathcal{L}_E = \{F \in C(X) : E \subset F\}$ is closed and connected in $C(X)$. We will also use this fact in several proofs.

Lemma 1.2. **[15]** *(a)* $\overline{< U_1, ..., U_n >} = < \overline{U}_1, ..., \overline{U}_n >$.

(b) $< V_1, ..., V_m > \subset < U_1, ..., U_n >$ *if and only if* $\cup_{j=1}^m V_j \subset \cup_{i=1}^n U_n$ *and for each* U_i *there is a* V_j *such that* $V_j \subset U_i$.

(c) *Let X be a space. Then for each $\mathcal{B} \in \mathcal{K}(\mathcal{K}(X))$, $\cup\mathcal{B} = \cup\{E : E \in \mathcal{B}\} \in \mathcal{K}(X)$.*

(d) *If $\mathcal{B}$ is a connected subset of 2^X which also contains at least one connected element, then $\cup\mathcal{B}$ is connected in X.*

(e) *The space X is compact if and only if 2^X is compact.*

Lemma 1.3. *(a) Suppose X is a locally connected regular space. If $\mathcal{U}$ is an open set in the subspace $C(X)$, then $\cup\mathcal{U}$ is open in X.*

(b) **[2, Theorem 2.2]** *If $\mathcal{U}$ is an open set in the subspace $\mathcal{K}(X)$, then $\cup\mathcal{U}$ is open in X.*

(c) **[2, Theorem 2.2]** *If $\mathcal{O}$ is an open set in 2^X, then $\cup\mathcal{O}$ is open in X.*

Proof: (a) Let $x \in U = \cup\mathcal{U}$. Without loss of generality let $\mathcal{U} = < U_1, ..., U_n > \cap C(X)$. Then there is an element $E \in \mathcal{U}$ such that $x \in E$. Suppose $x \in U_i$ for some i. Since X is regular and locally connected at x, there is a connected neighborhood V of x in X such that $V \subset \overline{V} \subset U_i$. Then $E \cup \overline{V}$ is closed and connected and $E \cup \overline{V} \in < U_1, ..., U_n > \cap C(X)$. Hence $V \subset E \cup \overline{V} \subset U$. Thus U is open in X.

Remarks. If any one of the spaces 2^X, $\mathcal{K}(X)$, $C(X)$, or $C_K(X)$ is Hausdorff, then X is Hausdorff, since X is homeomorphic to $\mathcal{F}_1(X)$. On the other hand, if X is a Hausdorff space, then $\mathcal{K}(X)$ is Hausdorff by [15, 4.9.8], so $C_K(X)$ is Hausdorff. If X is a locally compact Hausdorff space, then X is regular, so 2^X is Hausdorff by [15, 4.9.3]. Thus $C(X)$ is Hausdorff.

Proposition 1.4. *Let X be a normal space. Then $C(X)$ is closed in 2^X.*

Proof: Suppose $E \in 2^X$ is a limit point of $C(X)$ such that $E \in 2^X \backslash C(X)$. Let $E \in < U_1, ..., U_n >$, and $U = \cup_{i=1}^n U_i$. Since E is disconnected and closed, E is the union of two nonempty disjoint closed subsets E_1 and E_2. Since X is normal, there exist two disjoint nonempty open sets W_1 and W_2 containing E_1 and E_2 respectively such that $W_1 \cup W_2 \subset U$. Let $\{U^1_{i_1}, ..., U^1_{i_k}\}$ be the collection of all $U_i \in \{U_1, ..., U_n\}$ such that $U_i \cap E_1 \neq \emptyset$, and $\{U^2_{i_1}, ..., U^2_{i_p}\}$ be the collection of all $U_i \in \{U_1, ..., U_n\}$ such that $U_i \cap E_2 \neq \emptyset$. Now let $V^1_j = W_1 \cap U^1_{i_j}$ for $j = 1, ..., k$, and $V_l^2 = W_2 \cap U_{i_l}^2$ for $l = 1, ..., p$. Then $E_1 \subset \cup_{j=1}^{k} V_j^1 = V^1$ and $E_2 \subset \cup_{l=1}^{p} V_l^2 = V^2$ and $V^1 \cap V^2 = \emptyset$. It is easy to see that $E \in < V_1^1, ...V_k^1, V_1^2, ...V_p^2 > \subset < U_1, ..., U_n >$. Since E is a limit point of $C(X)$, there exists an element $C \in C(X)$ such that $C \in < V_1^1, ..., V_k^1, V_1^2, ..., V_p^2 >$. This would mean that $C \subset V^1 \cup V^2$ and $C \cap V^i \neq \emptyset$ for each $i = 1, 2$ which contradicts the connectedness of C. So $C(X)$ is closed in 2^X.

Proposition 1.5. *Let X be a locally compact Hausdorff space. Then $C_K(X)$ is open in $C(X)$.*

Proof: Let $E \in C_K(X)$. Let $U_1, ..., U_n$ be open sets in X with each having compact closure and $E \in < U_1, ..., U_n > \cap C(X)$. Let $A \in < U_1, .., U_n > \cap C(X)$. Then A is contained in the compact set $\cup_{i=1}^{n} \overline{U}_i$. Thus A is compact. Hence $A \in C_K(X)$. This proves that $< U_1, ..., U_n > \cap C(X) \subset C_K(X)$. It follows that $C_K(X)$ is open in $C(X)$.

Proposition 1.6. *Let X be a locally compact Hausdorff space. If X is locally connected, then $C_K(X)$ is dense in $C(X)$.*

Proof: Let $E \in C(X) \backslash C_K(X)$ and let $< U_1, ..., U_n > \cap C(X)$ be a basic open set in $C(X)$ containing E. Let $U = \cup_{i=1}^{n} U_i$. For each $x \in E \cap U$, let V_x be a connected neighborhood of x such that $\overline{V}_x$ is compact and is contained in U. The collection $\mathcal{V}$ of all such V_x covers E. Pick a point $a_i \in E \cap U_i$ for each $i = 1, .., n$. Let $\mathcal{A}_i = \{V_{x_{i_1}}, ..., V_{x_{i_{k_i}}}\}$ be a simple chain in $\mathcal{V}$ from a_1 to a_i [9, Theorem 3-4] $i = 2, ..., n$. Let $K_i = \cup_{j=1}^{k_i} \overline{V}_{x_{i_j}}$ for each $i = 2, .., n$. Since each K_i is a continuum containing a_1, $M = \cup_{i=2}^{n} K_i$ is a continuum. Since $M \cap U_i \neq \emptyset$ for each i and $M \subset U$, $M \in < U_1, ..., U_n > \cap C(X)$. This proves that $C_K(X)$ is dense in $C(X)$.

Lemma 1.7. *Let X be a locally compact Hausdorff space. (a) If X is connected and locally connected, then each compact subset of X is contained in the interior of some subcontinuum. (b) Let $E \in C_K(X)$. If $< U_1, ..., U_n >$ is a basic open set in 2^X containing E, then there is a compact set M such that*

$E \subset Int(M) \subset M \subset \cup_{i=1}^n U_i$. Furthermore, there is an open set $< W_1, ..., W_m >$, with each W_i having compact closure, such that $E \in < W_1, ..., W_m > \subset \overline{< W_1, ..., W_m >} \subset < U_1, ..., U_n >$.

Proof: (a) Since X is locally connected and locally compact, for each point $x \in X$, let U_x be a connected neighborhood of x with compact closure. Then the collection $\mathcal{U}$ of all such U_x, $x \in X$, is an open covering of X. Now let A be a compact subset of X. Let $a \in A$ be fixed. For each $b \in A$, let $\mathcal{U}_b$ be a simple chain of elements of $\mathcal{U}$ from a to b. We denote the last element of $\mathcal{U}_b$ by U_b. Then the collection $\mathcal{V}$ of all such U_b covers the compact set A. Let $U_{b_1}, ..., U_{b_n}$ be a finite subcollection of $\mathcal{V}$ which covers A. For each $i = 1, ..., n$, let $\mathcal{U}_{b_i}$ be a simple chain from a to b_i whose last element is U_{b_i}. Let $K_i = \cup\{\overline{U} : U \in \mathcal{U}_{b_i}\}$ for each $i = 1, 2, ..., n$. Since each $\mathcal{U}_{b_i}$ contains only a finite number of elements of $\mathcal{U}$, each K_i is a continuum and $a \in K_i$. So $\cup_{i=1}^n K_i$ is a compact and connected set containing A in its interior.

(b) For each $i = 1, ..., n$ and each $x \in E \cap U_j$, let V_x be a neighborhood of x with compact closure such that $\overline{V}_x \subset U_j$. Let $\mathcal{V}$ be the collection of all such V_x. Then $\mathcal{V}$ covers the compact set E. Let $\{V_{x_1}, ..., V_{x_k}\}$ be a finite subcollection of $\mathcal{V}$ which covers E. For each $i = 1, ..., n$, let $y_i \in E \cap U_i$. Let $M = (\cup_{j=1}^k \overline{V}_{x_j}) \cup (\cup_{i=1}^n \overline{V}_{y_i})$. Then M is compact and $E \subset Int(M) \subset \cup_{i=1}^n U_i$. Furthermore, $E \in < V_{x_1}, ..., V_{x_k}, V_{y_1}, ..., V_{y_n} > \subset \overline{< V_{x_1}, ..., V_{x_k}, V_{y_1}, ..., V_{y_n} >} = < \overline{V}_{x_1}, ..., \overline{V}_{x_k}, \overline{V}_{y_1}, ..., \overline{V}_{y_n} > \subset < U_1, ..., U_n >$.

2. Local Connectedness and Local Compactness as Pointwise Properties

Proposition 2.1. *Let X be a Hausdorff space. Let $x \in X$. Then X is connected im kleinen at x if and only if $C_K(X)$ is connected im kleinen at $\{x\}$.*

Proof: Suppose that X is connected im kleinen at x. Let $< U > \cap C_K(X)$ be a basic open set in $C_K(X)$ containing

$\{x\}$. Then $x \in U$ and there exists a component M of U which contains x in its interior. Let $W = Int(M)$. Then $\{x\} \in < W > \cap C_K(X) \subset < U > \cap C_K(X)$. If $E \in < W > \cap C_K(X)$, then $C(E)$ is a connected subset of $< U > \cap C_K(X)$ and $C(E) \cap \mathcal{F}_1(M) \neq \emptyset$. Since $\mathcal{F}_1(M)$ is connected, it follows that there is a connected subset of $< U > \cap C_K(X)$ which contains $\{x\}$ in its interior. Thus $C_K(X)$ is connected im kleinen at $\{x\}$.

Conversely, suppose that $C_K(X)$ is connected im kleinen at $\{x\}$. Let U be an open set in X such that $x \in U$. Then $\{x\} \in < U > \cap C_K(X)$. Since $C_K(X)$ is connected im kleinen at $\{x\}$, there exists an open set $< V > \cap C_K(X)$, $\{x\} \in < V > \cap C_K(X) \subset < U > \cap C_K(X)$, with the property that if $E \in < V > \cap C_K(X)$, then $< U > \cap C_K(X)$ contains a connected set containing E and $\{x\}$. Now $x \in V \subset U$. Let $y \in V$. Then $\{y\} \in < V > \cap C_K(X)$, so $< U > \cap C_K(X)$ contains a connected set $\mathcal{L}$ containing $\{x\}$ and $\{y\}$. Then $\cup\mathcal{L}$ is a connected subset of U containing x and y. It follows that X is connected im kleinen at x.

Corollary 2.1.1. *Let X be a Hausdorff space. Let $x \in X$. Then the following statements are equivalent:*

(1) *X is connected im kleinen at x*
(2) *2^X is connected im kleinen at $\{x\}$*
(3) *$\mathcal{K}(X)$ is connected im kleinen at $\{x\}$*
(4) *$C_K(X)$ is connected im kleinen at $\{x\}$.*

Proof: The equivalence of (1), (2), and (3) follows from [2, Theorem 2.10] and the equivalence of (1) and (4) is Proposition 2.1.

Remark. The equivalence of (1), (2), and (3) above when "connected im kleinen" is replaced by "locally connected" is given in [2, Theorem 2.10].

Proposition 2.2. (see [6, Corollary 4]) *Let X be a locally compact Hausdorff space. Let $x \in X$. Then X is connected* *im kleinen at x if and only if $C(X)$ is connected im kleinen at $\{x\}$.*

Proof: Suppose that X is connected im kleinen at x. Let $< U > \cap C(X)$ be a basic open set in $C(X)$ containing $\{x\}$. Let V be a neighborhood of x with compact closure such that $\overline{V} \subset U$. Then there exists a component M of V which contains x in its interior. Let $W = Int(M)$. Then $\{x\} \in < W > \cap C(X) \subset < V > \cap C(X) \subset < U > \cap C(X)$.If $E \in < W > \cap C(X)$, then $E \in C_K(X)$. Then $C(E)$ is a connected subset of $< V > \cap C(X)$ by Theorem 1.1. Since $C(E) \cap \mathcal{F}_1(M) \neq \emptyset$, $C(E) \cup \mathcal{F}_1(M)$ is connected and is contained in $< V > \cap C(X)$. It follows that there is a connected subset of $< V > \cap C(X)$ which contains $\{x\}$ in its interior.

The proof of the converse is the same as the corresponding proof in [6, Corollary 4].

Corollary 2.2.1 *Let X be a locally compact Hausdorff space. Let $x \in X$. Then the following statements are equivalent:*

(1) *X is connected im kleinen at x*
(2) *2^X is connected im kleinen at $\{x\}$*
(3) *$\mathcal{K}(X)$ is connected im kleinen at $\{x\}$*
(4) *$C_K(X)$ is connected im kleinen at $\{x\}$*
(5) *$C(X)$ is connected im kleinen at $\{x\}$.*

Proof: Use Corollary 2.1.1 and Proposition 2.2.

Proposition 2.3. *Let X be a Hausdorff space. Let $x \in X$. If X is locally connected at x, then $C_K(X)$ is locally connected at $\{x\}$.*

Proof: Suppose that X is locally connected at x. Let $< U > \cap C_K(X)$ be a basic open set in $C_K(X)$ containing $\{x\}$. Then $x \in U$ and there is a connected open set V such that $x \in V \subset U$. So $\{x\} \in < V > \cap C_K(X) \subset < U > \cap C_K(X)$. Let $E \in < V > \cap C_K(X)$. Then $C(E)$ is a connected subset of $< V > \cap C_K(X)$ and $C(E) \cap \mathcal{F}_1(V) \neq \emptyset$. Since $\mathcal{F}_1(V)$ is connected, it follows that $< V > \cap C_K(X)$ is connected. Hence $C_K(X)$ is locally connected at $\{x\}$.

Proposition 2.4. *Let X be a locally compact Hausdorff space. Let $x \in X$. If X is locally connected at x, then $C(X)$ is locally connected at $\{x\}$.*

Proof: Suppose that X is locally connected at x. Let $< U > \cap C(X)$ be a basic open set in $C(X)$ containing $\{x\}$. Let V be a connected neighborhood of x with compact closure such that $\overline{V} \subset U$. Then $< V > \cap C(X) = < V > \cap C_K(X)$. Let $E \in < V > \cap C(X)$. Then $C(E)$ is connected by Theorem 1.1 and is contained in $< V > \cap C(X)$. Since $C(E) \cap \mathcal{F}_1(V) \neq \emptyset$, $C(E) \cup \mathcal{F}_1(V)$ is a connected subset of $< V > \cap C(X)$. It follows that $< V > \cap C(X)$ is connected.

Remark. The converse of Proposition 2.3 and the converse of Proposition 2.4 are false by [6, Example 1].

Proposition 2.5. (see [6, Theorem 3]) *Let X be a locally compact Hausdorff space. Let $E \in C_K(X)$. If 2^X is connected im kleinen at E, then $C(X)$ is connected im kleinen at E.*

Proof: Let $< U_1, ..., U_n > \cap C(X)$ be a basic open set containing E. By Lemma 1.7(b), there exists a basic open set $< V_1, ..., V_m >$, with each V_i having a compact closure, such that $E \in < V_1, ..., V_m > \subset \overline{< V_1, ..., V_m >} \subset < U_1, ..., U_n >$. Let $V = \cup_{i=1}^m V_i$. By [6, Theorem 1], there is a component M of V which contains E in its interior. For each $i = 1, ..., m$, let $W_i = V_i \cap Int(M)$. Then $E \in < W_1, ..., W_m > \subset < V_1, ..., V_m >$. If $A \in < W_1, ..., W_m > \cap C(X)$, then $A \subset \overline{M}$, and $A, \overline{M} \in < \overline{V}_1, ..., \overline{V}_m > \cap C(X) = \overline{< V_1, ..., V_m >} \cap C(X)$ $\subset < U_1, ..., U_n > \cap C(X)$. Since $\overline{M}$ is compact and connected, $\mathcal{L}_A = \{B \in C(\overline{M}) : A \subset B\}$ is connected. For each $B \in \mathcal{L}_A$, $A \subset B \subset \overline{M}$, so $\mathcal{L}_A \subset < U_1, ..., U_n > \cap C(X)$. It follows that there is a component of $< U_1, ..., U_n >$ which contains E in its interior. Hence $C(X)$ is connected im kleinen at E.

Corollary 2.5.1. Let X be a locally compact Hausdorff space. Let $E \in C_K(X)$. If 2^X is connected im kleinen at E, then each of $\mathcal{K}(X)$, $C(X)$, and $C_K(X)$ is connected im kleinen at E.

Proof: Use [2, Theorem 2.1], Proposition 2.5, and Proposition 1.5.

Proposition 2.6. *Let X be a Hausdorff space. Let $x \in X$. Then the following statements are equivalent:*

(1) *X is locally compact at x*
(2) *2^X is locally compact at $\{x\}$*
(3) *$\mathcal{K}(X)$ is locally compact at $\{x\}$*
(4) *$C_K(X)$ is locally compact at $\{x\}$*
(5) *$C(X)$ is locally compact at $\{x\}$.*

Proof: (1) $\Rightarrow$ (2). Suppose that X is locally compact at x. Let $< U >$ be an open set such that $\{x\} \in < U >$. Then $x \in U$. Since X is locally compact at x, there exists an open set V with compact closure such that $x \in V \subset \overline{V} \subset U$. Then $\{x\} \in < V > \subset < \overline{V} > \subset \overline{< V >} \subset < U >$. Since $< \overline{V} > = 2^{\overline{V}}$ and $\overline{V}$ is compact, $2^{\overline{V}}$ is compact. Thus $< \overline{V} >$ is compact. It follows that 2^X is locally compact at $\{x\}$.

(2) $\Rightarrow$ (3). Suppose that 2^X is locally compact at $\{x\}$. Let $< U > \cap \mathcal{K}(X)$ be a basic open set such that $\{x\} \in < U > \cap \mathcal{K}(X)$. Then $\{x\} \in < U >$. Since 2^X is locally compact at $\{x\}$, there exists an open set $\mathcal{U}$ with compact closure such that $\{x\} \in \mathcal{U} \subset \overline{\mathcal{U}} \subset < U >$. Let V be an open set such that $\{x\} \in < V > \subset \mathcal{U}$. Then $\{x\} \in < V > \subset < \overline{V} > \subset \overline{\mathcal{U}} \subset < U >$ and $< \overline{V} >$ is compact. Thus $< \overline{V} > = 2^{\overline{V}} = \mathcal{K}(\overline{V})$. So $< \overline{V} > \cap \mathcal{K}(X) = < \overline{V} >$. Hence $\{x\} \in < V > \cap \mathcal{K}(X) \subset < \overline{V} > \cap \mathcal{K}(X) = < \overline{V} > \subset < U > \cap \mathcal{K}(X)$. It follows that $\mathcal{K}(X)$ is locally compact at $\{x\}$.

(3) $\Rightarrow$ (1). Suppose that $\mathcal{K}(X)$ is locally compact at $\{x\}$. Let U be an open set such that $x \in U$. Then $\{x\} \in < U > \cap \mathcal{K}(X)$. Since $\mathcal{K}(X)$ is locally compact at $\{x\}$, there exists an open set $\mathcal{U}$ in $\mathcal{K}(X)$ such that $\overline{\mathcal{U}} \cap \mathcal{K}(X)$ is compact and $\{x\} \in \mathcal{U} \subset \overline{\mathcal{U}} \cap \mathcal{K}(X) \subset < U > \cap \mathcal{K}(X)$. Let V be an open set such that $\{x\} \in < V > \cap \mathcal{K}(X) \subset \mathcal{U} \subset \overline{\mathcal{U}} \cap \mathcal{K}(X) \subset < U > \cap \mathcal{K}(X)$. Then $x \in V \subset \cup\{E : E \in \overline{\mathcal{U}} \cap \mathcal{K}(X)\} \subset U$. By Lemma 1.2(c), $\cup\{E : E \in \overline{\mathcal{U}} \cap \mathcal{K}(X)\}$ is compact. It follows that $\overline{V}$ is

compact and $x \in V \subset \overline{V} \subset U$. Hence X is locally compact at x.

(1) $\Rightarrow$ (4). Suppose that X is locally compact at x. Let $< U > \cap C_K(X)$ be a basic open set such that $\{x\} \in < U > \cap C_K(X)$. Then $x \in U$. Since X is locally compact at x, there exists an open set V with compact closure such that $x \in V \subset \overline{V} \subset U$. Then $\{x\} \in < V > \cap C_K(X) \subset < \overline{V} > \cap C_K(X) \subset < U > \cap C_K(X)$. Since $\overline{V}$ is compact, $C(\overline{V})$ is compact by Lemma 1.2(e) and Proposition 1.4. Since $< V > \cap C_K(X) \subset C(\overline{V})$ and $C(\overline{V})$ is closed, $\overline{< V > \cap C_K(X)} \subset C(\overline{V})$. Thus $Cl(< V > \cap C_K(X), C_K(X)) = \overline{< V > \cap C_K(X)} \cap C_K(X) = \overline{< V > \cap C_K(X)}$, since $\overline{< V > \cap C_K(X)} \subset C(\overline{V}) \subset C_K(X)$. Hence $Cl(< V > \cap C_K(X), C_K(X))$ is compact. Thus $C_K(X)$ is locally compact at $\{x\}$.

(4) $\Rightarrow$ (1). Suppose that $C_K(X)$ is locally compact at $\{x\}$. Let U be an open set such that $x \in U$. Then $\{x\} \in < U > \cap C_K(X)$. Since $C_K(X)$ is locally compact at $\{x\}$, there exists an open set $\mathcal{U}$ in $C_K(X)$ such that $\overline{\mathcal{U}} \cap C_K(X)$ is compact and $\{x\} \in \mathcal{U} \subset \overline{\mathcal{U}} \cap C_K(X) \subset < U > \cap C_K(X)$. Let V be an open set such that $\{x\} \in < V > \cap C_K(X) \subset \mathcal{U} \subset \overline{\mathcal{U}} \cap C_K(X) \subset < U > \cap C_K(X)$. Then $x \in V \subset \cup\{E : E \in \overline{\mathcal{U}} \cap C_K(X)\} \subset U$. By Lemma 1.2(c), $\cup\{E : E \in \overline{\mathcal{U}} \cap C_K(X)\}$ is compact. It follows that $\overline{V}$ is compact and $x \in V \subset \overline{V} \subset U$. Hence X is locally compact at x.

(1) $\Rightarrow$ (5). Suppose that X is locally compact at x. Let $< U > \cap C(X)$ be a basic open set such that $\{x\} \in < U > \cap C(X)$. Then $x \in U$. Since X is locally compact at x, there exists an open set V with compact closure such that $x \in V \subset \overline{V} \subset U$. Then $\{x\} \in < V > \cap C(X) \subset < \overline{V} > \cap C(X) \subset < U > \cap C(X)$. Since $\overline{V}$ is compact, $C(\overline{V})$ is compact by Lemma 1.2(e) and Proposition 1.4. Since $< V > \cap C(X) \subset C(\overline{V})$ and $C(\overline{V})$ is closed, $\overline{< V > \cap C(X)} \subset C(\overline{V})$. Thus $Cl(< V > \cap C(X), C(X)) = \overline{< V > \cap C(X)} \cap C(X) = \overline{< V > \cap C(X)}$, since $\overline{< V > \cap C(X)} \subset C(\overline{V}) \subset C(X)$. Hence $Cl(< V > \cap$

$C(X), C(X))$ is compact. Thus $C(X)$ is locally compact at $\{x\}$.

(5) $\Rightarrow$ (1). Suppose that $C(X)$ is locally compact at $\{x\}$. Let $\mathcal{U}$ be a neighborhood of $\{x\}$ in $C(X)$ such that $Cl(\mathcal{U}, C(X)) = \overline{\mathcal{U}} \cap C(X)$ is compact. Let V be a neighborhood of x such that $< V > \cap\, C(X) \subset \mathcal{U}$. Since $\mathcal{F}_1(X)$ is closed in 2^X and $\mathcal{F}_1(X) \subset C(X)$, $Cl(\mathcal{F}_1(V), C(X)) = \overline{\mathcal{F}_1(V)} \cap C(X) = \overline{\mathcal{F}_1(V)}$. Since $\mathcal{F}_1(V) \subset\, < V > \cap\, C(X)$, $Cl(\mathcal{F}_1(V), C(X)) \subset Cl(\mathcal{U}, C(X)) = \overline{\mathcal{U}} \cap C(X)$. It follows that $\overline{\mathcal{F}_1(V)}$ is compact. It is easy to see that $\overline{\mathcal{F}_1(V)} = \mathcal{F}_1(\overline{V})$. Since $\mathcal{F}_1(\overline{V})$ is homeomorphic to $\overline{V}$, $\overline{V}$ is compact. Hence X is locally compact at x.

3. Local Connectedness and Local Compactness as Global Properties

Proposition 3.1. *Let X be a Hausdorff space. Then X is compact if and only if $C(X)$ is compact.*

Proof. Suppose that X is compact. Then $C(X)$ is closed in 2^X by Proposition 1.4. Since 2^X is compact by Lemma 1.2(e), $C(X)$ is compact.

Conversely, suppose that $C(X)$ is compact. Then the closed subspace $\mathcal{F}_1(X)$ of $C(X)$ is compact. Since X and $\mathcal{F}_1(X)$ are homeomorphic, X is compact.

Corollary 3.1.1. *Let X be a Hausdorff space. Then the following statements are equivalent:*

(1) *X is compact*
(2) *2^X is compact*
(3) *$C(X)$ is compact.*

Proof: Use Lemma 1.2(e) and Proposition 3.1.

Remark. When X is compact, $2^X = \mathcal{K}(X)$ and $C(X) = C_K(X)$.

Proposition 3.2. *Let X be a Hausdorff space. Then X is connected if and only if $C_K(X)$ is connected.*

Proof: Suppose that X is connected. Then $\mathcal{F}_1(X)$ is connected and is contained in $C_K(X)$. For each $A \in C_K(X)$, $C(A) = C_K(A)$ is compact by Proposition 3.1 and is connected by Theorem 1.1, and $C(A) \subset C_K(X)$ and $C(A) \cap \mathcal{F}_1(X) \neq \emptyset$. Hence $\cup\{C(A) : A \in C_K(X)\} = C_K(X)$ is connected.

Conversely, suppose that $C_K(X)$ is connected. Then, the connectedness of $C_K(X)$ implies that $\cup C_K(X) = X$ is connected by Lemma 1.2(d).

Corollary 3.2.1. *Let X be a Hausdorff space. Then the following statements are equivalent:*

(1) *X is connected*
(2) *$C_K(X)$ is connected*
(3) *$\mathcal{K}(X)$ is connected*
(4) *2^X is connected.*

Proof: Use Proposition 3.2 and [15, Theorem 4.10].

Corollary 3.2.2. **[12, Theorem 2]** *If X is a locally compact, connected, locally connected Hausdorff space, then $C(X)$ is connected.*

Proof: Use Proposition 1.6 and Proposition 3.2.

Remarks. If X is connected, then $C(X)$ need not be connected (see [12, Example C]). However, if $C(X)$ is connected, then X is connected by Lemma 1.2(d).

Proposition 3.3. *Let X be a Hausdorff space. Then the following statements are equivalent:*

(1) *X is locally compact*
(2) *$\mathcal{K}(X)$ is locally compact*
(3) *$C_K(X)$ is locally compact.*

Proof: (1) $\Leftrightarrow$ (2). Suppose that X is locally compact. Let $E \in \mathcal{K}(X)$. Since E is a compact subset of a locally compact space X, there exists an open set U containing E such that $\overline{U}$ is compact. Then $E \in \ < U > \cap \mathcal{K}(X)$. Since each element of $< U >$ is compact in X, $< U > \subset \mathcal{K}(X)$. Also $\overline{< U >} = < \overline{U} >$ and $2^{\overline{U}} = < \overline{U} >$. Hence $< \overline{U} >$ is compact by Lemma 1.2(e). Therefore $\mathcal{K}(X)$ is locally compact at E. Hence $\mathcal{K}(X)$ is locally compact.

Conversely, suppose that $\mathcal{K}(X)$ is locally compact. Then, for each $x \in X$, $\mathcal{K}(X)$ is locally compact at $\{x\}$. Thus, by Proposition 2.6, for each $x \in X$, X is locally compact at x. Hence X is locally compact.

(1) $\Leftrightarrow$ (3). Suppose that X is locally compact. Let $E \in C_K(X)$. Let $< U_1, ..., U_n > \cap C_K(X)$ be a basic open set containing E. By Lemma 1.7(b), there is an open set $< W_1, ..., W_m >$, where each W_i has compact closure, such that $E \in < W_1, ..., W_m > \subset \overline{< W_1, ..., W_m >} = < \overline{W_1}, ..., \overline{W_m} > \subset < U_1, ..., U_n >$. Then $E \in < W_1, ..., W_m > \cap C_K(X) \subset \overline{< W_1, ..., W_m >} \cap C_K(X) \subset < U_1, ..., U_n > \cap C_K(X)$. Let $M = \cup_{i=1}^{m} \overline{W_i}$. Then M is compact and, by Proposition 3.1, $C(M)$ is compact. Since $< W_1, ..., W_m > \cap C_K(X) \subset C(M)$ and $C(M)$ is closed, $\overline{< W_1, ..., W_m > \cap C_K(X)} \subset C(M)$. Thus $Cl(< W_1, ..., W_m > \cap C_K(X), C_K(X)) = \overline{< W_1, ..., W_m > \cap C_K(X)} \cap C_K(X) = \overline{< W_1, ..., W_m > \cap C_K(X)}$, since $\overline{< W_1, ..., W_m > \cap C_K(X)} \subset C(M) \subset C_K(X)$. Hence $Cl(< W_1, ..., W_m > \cap C_K(X), C_K(X))$ is compact. Thus $C_K(X)$ is locally compact at E. Hence $C_K(X)$ is locally compact.

Conversely, suppose that $C_K(X)$ is locally compact. Then, for each $x \in X$, $C_K(X)$ is locally compact at $\{x\}$. Thus, by Proposition 2.6, for each $x \in X$, X is locally compact at x. Hence X is locally compact.

Remarks. The equivalence of (1) and (2) appears in [15, 4.9.12], but the proof depends on an incorrect result [15, 4.4.1]), as noted in [20]. In [20] Xie proved that the statements 2^X is locally compact, X is compact, and 2^X is compact are all equivalent.

Proposition 3.4. *Let X be a locally compact Hausdorff space. Then the following statements are equivalent:*

(1) *X is locally connected*
(2) *$C(X)$ is locally connected*
(3) *$C_K(X)$ is locally connected.*

Proof: (1) $\Rightarrow$ (2). Suppose X is locally connected. Let $E \in C(X)$. Let $< U_1, ..., U_n > \cap C(X)$ be a basic open set containing E. For each $i = 1, ..., n$ and each $x \in E \cap U_i$, let V_x be a connected neighborhood of x such that $\overline{V}_x \subset U_i$ and $\overline{V}_x$ is compact. Let $\mathcal{V}$ be the collection of all such V_x. Then $\mathcal{V}$ covers E and $V = \cup\mathcal{V} \subset \cup_{i=1}^{n} U_i$ and V is connected and open. Now pick one $V_{x_i} \in \mathcal{V}$ such that $V_{x_i} \subset U_i$ for each $i = 1, ..., n$. Then $E \in < V_{x_1}, ..., V_{x_n}, V > \subset < U_1, ..., U_n >$ by Lemma 1.2(b) and $< V_{x_1}, ..., V_{x_n}, V > \cap C(X) \subset < U_1, ..., U_n > \cap C(X)$. Since $C_K(X)$ is open and dense in $C(X)$ by Proposition 1.5 and Proposition 1.6, $< V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$ is dense in $< V_{x_1}, ..., V_{x_n}, V > \cap C(X)$. So for proving the connectedness of $< V_{x_1}, ..., V_{x_n}, V > \cap C(X)$, it is sufficient to show that $< V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$ is connected.

Let $F, E_0 \in < V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$. Since $E_0 \cup F$ is a compact subset of the connected, locally compact, locally connected open set V, by Lemma 1.7(a) there exists a continuum M containing $E_0 \cup F$ such that $M \subset V$. Since $E_0 \subset M$ implies $M \cap V_{x_i} \neq \emptyset$ for each i and $M \subset V$, $M \in < V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$. Let $\mathcal{L}_F = \{G \in C(M) : F \subset G\}$ and $\mathcal{L}_{E_0} = \{G \in C(M) : E_0 \subset G\}$. Then each of $\mathcal{L}_{E_0}$ and $\mathcal{L}_F$ is connected and $E_0, M \in \mathcal{L}_{E_0}$ and $F, M \in \mathcal{L}_F$. Hence $\mathcal{L}_{E_0} \cup \mathcal{L}_F$ is connected. Let $G \in \mathcal{L}_F$. Then $F \subset G \subset M$ implies that $G \in < V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$. Thus $\mathcal{L}_F \subset < V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$. Similarly $\mathcal{L}_{E_0} \subset < V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$. This proves that $< V_{x_1}, ..., V_{x_n}, V > \cap C_K(X)$ is connected. Hence $C(X)$ is locally connected at E. It follows that $C(X)$ is locally connected.

(2) $\Rightarrow$ (3). Suppose that $C(X)$ is locally connected. By Proposition 1.5, $C_K(X)$ is open in $C(X)$. Since an open subspace of a locally connected space is locally connected, $C_K(X)$ is locally connected.

(3) $\Rightarrow$ (1). Suppose that $C_K(X)$ is locally connected. Then, for each $x \in X$, $C_K(X)$ is connected im kleinen at $\{x\}$. Thus, by Corollary 2.2.1, for each $x \in X$, X is connected im kleinen at x. Hence X is locally connected.

Corollary 3.4.1. *Let X be a locally compact Hausdorff space. Then the following statements are equivalent:*

(1) *X is locally connected*
(2) *$C(X)$ is locally connected*
(3) *$C_K(X)$ is locally connected*
(4) *$\mathcal{K}(X)$ is locally connected*
(5) *$C(X)$ is locally connected at each $E \in C_K(X)$*
(6) *$\mathcal{K}(X)$ is locally connected at each $E \in C_K(X)$.*

Proof: By Proposition 3.4, (1), (2), and (3) are equivalent. By [15, Theorem 4.12], (1) $\Leftrightarrow$ (4). Clearly, (2) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6). The proofs that (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1) are analogous to the proof in Proposition 3.4 that (3) $\Rightarrow$ (1).

Remarks. If X is locally connected, then 2^X need not be locally connected (see [15, p.166]). However, if 2^X is locally connected, then X is locally connected by Corollary 2.1.1.

Our last corollary is a generalization of Wojdyslawski's result [19].

Corollary 3.4.2. *Let X be a compact Hausdorff space. Then each of 2^X and $C(X)$ is locally connected if and only if X is locally connected.*

Proof: Since X is compact, $2^X = \mathcal{K}(X)$ and $C(X) = C_K(X)$. Then the conclusion follows from Corollary 3.4.1.

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