Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 23, 1998

EXTENSIONS OF PARTITIONS OF UNITY

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ABSTRACT. In [5], Dydak proved some theorems concerning extensions of partitions of unity and extensions of continuous maps with metric simplicial complex values. In his paper, a subspace A of a space X is said to be P(locally-finite)-embedded in X if every locally finite partition of unity on A can be extended to a locally finite partition of unity on X. And a problem was posed there whether $A \times [0,1]$ is P(locally-finite)-embedded in $X \times [0,1]$ if A is P(locally-finite)-embedded in X. In this paper, under a set-theoretic viewpoint, we prove that A is P(locally-finite)-embedded in X if and only if every locally finite cover of cozero-sets of A can be extended to a locally finite cover of cozero-sets of X. This extends Przymusiński and Wage's theorem [13] in the case that X is normal and A is its closed subspace. As an application, we also give an affirmative answer to the problem above. Moreover by using continuous maps with metric simplicial complex values or partitions of unity we characterize well-known z_γ- or z-embedding.

1. INTRODUCTION

Throughout this paper, a space means a topological space. And γ denotes an infinite cardinal number. Let X be a space and A its subspace. For a collection $\mathcal{V} = \{V_\alpha : \alpha \in \Omega\}$ of subsets of X and a collection $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ of subsets of A,

$\mathcal{V}$ is said to be an *extension* of (or to *extend*) $\mathcal{U}$ if $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$. A is said to be *P^γ-embedded* in X if for every normal open cover $\mathcal{U}$ of A with Card $\leq \gamma$, there exists a normal open cover $\mathcal{V}$ of X such that $\mathcal{V} \wedge A < \mathcal{U}$ $\big(= \{V \cap A : V \in \mathcal{V}\}$ refines $\mathcal{U}\big)$. A is said to be *P-embedded* in X if A is P^γ-embedded in X for every γ. A is said to be *z_γ-embedded* in X if for every normal open cover $\mathcal{U}$ of A with Card $\leq \gamma$, there exist a cozero-set G of X containing A and a normal open cover $\mathcal{V}$ of G such that $\mathcal{V} \wedge A < \mathcal{U}$ [3]. If A is z_γ-embedded in X for every γ, A is said to be *z_∞-embedded* in X. Clearly, P^γ- (resp. P-)embedding implies z_γ- (resp. z_∞-)embedding, and it is known that z_ω-embedding or P^ω-embedding coincides with z-embedding or C-embedding, respectively; where A is said to be *z-embedded* in X if every zero-set in A is the intersection of A with some zero-set in X and A is said to be *C-embedded* in X if every real-valued continuous function on A can be extended over X (See [1], [3]).

In [5], Dydak investigated an extension theory by continuous functions with values in metric simplicial complexes or CW-complexes. He proved some interesting theorems characterizing several notions of embeddings defined in terms of extensions of partitions of unity, and showed that these results are closely related to P^γ-embedding. As one of such notions, it is defined in [5] that A is *P^γ(locally-finite)-embedded* in X if every locally finite partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on A with Card $\leq \gamma$ can be extended to a locally finite partition $\{g_\alpha : \alpha \in \Omega\}$ of unity on X, where "extended" means $g_\alpha | A = f_\alpha$ for every $\alpha \in \Omega$. If A is P^γ(locally-finite)-embedded in X for every γ, A is said to be *P(locally-finite)-embedded* in X. It is also shown in [5] that every closed subspace of a paracompact T_2 space X is P(locally-finite)-embedded in X.

From a set-theoretic viewpoint, we remind that the notion of P(locally-finite)-embedding originally relates to Katětov [10] and Przymusiński and Wage [13]. Katětov [10] proved that every collectionwise normal and countably paracompact space X satisfies the property that X is normal and this property was named *functionally Katětov* later by Przymusiński and Wage [13]. It was also shown in [13] that a T_1-space X is functionally Katětov if and only if every locally finite partition of unity on any closed subspace A of X can be extended to a locally finite partition of unity on X, that is A is P(locally-finite)-embedded in X for every cosed subspace A of X. In their proof of the "only if" part of this result, the normality of X and the closedness of A are essential. Also they proved that A is P-embedded in X if and only if every locally finite partition of unity on A can be extended to a (not necessarily locally finite) partition of unity on X [13, Theorem 1*]. Thus, it is natural to ask whether a subspace A of a space X is P(locally-finite)-embedded in X if and only if every locally finite cover of cozero-sets of A can be extended to a locally finite cover of cozero-sets of X. In Section 3 of our paper, we prove this equivalence, and apply it to answer affirmatively to a problem posed by Dydak [5] concerning product spaces.

In Section 4, we are concerned to describe P^γ-, z_γ-, C- or z-embedding, in terms of maps with metric simplicial complexes values. Since any continuous map on X with metric simplicial complex values corresponds to a point finite partition of unity on X(cf. [5]), extensions of maps with metric simplicial complex values can be regarded as extensions of point finite partitions of unity. In [5], Dydak characterized P^γ(point-finite)-embedding (see Section 2 for the definition) by using maps with values in contractible metric simplicial complexes (see Proposition 2.4). In [3], [4], [14] or [15], characterizations of z_γ-embedding in terms of continuous functions with values into the hedgehog with γ spines were given. Relating to these results, in Section 4 we first prove a key result including that A is z_γ-embedded in X if and only if for every continuous map $f : A \to Y$ into any finite dimensional metric simplicial complex with weight $\leq \gamma$ there exist a cozero-set G of X containing A and a continuous map $g : X \to Y$ such that $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ for each $a \in A$. Using this result, we give corresponding results for P^γ- or C-embedding. Moreover, we give another type of characterizations of z- or z_γ-embedding by approximations of these continuous maps $f : A \to Y$ above; they include the real-valued case (Blair and Hager [4]) or the hedgehog-valued case ([14], [15]).

2. Preliminaries

A collection $\{f_\alpha : \alpha \in \Omega\}$ of continuous functions from a space X into $[0,1]$ is said to be a *partition of unity* on X if $\sum_{\alpha\in\Omega} f_\alpha(x) = 1$ for every $x \in X$, where $\sum_{\alpha\in\Omega} f_\alpha(x)$ means the least upper bound of all sums of finitely many $f_\alpha(x)$' s. A partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on X is said to be *locally finite* (resp. *point finite*) if $\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\}$ is locally finite (resp. point finite) in X.

A subspace A of a space X is said to be P^γ*(point-finite)-embedded* in X if every point finite partition of unuty on A with Card $\leq \gamma$ can be extended to a point finite partition of unity on X [5]. If A is P^γ(point-finite)-embedded in X for every γ, A is said to be P*(point-finite)-embedded* in X. We note that P^γ(locally-finite)-embedding or P^γ(point-finite)-embedding implies P^γ-embedding.

Let us recall the hedgehog with γ spines. Let $I_\xi = [0,1]\times\{\xi\}$ for every $\xi \in \gamma$. We define the equivalence relation E on $\bigcup_{\xi\in\gamma} I_\xi$ such as $(x,\xi_1)E(y,\xi_2)$ whenever $x = y = 0$ or ($x = y$ and $\xi_1 = \xi_2$). We denote $J(\gamma)$ the set of all equivalence classes of E and define a metric on $J(\gamma)$ as follows:

$$\rho((x,\xi_1),(y,\xi_2)) = \begin{cases} |x-y| & \text{if } \xi_1 = \xi_2 \\ x+y & \text{if } \xi_1 \neq \xi_2 \end{cases}$$

for every (x,ξ_1), $(y,\xi_2) \in J(\gamma)$. θ denotes the class of $J(\gamma)$ consisting of $(0,\xi)$, $\xi \in \gamma$. We call this space with the metric topology associated by ρ the *hedgehog with γ spines*, and also denote it by $J(\gamma)$.

The set of all vertices of a simplex often means itself a simplex. Let Y be a simplicial complex. Then we denote by $|Y|$

the polyhedron of Y and by $\lambda_s(y)$ the barycentric s'th coordinate of a point y of $|Y|$. The *metric simplicial complex* (Y, d) is a pair of the simplicial complex Y and a metric defined by d, where $d(y,z) = \sum_{s\in\Lambda} |\lambda_s(y) - \lambda_s(z)|$ for each $y, z \in |Y|$ and Λ denotes the set of all vertices of Y. For a simplicial complex Y and $y \in |Y|$, the *carrier* of y is the smallest simplex of Y containing y, and is denoted by $\mathrm{carr}(y)$.

For a set S, let Σ_S is a set of all nonnegative functions $v : S \to [0,1]$ such that $\sum_{s\in S} v(s) = 1$ equipped with the norm. Then Σ_S is naturally a Banach space (see [5, Definition 5.1]). For a simplicial complex Y, Σ_S is denoted by Σ_Y, where S is the set of all vertices of Y.

Especially, if we regard $J(\gamma)$ as a simplicial complex, we denote $J(\gamma)$ with the simplicial complex metric d defined above by $(J(\gamma), d)$. Note that $(J(\gamma), \rho)$ and $(J(\gamma), d)$ have the same topology, in fact, $\rho \leq d \leq 2\rho$.

$\mathbb{N}$ denotes the set of all natural numbers. Other terminology are refered to [1], [6], [7] or [8].

Let us recall two examples.

Example 2.1. (1) ([13, Example 3]) There exists a space containing a P-embedded but not P^ω(locally-finite)-embedded closed subspace.

(2) ([5, Theorem 12.13 and Remark 12.14]) There exists a space containing a P(locally-finite)-embedded but not P^ω(point-finite)-embedded closed subspace.

The following propositions will be used in Section 3 or 4. (1) $\Leftrightarrow$ (2) is in [3, Theorem 3.8], (1) $\Leftrightarrow$ (3) is in [15, Lemma 2.2] and (1) $\Leftrightarrow$ (4) is in [14] or [15, Theorem 4.9].

Proposition 2.2. *Let X be a space and A its subspace. Then the following statements are equivalent*:

(1) *A is z_γ-embedded in X*;

(2) *for every continuous map $f : A \to J(\gamma)$, there exists a continuous map $g : X \to J(\gamma)$ such that $g^{-1}((0,1] \times \{\xi\}) \cap A = f^{-1}((0,1] \times \{\xi\})$ for every $\xi \in \gamma$*;

(3) *for every disjoint collection* $\{G_\alpha | \alpha \in \Omega\}$ *of cozero-sets of* A *with* Card $\Omega \le \gamma$ *satisfying* $\bigcup\{G_\alpha | \alpha \in \Omega\}$ *is a cozero-set of* A*, there exists a disjoint collection* $\{H_\alpha | \alpha \in \Omega\}$ *of cozero-sets of* X *such that* $\bigcup\{H_\alpha | \alpha \in \Omega\}$ *is a cozero-set of* X *and* $H_\alpha \cap A = G_\alpha$ *for each* $\alpha \in \Omega$;

(4) *for every continuous map* $f : A \to J(\gamma)$ *and any* $\varepsilon > 0$, *there exist a cozero-set* G *of* X *containing* A *and a continuous function* $g : G \to J(\gamma)$ *such that* $\rho(g(a), f(a)) < \varepsilon$ *for every* $a \in A$.

A subspace A of a space X is said to be *well-embedded* in X if A is completely separated from any zero-set of X disjoint from A. It is known that A is P^γ-embedded in X if and only if A is z_γ- and well-embedded in X [3]. The following also describes P^γ-embedding; (1) $\Leftrightarrow$ (2) is in [11, Corollary 10], (1) $\Leftrightarrow$ (3) is in [12, Theorem 2], (1) $\Leftrightarrow$ (4) is in [15, Theorem 4.7], (1) $\Leftrightarrow$ (5) is in [13, Theorem 1*] and [5, Proposition 12.8], (1) $\Leftrightarrow$ (6) easily follows from the definition of P^γ-embedding using [7, Theorems 1.2]. Concerning (2) or (3), similar characterizations are seen in [1], [6] or [12].

Proposition 2.3. *Let* X *be a space and* A *its subspace. Then the following statements are equivalent*:

(1) A *is* P^γ*-embedded in* X;

(2) *every continuous map* $f : A \to Y$ *into any Čech complete AR (=AR for metrizable spaces) with weight* $\le \gamma$ *can be extended over* X;

(3) *every continuous map* $f : A \to J(\gamma)$ *can be extended over* X;

(4) *for every continuous map* $f : A \to J(\gamma)$ *and any* $\varepsilon > 0$, *there exists a continuous map* $g : X \to J(\gamma)$ *such that* $\rho(g(a), f(a)) < \varepsilon$ *for every* $a \in A$;

(5) *every partition (or locally finite partition, point finite partition) of unity on* A *with Card* $\le \gamma$ *can be extended to a partition of unity on* X;

(6) *for every partition (or locally finite partition, point finite partition)* $\{f_\alpha : \alpha \in \Omega\}$ *of unity on* A *with Card* $\le \gamma$,

there exists a partition $\{g_\alpha : \alpha \in \Omega\}$ of unity on X such that ${g_\alpha}^{-1}\big((0,1]\big) \cap A \subset f_\alpha^{-1}\big((0,1]\big)$ for every $\alpha \in \Omega$.

The following due to Dydak [5] characterizes P^γ(point-finite)-embedding.

Theorem 2.4. [5, Theorem 9.1] *A subspace A of a space X is P^γ(point-finite)-embedded in X if and only if every continuous map from A into any contractible metric simplicial complex Y with weight $\leq \gamma$ can be extended over X.*

3. Extensions of locally finite partitions of unity

As is stated in the introduction, we now prove the following theorem:

Theorem 3.1. *Let X be a space and A its subspace. Then A is P^γ(locally-finite)-embedded in X if and only if every locally finite cover of cozero-sets of A with Card $\leq \gamma$ can be extended to a locally finite cover of cozero-sets of X.*

For the proof of Theorem 3.1, we need a lemma. By Ishii and Ohta [9], a subspace A of a space X is said to be C_1-*embedded* in X if any zero-set Z_1 of X and any zero-set Z_2 of A disjoint from Z_1 are completely separated in X. Note that A is C-embedded in X if and only if A is C^*- and C_1-embedded in X [9].

Lemma 3.2. *Let X be a space and A its subspace. Then, A is C-embedded in X if and only if for every continuous function $f : A \to [0,1]$ and any cozero-set U of X satisfying $U \cap A = f^{-1}\big((0,1]\big)$, there exists a continuous function $g : X \to [0,1]$ such that $g|A = f$ and $g^{-1}\big((0,1]\big) \subset U$.*

Proof: It is not hard to see that the assumption of the if part implies C^*- and C_1-embedding of A over X. To see the only if part, assume A is C-embedded in X. Let $f : A \to [0,1]$ be a continuous function and U be a cozero-set of X satisfying $U \cap A = f^{-1}\big((0,1]\big)$. By induction, we construct a continuous real-valued function h_n on X for each $n \in \mathbb{N}$ which satisfies the following three conditions:

$(1)_n$ $\quad |h_n| \leq \dfrac{1}{2^{n-1}}$;

$(2)_n$ $\quad h_n^{-1}\Big([-\dfrac{1}{2^{n-1}},0)\cup(0,\dfrac{1}{2^{n-1}}]\Big)\subset U$;

$(3)_n$ $\quad |f-\displaystyle\sum_{i=1}^{n}(h_i|A)|<\dfrac{1}{2^n}$.

Since $f^{-1}([1/2,1])$ is a zero-set of A contained U and A is C_1-embedded in X, $f^{-1}([1/2,1])$ and $X-U$ are completely separated in X. Take a zero-set Z_1 of X such that

$$f^{-1}([\frac{1}{2},1])\subset Z_1 \quad\text{and}\quad Z_1\subset U.$$

There exists a continuous function $g_1 : X \to [0,1]$ such that

$$g_1^{-1}(\{1\}) = Z_1 \quad\text{and}\quad g_1^{-1}(\{0\}) = X-U.$$

Since A is C^*-embedded in X, there exists a continuous function $f_1 : X \to [0,1]$ such that $f_1|A = f$. Define a continuous function $h_1 : X \to [0,1]$ by $h_1 = f_1\cdot g_1$. Then, the conditions $(1)_1$, $(2)_1$ and $(3)_1$ follow immediately.

Next assume continuous functions $h_1,\ldots,h_n$ satisfying $(1)_i$, $(2)_i$ and $(3)_i$ are defined for each $i=1,\ldots,n$. We put $\varphi = f-\sum_{i=1}^{n}(h_i|A)$. Then, by $(3)_n$, $\varphi : A \to [-1/2^n,1/2^n]$ is continuous. Put

$$Z=\varphi^{-1}\Big([-\frac{1}{2^n},-\frac{1}{2^{n+1}}]\cup[\frac{1}{2^{n+1}},\frac{1}{2^n}]\Big).$$

Then, we have that $Z\subset U$. Thus, $Z\subset U$. Since A is C_1-embedded in X, there exists a zero-set Z^* of X such that $Z\subset Z^*$ and $Z^*\subset U$. Hence there exists a continuous function $g' : X \to [0,1]$ such that ${g'}^{-1}(\{1\}) = Z^*$ and ${g'}^{-1}(\{0\}) = X-U$. Since A is C^*-embedded in X, there exists a continuous function $f' : X\to[-1/2^n,1/2^n]$ such that $f'|A=\varphi$. Define a continuous function h_{n+1} by $h_{n+1}=f'\cdot g'$. Then $(1)_{n+1}$, $(2)_{n+1}$

and $(3)_{n+1}$ follow immediately, it completes the proof of the induction.

Put

$$g = \Big(\big(\sum_{i \in \mathbb{N}} h_i \big) \wedge 1 \Big) \vee 0.$$

Then g is continuous and $g|A = f$. By the way, for $x \notin U$, it follows from $(2)_n$ that $h_n(x) = 0$ for every $n \in \mathbb{N}$, which imply $g(x) = 0$. This proves $g^{-1}\big((0,1]\big) \subset U$. It completes the proof. $\square$

Proof of Theorem 3.1: Since the "only if" part is easy to show, we only prove the "if" part. Let the assumption of the if part be satisfied. Let $\{f_\alpha : \alpha \in \Omega\}$ be a locally finite partition of unity on A with Card $\Omega \le \gamma$. Since $\{f_\alpha^{-1}\big((0,1]\big) : \alpha \in \Omega\}$ is a locally finite cover of cozero-sets of A, by the assumption of the theorem, there exists a locally finite cover $\{U_\alpha : \alpha \in \Omega\}$ of cozero-sets of X such that

$$U_\alpha \cap A = f_\alpha^{-1}\big((0,1]\big)$$

for each $\alpha \in \Omega$. Since A is C-embedded in X, from Lemma 3.2, for every $\alpha \in \Omega$ there exists a continuous function $h_\alpha : X \to [0,1]$ such that

$$h_\alpha|A = f_\alpha \text{ and } {h_\alpha}^{-1}\big((0,1]\big) \subset U_\alpha.$$

Then $\{{h_\alpha}^{-1}\big((0,1]\big) : \alpha \in \Omega\}$ is a locally finite collection of cozero-sets of X and covers A. Since A is well-embedded in X and $\bigcup\{{h_\alpha}^{-1}\big((0,1]\big) : \alpha \in \Omega\}$ is a cozero-set of X, there exists a continuous function $h_0 : X \to [0,1]$ such that

$$A \subset h_0^{-1}(\{0\}) \subset \bigcup\{{h_\alpha}^{-1}\big((0,1]\big) : \alpha \in \Omega\}.$$

Since $\{{h_\alpha}^{-1}\big((0,1]\big) : \alpha \in \Omega \cup \{0\}\}$ is a locally finite cover of X, $\sum_{\beta \in \Omega \cup \{0\}} h_\beta$ is continuous. Fix an $\alpha_* \in \Omega$ arbitrarily. Define a continuous function $g_\alpha : X \to [0,1]$ for each $\alpha \in \Omega$ as follows:

$$g_\alpha = \begin{cases} (h_{\alpha_*} + h_0) \Big/ \sum_{\beta \in \Omega \cup \{0\}} h_\beta & \text{if } \alpha = \alpha_* \\ \\ h_\alpha \Big/ \sum_{\beta \in \Omega \cup \{0\}} h_\beta & \text{otherwise.} \end{cases}$$

Then, $\{g_\alpha : \alpha \in \Omega\}$ is the desired locally finite partition of unity on X that extends $\{f_\alpha : \alpha \in \Omega\}$. The proof of the theorem is completed. □

Corollary 3.3. *Let X be a space and A its subspace. Then A is P(locally-finite)-embedded in X if and only if every locally finite cover of cozero-sets of A can be extended to a locally finite cover of cozero-sets of X.*

Theorem 3.4. *Let X be a space, A its subspace and C a non-empty compact T_2-space with weight $\leq \gamma$. Then, $A \times C$ is P^γ(locally-finite)-embedded in $X \times C$ if and only if A is P^γ(locally-finite)-embedded in X.*

Proof: The "only if" part follows immediately. To prove the "if" part, we assume A is P^γ(locally-finite)-embedded in X. Then $A \times C$ is C-embedded in $X \times C$ (see [7]). Let $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ be a locally finite cover of cozero-sets of $A \times C$ with Card $\Omega \leq \gamma$. Since C is compact, $\{p_A(U_\alpha) : \alpha \in \Omega\}$ is a locally finite cover of cozero-sets of A, where $p_A : A \times C \to A$ is the projection. From the assumption and Theorem 3.1, there exists a locally finite cover $\{V_\alpha : \alpha \in \Omega\}$ of cozero-sets of X such that $V_\alpha \cap A = p_A(U_\alpha)$ for each $\alpha \in \Omega$. Since $A \times C$ is z-embedded in $X \times C$, there exists a cozero-set U_α^* of $X \times C$ such that $U_\alpha^* \cap (A \times C) = U_\alpha$ for each $\alpha \in \Omega$. Since $A \times C$ is well-embedded in $X \times C$, there exists a cozero-set W of X such that $A \cap W = \emptyset$ and

$$\bigcup \{(V_\alpha \times C) \cap U_\alpha^* : \alpha \in \Omega\} \cup W = X \times C.$$

If we fix an $\alpha_0 \in \Omega$ arbitrarily,

$$\{(V_\alpha \times C) \cap U_\alpha^* : \alpha \in \Omega - \{\alpha_0\}\} \cup \{((V_{\alpha_0} \times C) \cap U_{\alpha_0}^*) \cup W\}$$

is a locally finite cover of cozero-sets of $X \times C$ and extends $\mathcal{U}$. Hence $A \times C$ is P^γ(locally-finite)-embedded in $X \times C$ by Theorem 3.1. It completes the proof. $\square$

The following corollary contains an affirmative answer to [5, Problem 13.16] posed by Dydak when we put $C = [0,1]$.

Corollary 3.5. *Let X be a space, A its subspace and C a non-empty compact T_2-space. Then, $A \times C$ is P(locally-finite)-embedded in $X \times C$ if and only if A is P(locally-finite)-embedded in X.*

Remark 3.6. In view of (1) and (5) of Proposition 2.3 and Theorem 3.1, one can ask the following: Is it true that A is P^γ-embedded in X (=every partition (or locally finite partition, point finite partition) of unity on A with Card $\leq \gamma$ can be extended to a partition of unity on X) if and only if every cover (or locally finite cover, point finite cover) of cozero-sets of A with Card $\leq \gamma$ can be extended to a cover of cozero-sets of X ? On the case $\gamma = \omega$, this is affirmatively answered easily. However on the case of $\gamma > \omega$, this is negative. Indeed, in Bing's space H ([2, Example H]), there exists a closed subset A which is not P^γ-embedded in H. We have that $U \cup (H - A)$ is a cozero-set of H for every cozero-set U of A. Therefore every cover (or locally finite cover, point finite cover) of cozero-sets can be extended to a cover of cozero-sets of H.

Here, we call a subspace A of a space X L^γ-*embedded* in X if every locally finite collection $\{U_\alpha : \alpha \in \Omega\}$ of cozero-sets of A with Card $\leq \gamma$ there exists a locally finite collection $\{V_\alpha : \alpha \in \Omega\}$ of cozero-sets of X such that $U_\alpha \subset V_\alpha$ for each $\alpha \in \Omega$. Then, it is easily shown that A is P^γ- and L^γ-embedded (more generally, C- and L^γ-embedded) in X if and only if A is P^γ(locally-finite)-embedded in X. However the author does not know whether every P^γ- and L^ω-embedded subspace A of X is P^γ(locally-finite)-embedded in X or not.

4. Extensions of continuous maps with metric simplicial complex values

In this section, we study extensions of a continuous map with metric simplicial complex values. P^γ(point-finite)-embedding was characterized by using these functions with contractible metric simplicial complex values (see Theorem 2.4). And P^γ-embedding was also characterized by extensions of point finite partitions of unity (see Proposition 2.3(5)). This is naturally regarded to the following: Any continuous map $f : A \to Y$ into any metric simplicial complex with weight $\leq \gamma$ can be extended to a continuous map $g : X \to \Sigma_Y$. From these points of view, at first, in the following theorem we obtain several equivalent conditions to z_γ-embedding; (2) is by partitions of unity which seems intermediate between extensions of partitions of unity and extensions of covers of cozero-sets, (3) or (4) is by maps with (finite dimensional) metric simplicial complex values which are based on approximations, and (5) or (6) is related to Blair [3] (see Proposition 2.2(2)).

Theorem 4.1. *Let X be a space and A its subspace. Then the following statements are equivalent*:

(1) *A is z_γ-embedded in X;*
(2) *for every point finite partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on A with Card $\Omega \leq \gamma$, there exist a cozero-set G of X containing A and a partition $\{g_\alpha : \alpha \in \Omega\}$ of unity on G such that $g_\alpha^{-1}\big((0,1]\big) \cap A = f_\alpha^{-1}\big((0,1]\big)$ for each $\alpha \in \Omega$;*
(3) *for every continuous map $f : A \to Y$ into any metric simplicial complex with weight $\leq \gamma$, there exist a cozero-set G of X containing A and a continuous map $g : G \to \Sigma_Y$ such that* $\mathrm{carr}\big(f(a)\big) = \mathrm{carr}\big(g(a)\big)$ *for each $a \in A$;*
(4) *for every continuous map $f : A \to Y$ into any finite dimensional metric simplicial complex with weight $\leq \gamma$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that* $\mathrm{carr}\big(f(a)\big) = \mathrm{carr}\big(g(a)\big)$ *for each $a \in A$;*

(5) *for every continuous map* $f : A \to J(\gamma)$, *there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to J(\gamma)$ *such that* $g^{-1}\big((0,1) \times \{\xi\}\big) \cap A = f^{-1}\big((0,1) \times \{\xi\}\big)$ *and* $g^{-1}\big(\{1\} \times \{\xi\}\big) \cap A = f^{-1}\big(\{1\} \times \{\xi\}\big)$ *for every* $\xi \in \gamma$;

(6) *for every continuous map* $f : A \to J(\gamma)$, *there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to J(\gamma)$ *such that* $g^{-1}\big((0,1] \times \{\xi\}\big) \cap A = f^{-1}\big((0,1] \times \{\xi\}\big)$ *for every* $\xi \in \gamma$.

Proof: Note that (2) $\Leftrightarrow$ (3) is obvious (see [5, Theorem 6.5]). To prove (1) $\Rightarrow$ (3), assume A is z_γ-embedded in X. Let Y be a metric simplicial complex with weight $\leq \gamma$ and $f : A \to Y$ a continuous map. S denotes the set of all vertices of Y. For every $k < \omega$, we define

$$\Delta_k = \big\{\delta \in Y : \operatorname{Card} \delta = k+1\big\}.$$

Build a barycentric subdivision to Y and denote by λ'_δ its δ'th barycentric coordinate for every simplex $\delta \in Y$. Define, for every $k < \omega$ and $\delta \in \Delta_k$,

$$U_\delta = \big\{y \in |Y| : \lambda'_\delta(y) > 0\big\}.$$

It is easy to show that U_δ is open in $|Y|$, and we have

$$U_\delta = \Big\{y \in |Y| : \lambda_s(y) > 0 \ \text{ and } \ \lambda_s(y) > \lambda_t(y) \quad \text{for every } \ s \in \delta, \ t \in S - \delta\Big\}$$

and $\operatorname{int}\delta \subset U_\delta$ for every $\delta \in Y$.

Since cardinality of each member of Δ_k is just $k+1$, it is easy to show that $\{U_\delta : \delta \in \Delta_k\}$ is a disjoint for each $k < \omega$. Hence we have that $\{f^{-1}(U_\delta) : \delta \in \Delta_k\}$ is a disjoint collection of cozero-sets of A, $\bigcup\{f^{-1}(U_\delta) : \delta \in \Delta_k\}$ is a cozero-set of A and $\operatorname{Card} \Delta_k \leq \gamma$. From Proposition 2.2(3), there exists a disjoint collection $\{V_\delta : \delta \in \Delta_k\}$ of cozero-sets of X such that

$$V_\delta \cap A = f^{-1}(U_\delta)$$

for each $\delta \in \Delta_k$ and $\bigcup\{V_\delta : \delta \in \Delta_k\}$ is a cozero-set of X.

For each $k < \omega$, there exists a continuous function $g_k : X \to [0, 1/2^{k+1}]$ such that

$$\bigcup_{\delta \in \Delta_k} V_\delta = g_k^{-1}\big((0, 1/2^{k+1}]\big).$$

Put

$$G = \bigcup_{k<\omega} \bigcup_{\delta \in \Delta_k} V_\delta.$$

Then G is a cozero-set of X and $A \subset G$.

Next define a function $g_k^s : G \to [0, 1/2^{k+1}]$ by

$$g_k^s(x) = \begin{cases} \dfrac{g_k(x)}{k+1} & \text{if } x \in \bigcup\{V_\delta : s \in \delta, \delta \in \Delta_k\} \\ 0 & \text{otherwise} \end{cases}$$

for every $k < \omega$ and $s \in S$. To show the continuity of g_k^s, let $x \in G$ and $\varepsilon > 0$ arbitrarily. It suffices to show the case $x \notin \bigcup\{V_\delta : s \in \delta, \delta \in \Delta_k\}$. If $x \notin \bigcup_{\delta \in \Delta_k} V_\delta$, then $x \in g_k^{-1}\big([0, \varepsilon)\big) \cap G \subset g_k^{s\,-1}\big([0, \varepsilon)\big)$. Hence assume $x \in \bigcup_{\delta \in \Delta_k} V_\delta - \bigcup\{V_\delta : s \in \delta, \delta \in \Delta_k\}$. Then there exists $\delta' \in \Delta_k$ such that $s \notin \delta'$ and $x \in V_{\delta'}$. Since $\{V_\delta : \delta \in \Delta_k\}$ is a disjoint collection, we have

$$V_{\delta'} \cap \Big(\bigcup\{V_\delta : s \in \delta, \delta \in \Delta_k\}\Big) = \emptyset.$$

Hence it follows that $g_k^s(V_{\delta'}) = 0$, it shows the continuity of g_k^s.

For every $s \in S$, put

$$h_s = \sum_{k<\omega} g_k^s.$$

Then h_s is a continuous function from G into $[0, 1]$, because $g_k^s \leq g_k \leq 1/2^{k+1}$ is satisfied for every $k < \omega$. By the same way, $\sum_{i<\omega} g_i$ is continuous and positive on G. Define a continuous function $\ell_s : G \to [0, 1]$ by

$$\ell_s = h_s \Big/ \textstyle\sum_{i<\omega} g_i$$

for every $s \in S$.

Here we shall show that $\{\ell_s : s \in S\}$ is a partition of unity on G. To see this, pick $x \in G$ arbitrarily. First let us show that

$\sum_{s\in S} g_k^s(x) = g_k(x)$ for every $k < \omega$. Fix $k < \omega$ arbitrarily. At first, assume that there exists $\delta \in \Delta_k$ such that $x \in V_\delta$. Then we have $g_k^s(x) = g_k(x)/(k+1)$ for each $s \in \delta$. Note that

$$x \notin \bigcup\{V_{\delta'} : t \in \delta', \delta' \in \Delta_k\}$$

for every $t \notin \delta$. Indeed, for $\delta, \delta' \in \Delta_k$ satisfying $t \notin \delta$ and $t \in \delta'$, we have $\delta \neq \delta'$; it shows that $V_{\delta'} \cap V_\delta = \emptyset$. It follows that $g_k^t(x) = 0$ for every $t \in S - \delta$. Hence it follows that

$$\sum_{s\in S} g_k^s(x) = \sum_{s\in\delta} \frac{g_k(x)}{k+1} = (k+1)\cdot\frac{g_k(x)}{k+1} = g_k(x).$$

On the other hand, assume $x \notin \bigcup_{\delta\in\Delta_k} V_\delta$. Since $g_k^s(x) = 0$ for every $s \in S$, we have

$$\sum_{s\in S} g_k^s(x) = 0 = g_k(x),$$

it completes the proof of $\sum_{s\in S} g_k^s(x) = g_k(x)$. Since $\sum_{k<\omega} g_k^s(x) = h_s(x)$, the set $\sum_{s\in S}\sum_{k<\omega} g_k^s(x)$ can be defined and is equal to $\sum_{k<\omega}\sum_{s\in S} g_k^s(x)$. Hence we have

$$\begin{aligned} 1 &= \frac{\sum_{k<\omega} g_k(x)}{\sum_{i<\omega} g_i(x)} = \frac{\sum_{k<\omega}\sum_{s\in S} g_k^s(x)}{\sum_{i<\omega} g_i(x)} = \frac{\sum_{s\in S}\sum_{k<\omega} g_k^s(x)}{\sum_{i<\omega} g_i(x)} \\ &= \frac{\sum_{s\in S} h_s(x)}{\sum_{i<\omega} g_i(x)} = \sum_{s\in S}\frac{h_s(x)}{\sum_{i<\omega} g_i(x)} = \sum_{s\in S} \ell_s(x). \end{aligned}$$

It completes the proof that $\{\ell_s : s \in S\}$ is a partition of unity on G.

Define a function $g : G \to \Sigma_Y$ by $\lambda_s \circ g = \ell_s$ for every $s \in S$, where $\lambda_s \circ v = v(s)$ for every $v \in \Sigma_Y$. By [5, Proposition 5.4], g is continuous.

Finally it suffices to show that $\mathrm{carr}(f(a))$ $=\mathrm{carr}(g(a))$ for each $a \in A$. Pick $a \in A$ arbitrarily. Let $\delta \in Y$ satisfying that

$\mathrm{carr}(f(a)) = \delta$. It suffices to show that $\lambda_t \circ g(a) = 0$ for every $t \in S - \delta$ and $\lambda_s \circ g(a) > 0$ for every $s \in \delta$. First assume $t \in S - \delta$ arbitrarily. Since $t \notin \delta$, it follows that $\lambda_t \circ f(a) = 0$. For every $\delta' \in Y$ satisfying $t \in \delta'$, it follows from the definition of $U_{\delta'}$ that $f(a) \notin U_{\delta'}$. Hence it is true that $f(a) \notin \bigcup_{t\in\delta'} U_{\delta'}$. Thus

$$a \notin \bigcup\{V_{\delta'} : t \in \delta', \delta' \in \Delta_k\}$$

for every $k < \omega$. So we have $g_k^t(a) = 0$ for every $k < \omega$, it follows that

$$\lambda_t \circ g(a) = \ell_t(a) = \frac{h_t(a)}{\sum_{i<\omega} g_i(a)} = \frac{\sum_{k<\omega} g_k^t(a)}{\sum_{i<\omega} g_i(a)} = 0.$$

Next, let $s \in \delta$ arbitrarily. Since $f(a) \in U_\delta$, we have $a \in V_\delta$. Put $k = \mathrm{Card}\delta - 1$. Then we have $\delta \in \Delta_k$. The fact

$$h_s(a) \geq g_k^s(a) = \frac{g_k(a)}{k+1} > 0$$

implies that

$$\lambda_s \circ g(a) = \ell_s(a) = \frac{h_s(a)}{\sum_{i<\omega} g_i(a)} > 0.$$

It completes the proof that $\mathrm{carr}(f(a)) = \mathrm{carr}(g(a))$ for each $a \in A$. The proof of (1) $\Rightarrow$ (3) is completed.

To prove (3) $\Rightarrow$ (4), we assume (3) to be satisfied. Let Y be a finite dimensional metric simplicial complex with weight $\leq \gamma$ and $f : A \to Y$ be a continuous map. From the assumption, there exists a cozero-set G of X containing A and a continuous map $g : G \to \Sigma_Y$ such that $\mathrm{carr}(f(a)) = \mathrm{carr}(g(a))$ for each $a \in A$. Since Y is ANR and is closed in Σ_Y, there exist an open subset W of Σ_Y containing Y and a retraction $r : W \to Y$. Since $g^{-1}(W)$ is a cozero-set of G and G is a cozero-set of X, $g^{-1}(W)$ is a cozero-set of X. Hence $r \circ g : g^{-1}(W) \to Y$ is the required continuous map such that $\mathrm{carr}(f(a)) = \mathrm{carr}((r \circ g)(a))$ for every $a \in A$.

The proofs of (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) follow immediately.

To prove (6) $\Rightarrow$ (1), assume (6) to be satisfied. Let $f : A \to J(\gamma)$ be a continuous map. Let us show Proposition 2.2(2). Since (6) is assumed, there exist a cozero-set G of X containing A and a continuous map $g' : G \to J(\gamma)$ such that $g'^{-1}\big((0,1] \times \{\xi\}\big) \cap A = f^{-1}\big((0,1] \times \{\xi\}\big)$ for every $\xi \in \gamma$. Then, as was essentially proved by Blair [3, Theorem 3.8], we can get a continuous map $g : X \to J(\gamma)$ satisfying that $g^{-1}\big((0,1] \times \{\xi\}\big) \cap A = f^{-1}\big((0,1] \times \{\xi\}\big)$ for every $\xi \in \gamma$. Let us give its proof for the completeness. Take a continuous function $h : X \to [0,1]$ satisfying $G = h^{-1}\big((0,1]\big)$ and define a continuous map $g : X \to J(\gamma)$ by

$$g(x) = \begin{cases} \big(h(x), (j \circ g')(x)\big) & \text{if } x \in G \\ \theta & \text{otherwise} \end{cases}$$

for every $x \in X$, where $j : (J(\gamma) - \{\theta\}) \to \gamma$ is the natural projection. It is the required map. Hence, it follows from Proposition 2.2(2) that A is z_γ-embedded in X. The proof of the theorem is completed. □

Corollary 4.2. *Let X be a space and A its subspace. Then the following statements are equivalent:*

(1) *A is z-embedded in X;*
(2) *for every countable point finite partition $\{f_i : i < \omega\}$ of unity on A, there exist a cozero-set G of X containing A and a partition $\{g_i : i < \omega\}$ of unity on G such that $g_i^{-1}\big((0,1]\big) \cap A = f_i^{-1}\big((0,1]\big)$ for each $i < \omega$;*
(3) *for every continuous map $f : A \to Y$ into any separable metric simplicial complex, there exist a cozero-set G of X containing A and a continuous map $g : G \to \Sigma_Y$ such that* $\operatorname{carr}\big(f(a)\big) = \operatorname{carr}\big(g(a)\big)$ *for each $a \in A$;*
(4) *for every continuous map $f : A \to Y$ into any finite dimensional separable metric simplicial complex with weight $\leq \gamma$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that* $\operatorname{carr}\big(f(a)\big) = \operatorname{carr}\big(g(a)\big)$ *for each $a \in A$;*

(5) *for every continuous map* $f : A \to J(\omega)$, *there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to J(\omega)$ *such that* $g^{-1}\big((0,1) \times \{\xi\}\big) \cap A = f^{-1}\big((0,1) \times \{\xi\}\big)$ *and* $g^{-1}(\{1\} \times \{\xi\}) \cap A = f^{-1}(\{1\} \times \{\xi\})$ *for every* $\xi \in \omega$;

(6) *for every continuous map* $f : A \to J(\omega)$, *there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to J(\omega)$ *such that* $g^{-1}\big((0,1]\times\{\xi\}\big)\cap A = f^{-1}\big((0,1]\times\{\xi\}\big)$ *for every* $\xi \in \omega$.

Corollary 4.3. *Let* X *be a space and* A *its subspace. Then the following statements are equivalent*:

(1) A *is* z_∞*-embedded in* X;

(2) *for every point finite partition* $\{f_\alpha : \alpha \in \Omega\}$ *of unity on* A, *there exist a cozero-set* G *of* X *containing* A *and a partition* $\{g_\alpha : \alpha \in \Omega\}$ *of unity on* G *such that* $g_\alpha^{-1}\big((0,1]\big)\cap A = f_\alpha^{-1}\big((0,1]\big)$ *for each* $\alpha \in \Omega$;

(3) *for every continuous map* $f : A \to Y$ *into any metric simplicial complex, there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to \Sigma_Y$ *such that* $\operatorname{carr}\big(f(a)\big) = \operatorname{carr}\big(g(a)\big)$ *for each* $a \in A$;

(4) *for every continuous map* $f : A \to Y$ *into any finite dimensional metric simplicial complex, there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to Y$ *such that* $\operatorname{carr}\big(f(a)\big) = \operatorname{carr}\big(g(a)\big)$ *for each* $a \in A$;

(5) *for every* γ *and any continuous map* $f : A \to J(\gamma)$, *there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to J(\gamma)$ *such that* $g^{-1}\big((0,1) \times \{\xi\}\big) \cap A = f^{-1}\big((0,1)\times\{\xi\}\big)$ *and* $g^{-1}(\{1\}\times\{\xi\})\cap A = f^{-1}(\{1\}\times\{\xi\})$ *for every* $\xi \in \gamma$;

(6) *for every* γ *and any continuous map* $f : A \to J(\gamma)$, *there exist a cozero-set* G *of* X *containing* A *and a continuous map* $g : G \to J(\gamma)$ *such that* $g^{-1}\big((0,1] \times \{\xi\}\big) \cap A = f^{-1}\big((0,1] \times \{\xi\}\big)$ *for every* $\xi \in \gamma$.

Remark 4.4. In Theorem 4.1(5), "G" can not be strengthen to "X". In other words, g in Proposition 2.2(2) can not be required to have further property $g^{-1}(\{1\}\times\{\xi\})\cap A = f^{-1}(\{1\}\times\{\xi\})$ for every $\xi \in \gamma$. (See the next theorem.)

From Theorem 4.1, we have the following characterizations of P^γ-embedding; (1) $\Leftrightarrow$ (2) is contained in Proposition 2.3.

Theorem 4.5. *Let X be a space and A its subspace. Then the following statements are equivalent*:

(1) *A is P^γ-embedded in X*;

(2) *for every point finite partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on A with Card $\Omega \leq \gamma$, there exists a partition $\{g_\alpha : \alpha \in \Omega\}$ of unity on X such that $g_\alpha^{-1}\big((0,1]\big) \cap A = f_\alpha^{-1}\big((0,1]\big)$ for each $\alpha \in \Omega$*;

(3) *for every continuous map $f : A \to Y$ into any finite dimensional contractible metric simplicial complex with weight $\leq \gamma$, there exists a continuous map $g : X \to Y$ such that $\mathrm{carr}\big(f(a)\big) = \mathrm{carr}\big(g(a)\big)$ for each $a \in A$*;

(4) *for every continuous map $f : A \to J(\gamma)$, there exists a continuous map $g : X \to J(\gamma)$ such that $g^{-1}\big((0,1) \times \{\xi\}\big) \cap A = f^{-1}\big((0,1) \times \{\xi\}\big)$ and $g^{-1}(\{1\} \times \{\xi\}) \cap A = f^{-1}(\{1\} \times \{\xi\})$ for every $\xi \in \gamma$.*

Proof of the following lemma is essentially in [15, Lemma 4.3] and omitted.

Lemma 4.6. *Let X be a space and A its subspace. Assume that for every continuous map $f : A \to J(\omega)$, there exists a continuous map $g : X \to J(\omega)$ such that $g^{-1}\big((0,1)\times\{\xi\}\big)\cap A = f^{-1}\big((0,1) \times \{$
$xi\}\big)$ and $g^{-1}(\{1\}\times\{\xi\})\cap A = f^{-1}(\{1\}\times\{\xi\})$ for every $\xi \in \omega$. Then, A is well-embedded in X.*

Proof of Theorem 4.5: To prove (1) $\Rightarrow$ (3), we assume (1) to be satisfied. Let Y be a finite dimensional contractible metric simplicial complex with weight $\leq \gamma$ and $f : A \to Y$ be a continuous map. By Theorem 4.1(4) and well-embeddedness of A, there exist a cozero-set G of X, a zero-set Z of X and a continuous map $h : G \to Y$ such that $A \subset Z \subset G$ and $\operatorname{carr}(f(a)) = \operatorname{carr}(h(a))$ for each $a \in A$. As was proved by Dydak [5, Section 8], there exists a continuous map $g : X \to Y$ such that $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ for each $a \in A$.

(3) $\Rightarrow$ (4) is obvious. (4) $\Rightarrow$ (1) follows from Theorem 4.1(5) and Lemma 4.6. It completes the proof. □

Corollary 4.7. *Let X be a space and A its subspace. Then the following statements are equivalent*:

(1) *A is C-embedded in X*;
(2) *for every continuous map $f : A \to Y$ into any finite dimensional separable contractible metric simplicial complex, there exists a continuous map $g : X \to Y$ such that* $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ *for each $a \in A$*;
(3) *for every continuous map $f : A \to J(\omega)$, there exists a continuous map $g : X \to J(\omega)$ such that $g^{-1}((0,1) \times \{\xi\}) \cap A = f^{-1}((0,1) \times \{\xi\})$ and $g^{-1}(\{1\} \times \{\xi\}) \cap A = f^{-1}(\{1\} \times \{\xi\})$ for every $\xi \in \omega$.*

Corollary 4.8. *Let X be a space and A its subspace. Then the following statements are equivalent*:

(1) *A is P-embedded in X*;
(2) *for every continuous map $f : A \to Y$ into any finite dimensional contractible metric simplicial complex, there exists a continuous map $g : X \to Y$ such that* $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ *for each $a \in A$*;
(3) *for every γ and every continuous map $f : A \to J(\gamma)$, there exists a continuous map $g : X \to J(\gamma)$ such that $g^{-1}((0,1) \times \{\xi\}) \cap A = f^{-1}((0,1) \times \{\xi\})$ and $g^{-1}(\{1\} \times \{\xi\}) \cap A = f^{-1}(\{1\} \times \{\xi\})$ for every $\xi \in \gamma$.*

Remark 4.9. (i) In Theorem 4.5(2), a partition $\{g_\alpha : \alpha \in \Omega\}$ of unity can not be replaced by a cover of cozero-sets (see Remark 3.6). (ii) In [15], $(P^*)^\gamma$-embedding was introduced so as to be coincide with z_γ- plus C^*-embedding and characterized by using hedgehog-valued functions. Related to $(P^*)^\gamma$-embedding, we can show the following: If A is a $(P^*)^\gamma$-embedded subspace of a space X, then for every continuous map $f : A \to Y$ into any finite dimensional metric simplicial complex with weight $\leq \gamma$ and every finite number simplices $\sigma_1, \dots, \sigma_n$ of $|Y|$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ for each $a \in A$ and $f(a) = g(a)$ for every $a \in f^{-1}\big(\bigcup_{i=1}^n \sigma_i\big)$. Its proof is complicated and but uses a similar technique to that of the proof of "(1) $\Rightarrow$ (3)" of Theorem 4.1 that we omit it.

Next, we study another type of characterization of z_γ-embedding by approximations. In [4, Theorem 2.2], Blair and Hager proved the following: A subspace A of a space X is z-embedded in X if and only if for every continuous real-valued (or bounded real-valued) function f on A and any $\varepsilon > 0$, there exist a cozero-set G of X containing A and a continuous function g on G such that $|g(a) - f(a)| < \varepsilon$ for each $a \in A$. Proposition 2.2(4) extends this result to the case of z_γ-embedding. Theorem 4.10 or Corollary 4.11 below extends these results to the case of maps with values into finite dimensional metric simplicial complexes.

Theorem 4.10. *Let X be a space and A its subspace. Then the following statements are equivalent*:

(1) *A is z_γ-embedded in X*;
(2) *for every continuous map $f : A \to (Y, d)$ into any finite dimensional metric simplicial complex with weight $\leq \gamma$ and any $\varepsilon > 0$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that $d(g(a), f(a)) < \varepsilon$ for each $a \in A$.*

Proof: To prove (1) $\Rightarrow$ (2), we assume A is z_γ-embedded in X. Let (Y, d) be a finite dimensional metric simplicial complex with weight $\leq \gamma$, $f : A \to Y$ be a continuous map and $\varepsilon > 0$. Let Y' be a finitely many fold iterated barycentric subdivision of Y such that $\operatorname{mesh} Y' < \varepsilon$ with respect to d. We denote by d' the metric of simplicial complex Y'. We note that (Y, d) and

(Y', d') are homeomorphic under the usual identity continuous map. From Theorem 4.1(2), there exist a cozero-set G of X containing A and a continuous map $g : G \to (Y', d')$ such that $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ (in Y') for each $a \in A$. Since every diameter of simplex of Y' is less than ε with respect to d, we have $d(g(a), f(a)) < \varepsilon$ for every $a \in A$.

To prove (2) $\Rightarrow$ (1), we assume (2) to be satisfied. Let $f : A \to (J(\gamma), \rho)$ be a continuous map and $\varepsilon > 0$. It suffices to show that there exist a cozero-set G of X containing A and a continuous map $g : G \to J(\gamma)$ such that $\rho(g(a), f(a)) < \varepsilon$ for every $a \in A$ because of Proposition 2.2(4). From the assumption, there exist a cozero-set G of X containing A and a continuous map $g : G \to (J(\gamma), d)$ such that $d(g(a), f(a)) < \varepsilon$. Since $\rho \leq d$, we have that $\rho(g(a), f(a)) < \varepsilon$ for each $a \in A$; the proof is completed. □

Corollary 4.11. *A subspace A of a space X is z-embedded in X if and only if for every continuous map $f : A \to (Y, d)$ into any finite dimensional separable metric simplicial complex and any $\varepsilon > 0$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that $d(g(a), f(a)) < \varepsilon$ for each $a \in A$.*

Corollary 4.12. *A subspace A of a space X is z_∞-embedded in X if and only if for every continuous map $f : A \to (Y, d)$ into any finite dimensional metric simplicial complex and any $\varepsilon > 0$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that $d(g(a), f(a)) < \varepsilon$ for each $a \in A$.*

Here is a problem whether $J(\gamma)$ in Proposition 2.2(4) can be changed into any Čech-complete AR or not; this was asked in [14] or [15, Problem 4.10]. Theorem 4.10 is a partial answer to this problem; more generally we pose the following:

Problem 4.13. *Let A be a z_γ-embedded subspace of a space X. Is it true that for every continuous map from A into any Čech-complete ANR Y with weight $\leq \gamma$, any (or some) metric ρ' on*

Y and any $\varepsilon > 0$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that $\rho'(g(a), f(a)) < \varepsilon$ for every $a \in A$?

Related to Theorem 4.10, we also have the folowing result: A subspace A of a space X is P^γ-embedded in X if and only if for every continuous map $f : A \to (Y, d)$ into any finite dimensional contractible metric simplicial complex and any $\varepsilon > 0$, there exists a continuous map $g : X \to Y$ such that $d(g(a), f(a)) < \varepsilon$ for every $a \in A$. However, if we consider (2) with (4) in Proposition 2.3, then we can conclude more general results as follows:

Proposition 4.14. *Let $\mathcal{C}$ and $\mathcal{C}'$ be subclasses of the class of all finite dimensional metric simplicial complexes with weight $\le \gamma$ satisfying $J(\gamma) \in \mathcal{C}, \mathcal{C}'$. Let X be a space and A its subspace. Then the following statements are equivalent*:

(1) *A is z_γ-embedded in X*;
(2) *for any $Y \in \mathcal{C}$ and any continuous map $f : A \to Y$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that* $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ *for each $a \in A$*;
(3) *for any $(Y, d) \in \mathcal{C}'$, any continuous map $f : A \to Y$ and any $\varepsilon > 0$, there exist a cozero-set G of X containing A and a continuous map $g : G \to Y$ such that $d(g(a), f(a)) < \varepsilon$ for each $a \in A$.*

Proposition 4.15. *Let $\mathcal{C}$ be a subclass of the class of all complete AR metric simplicial complexes with weight $\le \gamma$ satisfying $J(\gamma) \in \mathcal{C}$ and $\mathcal{C}'$ be a subclass of the class of all Čech-complete AR spaces with weight $\le \gamma$ satisfying $J(\gamma) \in \mathcal{C}'$. Then the following statements are equivalent*:

(1) *A is P^γ-embedded in X*;
(2) *for any $Y \in \mathcal{C}$ and any continuous map $f : A \to Y$, there exists a continuous map $g : X \to Y$ such that* $\operatorname{carr}(f(a)) = \operatorname{carr}(g(a))$ *for each $a \in A$*;

(3) *for any* $(Y, d) \in \mathcal{C}'$, *any continuous map* $f : A \to Y$ *and any* $\varepsilon > 0$, *there exists a continuous map* $g : X \to Y$ *such that* $d\big(g(a), f(a)\big) < \varepsilon$ *for each* $a \in A$.

Added in proof. Recently, Professor V. Gutev kindly sent the author an e-mail showing that Problem 4.13 is affirmative.

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