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GENERIC HOMEOMORPHISMS OF COMPACT MANIFOLDS

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ABSTRACT. We summarize results in our longer paper concerning the properties of a residual set of self homeomorphisms admitted by a compact manifold.

1. INTRODUCTION

This announcement outlines some of the main results from our longer paper [3] about homeomorphisms of compact metric spaces. Most of these results require some mild homogeneity of the space, so that certain local perturbations can be made. For simplicity we consider here only the case of a compact manifold M. This provides the needed homogeneity and also puts the discussion in a familiar setting. For a description of the general homogeneity conditions, see [12] or [3].

We use the term basic set to refer to a chain transitive component of f (i.e., a subset of the chain recurrent set which is chain transitive and is not a subset of any other chain transitive set). Other definitions are given below.

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Theorem 1. (Main Theorem) *For a generic homeomorphism f of a compact manifold M:*

1. *The chain recurrent set of f is a Cantor set, and the periodic points of f are dense in its chain recurrent set.*
2. *Every basic set that is either initial or terminal is a minimal set, and is either a periodic orbit or a generalized adding machine.*
3. *There is a collection $\mathcal{M}_1$ of basic sets which are both initial and terminal. Furthermore, the collection $\mathcal{M}_1$ is a residual subset of the space of all closed transitive sets endowed with the Hausdorff metric. The union of the elements of $\mathcal{M}_1$ is a residual subset of the chain recurrent set. Unless M is the circle, each element of $\mathcal{M}_1$ is a generalized adding machine.*
4. *Let $\mathcal{T}$ denote the set of all basic sets that are terminal but not initial, so $\mathcal{T}$ is disjoint from the residual set $\mathcal{M}_1$ described in (3). If we define*
$$P = \bigcup_{T\in\mathcal{T}} \{x \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty \text{ for some } y \in T\},$$
then P is a residual subset of M if we consider only the elements $T \in \mathcal{T}$ that are generalized adding machines. Similarly the set of points that are backward asymptotic to a point of some initial-but-not terminal basic set is residual.
5. *For P as in (4), any $x \in P$ is a chain continuity point of f (i.e., for each $\epsilon > 0$ there is a $\delta > 0$ such that the forward orbit of x ϵ-shadows any δ-chain beginning within δ of x).*
6. *Unless M is the circle, f has basic sets that are not minimal sets. In fact the set of non-minimal basic sets is dense in the set of all closed chain transitive invariant sets in the Hausdorff topology. On the circle, each basic set is a single periodic orbit.*

Many of our techniques grew out of results in [12,10,11,1] to which the reader is referred for more background. Our more general results and the full proofs can be found in [3]. Most of our genericity results extend to the case of continuous maps in place of homeomorphisms.

In the case of manifolds, some of the generic properties we describe were known previously to be properties of generic homeomorphisms in the C^0 closure of the set of diffeomorphisms on M. Generally the reason for this restriction in the earlier results is that the proofs relied on Shub's theorem that the set of Smale diffeomorphisms is C^0 dense in the set of all diffeomorphisms [16].

1.1. Notation and background.

- In this paper, M denotes a compact manifold without boundary, d denotes a metric on M, and $\mathcal{H}(M)$ denotes the set of homeomorphisms on M with the uniform topology. A subset of $\mathcal{H}(M)$ is *residual* if it contains a countable intersection of open, dense subsets of $\mathcal{H}(M)$. A property of homeomorphisms is called *generic* if the set of homeomorphisms possessing the property is residual.
- For $f \in \mathcal{H}(M)$, $x \in M$, the orbit of x under f is the set $\mathcal{O}(x; f) = \{f^n(x) : n \text{ is an integer}\}$, the *forward orbit* of x under f is the set $\mathcal{O}^+(x; f) = \{f^n(x) : n \geq 0\}$, and the *backward orbit* of x under f is the set $\mathcal{O}^-(x; f) = \{f^n(x) : n \leq 0\}$. The *$\omega$-limit set* or *forward limit set* of x under f is the set $\omega(x) = \{z \in M :$ there is an increasing sequence of integers $n_1, n_2, \ldots$ such that $\lim_{j\to\infty} f^{n_j}(x) = z\}$, and the *$\alpha$-limit set* or *backward limit set* of x under f is the set $\alpha(x) = \{z \in M :$ there is a decreasing sequence of integers $n_1, n_2, \ldots$ such that $\lim_{j\to\infty} f^{n_j}(x) = z\}$.
- A nonempty subset U of M is called a *trapping region* for f if $\overline{f(U)}$ is contained in the interior of U. When U is a trapping region, the set A defined by $A = \cap_{n\geq 0} f^n(U)$ is nonempty, compact, f-invariant, and is called an *attractor* for f. The *basin of attraction* of A is the set of all points whose ω-limit sets are contained in A; this is the same as

the union of the set of all inverse images of the trapping region U, and is open.

- A nonempty compact set Q is a *quasi-attractor* if it is the intersection of some family of attractors; its basin is the intersection of the basins of the attractors.
- A *repeller* for f is an attractor for f^{-1}; quasi-repellers are defined similarly.
- A compact, nonempty, f-invariant set X is *stable* if for each neighborhood V of X there is another neighborhood W of X such that $f^n(W) \subset V$ for each $n \geq 0$. The stable set is *indecomposable* if it is not the disjoint union of two smaller stable sets. Attractors and quasi-attractors are always stable.
- The set $D \subset M$ is a *k-absorbing disk* for $f \in \mathcal{H}(M)$ if (i) D is homeomorphic to the closed unit disk of the same dimension as M, (ii) $f^k(D)$ is contained in the interior of D, and (iii) $D \cap f^i(D) = \emptyset$ for $0 < i < k$. We say that D is an *absorbing disk* if it is k-absorbing disk for some $k \geq 1$. If D is a k-absorbing disk, then so is $f^i(D)$ for every i. Additionally, $\cup_{i=0}^{k-1} f^i(D)$ is a trapping region with an indecomposable associated attractor.
- For $\epsilon > 0$, an *ϵ-chain* for f is a sequence x_i of points of M with the property that $d(f(x_i), x_{i+1}) < \epsilon$ for all i. If an ϵ-chain is a finite sequence $x_0, x_1, \ldots, x_n$ then we say that it has *length* n; the *trivial* ϵ-chains are those of length 0. A point x is *chain recurrent* for f if for each $\epsilon > 0$ there is a nontrivial finite ϵ-chain that both begins and ends at x. We use $CR(f)$ to denote the *chain recurrent set* of f.
- A subset X of $CR(f)$ is *ϵ-chain transitive* if for any two points of X there is an infinite, periodic ϵ-chain that is contained in X and contains both of the points; X is *chain transitive* if it is ϵ-chain transitive for each $\epsilon > 0$. The collection of closed, nonempty, chain transitive sets in M under f is denoted $CT(f)$. We say that the subset X is a *basic set* of f if it is a maximal (by inclusion) chain recurrent set. Each basic set is closed and invariant, and

the basic sets form a partition of $CR(f)$. There is a partial order on the collection of basic sets defined as follows: we say the basic set X_1 *precedes* the basic set X_2 if for each $\epsilon > 0$ there is an ϵ-chain that begins in X_1 and ends in X_2. We call a basic set *initial* if it has no predecessors other than itself, and we call it *terminal* if it has no successors other than itself. A basic set is terminal if and only if it is a quasi-attractor; it is initial if and only if it is a quasi-repeller. In the literature, basic sets have often also been called *chain transitive components* or a *chain components*.

- The *Hausdorff metric* d_H makes the set 2^M of all closed, nonempty subsets of M into a compact metric space. The metric is defined by $d_H(X, Y) < \epsilon$ if and only if each of the two sets X, Y is contained in the ϵ-neighborhood of the other. The set $CT(f)$ of nonempty, closed, chain transitive invariant subsets of M is a closed subset of 2^M and so is a compact metric space.
- The *chain limit set of* x is denoted by $CL(x; f)$ and is the set of points y with the property that there is an ϵ-chain from x to y for each $\epsilon > 0$. Here we allow the trivial ϵ-chain $\{x\}$, so $x \in CL(x; f)$.
- The *prolongation of the orbit of* x *with respect to initial conditions* is denoted $PI(x; f)$ and is defined to be

$$PI(x; f) = \limsup_{y \to x} \overline{\mathcal{O}^+(y; f)}.$$

- *Conley's Theorem* describes the connection between the set of attractors of f and its chain recurrent set, namely that a point $x \in M$ is not chain recurrent if and only if there is an attractor A such that x is in the basin of A but is not in A. See 6.2.A on page 37 of [5], section 1 of [9], or 3.11 of [2].
- A generalized adding machine is a homeomorphism of a Cantor set that is defined as follows. For each $i \geq 1$, let k_i be a positive integer, let $N_i = \{0, 1, \dots, k_i\}$, and endow N_i with the discrete topology. Then let K denote the infinite product of the N_i's, and endow this set with the

product topology. A homeomorphism g on K is defined by the rule 'add 1 to the first coordinate, carrying to the right as necessary.' The dynamical system (K, g) is an *abstract adding machine* and is minimal in the sense that every g-orbit is dense in K. A *generalized adding machine* for a homeomorphism f on M is a compact f-invariant set on which the action of f is conjugate to some abstract adding machine g.

Theorem 2. *For a generic homeomorphism f :*

1. *No terminal basic set is an attractor, but each is a quasi-attractor.*
2. *Each attractor is the closure of its interior.*
3. *For each attractor A either the boundary of A is empty or else ∂A is a quasi-attractor.*
4. *Any neighborhood of any periodic orbit contains an attractor.*
5. *Every attractor contains a repeller.*
6. *The chain recurrent set $CR(f)$ is a perfect set.*

Proof: These results are proven in [12,10,11]. □

Theorem 3. *Let Q denote either 2^M with the Hausdorff metric or else the standard two-point compactification of the reals, and assume that Y is a complete metric space. If $\psi : Y \to Q$ is either upper semicontinuous or lower semicontinuous, then the set of continuity points of ψ is a residual subset of Y.*

Proof: See [17] or 7.19 of [2]. □

2. Basic results.

Theorem 4. *For a generic homeomorphism, the chain recurrent set is homeomorphic to the Cantor set and is the closure of the set of periodic points.*

Proof: In light of Theorem 2(6), in order to show that the chain recurrent set is a Cantor set it suffices to show that it is totally disconnected; this is done in [3]. The proof is based on some of the ideas used to prove Shub's Density Theorem. The fact that generically the periodic points are dense in the chain recurrent set is established in [11], based on earlier work in [14,6]. □

Theorem 5. *For a generic homeomorphism, each of its initial or terminal basic sets is either a single periodic orbit or else is a generalized adding machine. In particular, each of these sets is a minimal set (every orbit is dense).*

Proof: It is shown in [4,7] (without genericity hypotheses) that any compact indecomposable stable set either has a finite number of connected components which are cyclically permuted by the map, or else there are infinitely many connected components and the action of the map on them is a generalized adding machine. A terminal basic set T of a homeomorphism f is stable, and by Theorem 2, for generic f it will also be totally disconnected, so the results of [4,7] apply to the individual points of T. One can draw the same conclusions about the initial basic sets by considering the set of homeomorphisms whose inverses are in the appropriate residual set. □

3. Homeomorphisms of the circle

For homeomorphisms of the circle, Theorems 2 and 4 provide a fairly complete picture of the dynamics of a generic homeomorphism f. For simplicity we discuss only the case where f is orientation preserving. Since there are periodic points, the rotation number of f is rational and all of its periodic points have the same period m. Since the set of all period m points is closed, we see that it is the entire chain recurrent set of f, and so is a Cantor set. In order to simplify the discussion we assume that $m = 1$; the situation for higher values of m and for orientation reversing homeomorphisms is only slightly more complicated.

Fix an orientation on the circle; for two points a, b of the circle let $\alpha = (a, b)$ denote the arc that extends counterclockwise from a to b. If α is invariant under f, we say that f is *increasing along* α if $f(x) \in (x, b)$ for all $x \in \alpha$, and we say f is *decreasing along* α if $f(x) \in (a, x)$ for all $x \in \alpha$.

Given two fixed points a, b, either the arc $\alpha = (a, b)$ is disjoint from $CR(f)$ or not. In the latter case there is a fixed point $z \in \alpha$, and so by parts (4) and (5) of Theorem 2, α contains both an attractor and a repeller. This implies that α contains a pair of arcs complementary to the Cantor set $CR(f)$ with f increasing on one arc and decreasing on the other. It follows that for any two distinct components $\beta_i = (p_i, q_i)$ of the complement of $CR(f)$ there is a complementary arc in (q_1, p_2) on which f is increasing and another on which f is decreasing.

Since the chain recurrent set consists of fixed points and is totally disconnected, it is easy to see that each fixed point is an entire basic set. In fact, each fixed point is either an initial basic set or a terminal one, and all but countably many are both initial and terminal. To see this, suppose that $z \in CR(f)$ is not the endpoint of an arc complementary to $CR(f)$, and that y is some other fixed point. Then each of the two connected components of $S^1 \backslash \{y, z\}$ meets $CR(f)$, so by the above there is an invariant arc in (y, z) on which f is increasing and an invariant arc in (z, y) on which f is decreasing. It follows that for small ϵ there is no ϵ-chain from z to y. Since we can apply this argument with each fixed point $y \neq z$, we see that z is a terminal basic set. A similar argument shows that z is also an initial basic set. The same considerations show that if α is an endpoint of a complementary arc α, then either a is terminal (if f maps points of α toward a), or else a is initial (when f maps points of α away from a). Except for the countably many fixed points that are initial but not terminal, every point is in the basin of some terminal basic set.

Thus for generic homeomorphisms of the circle, the chain recurrent set is a Cantor set of periodic points, the sets of periodic orbits coincides with the set of basic sets, each periodic orbit is either initial or terminal, and most periodic orbits are both initial and terminal. Some of these features carry over to the higher dimensional case (most basic sets are both initial and terminal). However, we will see below that some of these results are special to the one-dimensional case. In particular, for dimension > 1, most basic sets are not periodic; instead they are minimal sets of generalized adding machine type and so do not even contain periodic points. Also, there are basic sets that are neither initial nor terminal, there are basic sets that are not minimal sets, and there are points of arbitrarily high period.

4. Density of Terminal Basic Sets in the Chain Recurrent Set

Lemma 6. *Suppose that the dimension of M is at least 2. For a generic homeomorphism f, if x is periodic, then any neighborhood of x contains periodic points of f of arbitrarily large periods. In light of Theorem 4, it follows that if y is a chain recurrent point of a generic homeomorphism, then any neighborhood of y contains periodic points of arbitrarily large periods.*

Proof: The lemma follows from a proof combining the orbit-doubling argument (Lemma 3.2 of [14]) with a Nitecki-Shub perturbation (Lemma 13 of [13]) and the stabilization arguments of [11]. The basic idea is that given a period k orbit we can perturb the map to create a nearby period $2k$ orbit that is attracting. See Proposition 3 of [11] for the details of a similar argument. □

Theorem 7. *Assume that* $\dim M \geq 2$. *For a generic homeomorphism on M, any neighborhood of any periodic orbit contains a terminal basic set of adding machine type, so the union of all these adding machine terminal basic sets is dense in the chain recurrent set. Similarly, for a generic homeomorphism the union of the initial basic sets of adding machine type is also dense in the chain recurrent set.*

Proof: Let γ_1 be a periodic orbit, and let k_1 denote its period. Given a periodic orbit γ_n with period k_n, cover it by disjoint open balls of diameter less than $1/n$ centered at each point of γ_n, and let W_n denote the union of these balls. By parts (4) and (5) of Theorem 2, W_n contains an attractor A_n that is not a periodic orbit, yet A_n contains periodic points since they are dense in the chain recurrent set. Theorem 2(2) and Lemma 6 show that A_n contains a periodic orbit γ_{n+1} whose period k_{n+1} is larger than k_n. Proceeding inductively, we obtain a nested sequence of attractors $A_{n+1} \subset A_n$ whose intersection Q is a terminal basic set (each A_n is covered by k_n disjoint balls whose diameters are at most $1/n$, so A_n is δ_n-chain transitive where $\delta_n = \sup\{d(f(x), f(y)) : d(x, y) \leq 1/n\}$ goes to 0 as $n \to \infty$). The same observations show that Q is totally disconnected, since no connected component of A_n has diameter greater than $2/n$. Clearly Q is not periodic, so Theorem 5 shows that the restriction of f to Q is an adding machine (this is also clear from the construction). □

With a little more work we can strengthen the last result by showing that the aperiodic basic sets that are both initial and terminal are residual not only in $CR(f)$, but also in the space $CT(f)$ of closed, nonempty, chain transitive invariant sets of f.

Theorem 8. *When the dimension of M is at least 2, for generic $f \in \mathcal{H}(M)$ each of the following sets is dense in $CR(f)$:*

1. *the set of periodic orbits of f,*
2. *the set $ITAM(f)$ consisting of basic sets of f that are both initial and terminal, and that are generalized adding machines, and*
3. *the collection of compact invariant sets that are chain transitive and shift extensions.*

Proof: For each $n \geq 1$ let $\mathcal{B}_n$ be a finite cover of M by open balls of radius $1/n$ and let $\mathcal{B}'_n$ be the set of finite nonempty subsets of $\mathcal{B}_n$. Given n and $I \in \mathcal{B}'_n$ define a map $P_{I,n}$ on $\mathcal{H}(M)$ by $P_{I,n}(h) = 1$ if h has a closed, absorbing disk D satisfying both (i) each forward iterate of D is contained in one of the balls in I, and (ii) each ball in I contains a forward iterate of D; define $P_{I,n}(h) = 0$ otherwise. It is easy to see that if D is an absorbing disk for h then it is also an absorbing disk for any g that is sufficiently close to h in $\mathcal{H}(M)$; thus each map $P_{I,n}$ is lower semicontinuous.

Now assume that f is a continuity point of $P_{I,n}$ for all (I,n), and let $X \in CT(f)$ and $n \geq 1$ be given. Let $I \in \mathcal{B}'_n$ be a collection of $1/n$ balls covering X chosen so that no subcollection of these balls also covers X. Since X is chain transitive, for each $\delta > 0$ there is a periodic δ-chain $\{x_n\}$ in X that meets each of the balls in I and that contains no unnecessary repetitions (in the sense that if $x_n = x_m$ then $n - m$ is a multiple of the period). As in Proposition 3 of [11] there is a homeomorphism g near f (how near depending upon δ) such that this periodic δ-chain for f is an attracting periodic orbit for f. It follows that $P_{I,n}(g) = 1$, so there is a sequence of homeomorphisms of $\mathcal{H}(M)$ converging to f, with $P_{I,n}$ identically 1 on the sequence. Since f is a continuity point, we see that $P_{I,n}(f) = 1$ as well. Let D be a k-absorbing disk satisfying (i) and (ii) above, and let W be the trapping region $\cup_{i=0}^{k-1} f^i(D)$. By the Brouwer Fixed Point Theorem, W contains a periodic orbit γ, necessarily in the interior of W. Applying Theorem 7, we see that W contains a terminal basic set Q of adding machine type. Each of Q, γ is invariant, so they each meet each $f^i(D)$, and we conclude that the Hausdorff distances $d_H(Q,X)$ and $d_H(\gamma,X)$ are each no more than $2/n$. This establishes both (1) and (2).

As for (3), a compact invariant chain transitive set X is a shift extension if there is a positive integer k such that there is a decomposition of X into k pairwise disjoint compact sets that are cyclically permuted by f and such that the action of f^k on any of these pieces is semiconjugate to the full 2-shift. The proof of the density of shift extensions in $CT(f)$ can be found in [3]. $\square$

Corollary 9. *For generic f, $CT(f)$ is a Cantor set.*

Proof: Since $CT(f)$ is closed in the space of all nonempty compact subsets of M, $CT(f)$ is a compact metric space. It has no isolated points because the previous lemma shows that there is a disjoint pair of dense subsets. To see that $CT(f)$ is totally disconnected, take $X \neq Y$ in $CT(f)$. Without loss of generality we can assume that there is a point x in $X \backslash Y$. For generic f, $CR(f)$ is a Cantor set, so there is an open and closed subset U of $CR(f)$ that contains Y but not X. It is easy to check that the set of all elements of $CT(f)$ that are contained in U is open and closed in $CT(f)$, and contains Y while its complement contains X. □

Corollary 10. *If the dimension of M is at least 2, then the generic homeomorphism f on M has basic sets that are not minimal sets. It follows that the subspace of basic sets is not closed in the space 2^M of closed subsets of M (unless M is the circle).*

Proof: A shift extension X is not minimal because the shift is not (a proper, closed, invariant subset of the shift pulls back to a proper, closed invariant subset of X). Since X is not minimal, it has a proper, compact, chain transitive invariant subset Y. By Theorem 8(2) there is a sequence of basic sets converging in $CT(f)$ to Y, which shows that the set of basic sets is not closed. □

Lemma 11. *Suppose M is not the circle. For generic $f \in \mathcal{H}(M)$ there is a residual subset $\mathcal{M}$ of $CT(f)$ consisting of sets that have infinitely many connected components.*

Proof: For each $k \geq 1$, let $\mathcal{C}'_k$ denote the collection of nonempty closed subsets of M that have at least k components. Given $X \in \mathcal{C}'_k$, we can partition X into exactly k disjoint closed pieces, and there is an $\epsilon > 0$ such that the minimum distance between any two of these pieces is greater than 2ϵ. If $Y \subset M$, is closed and $d_H(Y, X) < \epsilon$, then Y is similarly partitioned, so that $Y \in \mathcal{C}'_k$ as well. Thus $\mathcal{C}'_k$ is open in 2^M. Clearly $\mathcal{C}'_k$ contains the collection of adding machine basic sets, so $\mathcal{C}_k = \mathcal{C}'_k \cap CT(f)$

is open and dense in $CT(f)$ by Theorem 8(2), and so the intersection $\mathcal{C}_\infty(f) = \cap_k \mathcal{C}_k$ is residual in $CT(f)$. □

For $x \in CR(f)$ define $C(x)$ to be $CL(x; f) \cap CR(f)$; note that $x \in C(x)$ for every x. Define $R(x) \geq 0$ by $R(x) = \inf\{\epsilon > 0 : C(x)$ is ϵ-chain transitive$\}$.

Lemma 12. *The map R defined above is upper semicontinuous, and $R(x) = 0$ for all x in a residual subset of $CR(f)$. Any point x such that $R(x) = 0$ is contained in a terminal basic set of f.*

Proof: It is easy to verify that the map $x \to C(x)$ is upper semicontinuous, from which it follows that the map R is upper semicontinuous. (Given $x \in CR(f)$ and $\epsilon > R(x)$, $CL(x)$ is ϵ-chain transitive, so a small neighborhood U of $CL(x)$ is also ϵ-chain transitive, and for $y \in CR(f)$ sufficiently close to x, $CL(y) \subset U$, so that $R(y) \leq \epsilon$. Since R is upper semicontinuous, the set of its continuity points is residual in $CR(f)$. Note that $R(x) = 0$ if and only if x lies in a terminal basic set and the set of such points is dense, so R must be equal to 0 at any of its continuity points. These continuity points form a residual subset of $CR(f)$ consisting of points lying on some terminal basic set. □

Lemma 13. *When the dimension of M is at least 2, for generic f there are residual subsets $\mathcal{T}$, $\mathcal{I}$ such that each element of $\mathcal{T}$ is a terminal basic set, and each element of $\mathcal{I}$ is an initial basic set.*

Proof: Let R be the map from the last lemma. For $X \in CT(f)$, the set $C(x)$ is independent of the choice of $x \in X$, and so R is constant on X. Hence R extends to a map R' defined on $CT(f)$ by $R'(x) = R(x)$ for any $x \in X$. The map R' inherits the property of being upper semicontinuous from R, so its continuity points form a residual subset of $\mathcal{T} \subset CT(f)$. Theorem 8(2) and Corollary 10 show that R' is 0 on a dense subset of $CT(f)$, so R' is 0 at any element of $\mathcal{T}$, and any $X \in \mathcal{T}$ is a subset of some terminal basic set $C(X)$. By Theorem 5, $C(X)$ is a minimal set, so the fact that X is a nonempty invariant closed subset of $C(X)$ implies that $X = C(X)$ is in fact a basic set. The construction of $\mathcal{I}$ is analogous. $\square$

Theorem 14. *For generic f there is a residual subset $\mathcal{M}_1$ of $CT(f)$ consisting of basic sets that are both initial and terminal. If the dimension of M is at least 2, then $\mathcal{M}_1$ can be chosen so that each basic set in $\mathcal{M}_1$ is a generalized adding machine.*

Proof: For the circle this was established in Section 3. In the higher dimensional case, let $\mathcal{M}_1$ be the intersection of the residual sets $\mathcal{M}_1, \mathcal{T}$, and $\mathcal{I}$ given by Lemmas 9 and 11. This set consists of terminal basic sets of adding machine type. Being elements of $\mathcal{I}$ they are also initial. $\square$

Corollary 15. *There is a residual subset of $CR(f)$ consisting of the points that are contained in basic sets that are both initial and terminal.*

Proof: It is easy to see that the union of a collection of basic sets is dense in $CR(f)$ whenever the collection is dense in $CT(f)$. If X is a closed subset of $CT(f)$ then the set $\vee X$ that is the union in M of the elements in X is a closed subset of $CR(f)$. The complement of $\mathcal{M}_1$ is the union of a sequence X_n of closed, nowhere dense subsets of $CT(f)$. Each $\vee X_n$ is closed in $CR(f)$, and because the basic sets making up $\mathcal{M}_1$ are minimal, each $\vee X_n$ is disjoint from the dense set $\vee\mathcal{M}_1$. Thus each set $\vee X_n$ is closed and nowhere dense in M, and the union of these sets in the complement of $\vee\mathcal{M}_1$ is $CR(f)$. $\square$

5. Density of basins of terminal basic sets

We have already noted that the following theorem holds when $M = S^1$.

Theorem 16. *For a generic homeomorphism, the union of the basins of its terminal basic sets forms a residual subset of M. Similarly, the union of the basins of its initial basic sets is a residual subset of M.*

Proof: In the C^0 closure of the set of diffeomorphisms on M this result follows fairly easily from Shub's Theorem, as is shown in Section 6 of [8]. Following the outline of the argument from [8], it is enough to show that if $K \subset M$ is compact, $G \subset M$ is open and nonempty, and if the forward limit set $\omega(x)$ is a subset of K for every $x \in G$, then K contains a quasi-attractor. To avoid using Shub's Theorem, we rely on the following result from [15], where on pages 35-58 it is shown that a generic homeomorphism f has the following two properties: (i) $PI(x; f) = CL(x; f)$ for all x, and (ii) $PI(x; f) = \overline{O^+(x; f)}$ for residual $x \in M$ (see also [2], Exercise 7.40 and Proposition 7.22). We assume these properties for f.

Suppose that K and G are as above; there is no loss of generality in assuming that $K = \overline{\cup_{x \in G} \omega(x)}$, so that $K \subset CR(f)$. Since it is generically true that $CR(f)$ has empty interior, by replacing G with $G \backslash K$ if necessary we can assume that G and K are disjoint; by further shrinking G we can in fact assume that $\overline{G}$ and K are disjoint.

By (ii) above we can choose some $x \in G$ such that $PI(x; f) = \overline{O^+(x; f)}$ and so $PI(x; f) \subset K \cup O^+(x; f)$. For $\epsilon > 0$, let U_ϵ be the set of all points that can be reached via an ϵ-chain that begins in $\omega(x)$. It is well known, and easy to check, that the set U_ϵ is a trapping region for f; let A_ϵ denote the corresponding attractor. Clearly $A_\delta \subset A_\epsilon$ whenever $\delta < \epsilon$ so that $Q = \cap_{\epsilon > 0} A_\epsilon$ is a quasi-attractor. Further, Q is contained in $CL(x; f)$, and by (i) we have $PI(x; f) = CL(x; f)$. Thus,

$$Q \subset CL(x; f) = PI(x; f) \subset K \cup O^+(x; f),$$

so it suffices to show that no point on the forward orbit of x lies in Q. Since x is not chain recurrent, it is not contained in U_ϵ for sufficiently small ϵ, so $x \notin Q$. Since f is invertible, no point on the forward orbit of x is chain recurrent, so the same argument shows that $O^+(x; f) \cap Q = \emptyset$, which finishes the proof of the first assertion of the hypothesis. The same arguments with f^{-1} in place of f establish the second assertion. □

Theorem 17. *Suppose the dimension of M is at least 2. For a generic homeomorphism f, the set of points contained in the basin of attraction of some terminal basic set of adding machine type is residual in M. Similarly, the set of points in the basin of some initial basic set of adding machine type is residual, and so the set of points in the intersection of these two residual sets is also residual.*

Proof: The last two assertions follow easily from the first. To establish the first assertion it is enough (by Theorems 5 and 16) to show that almost any point of M is contained in the basin of an indecomposable attractor with an arbitrarily large number of connected components.

Given $m > 1$ and an open subset U of M, define a map $\Gamma_{m,U}$ on $\mathcal{H}(M)$ by $\Gamma_{m,U}(h) = 1$ if there is a $k \geq m$ and a k-absorbing disk D that meets the forward h-orbit of U; otherwise let $\Gamma_{m,U}(h) = 0$. It is easy to verify that each map $\Gamma_{m,U}$ is lower semicontinuous.

We claim that if f is a continuity point of $\Gamma_{m,U}$, then $\Gamma_{m,U}(f) = 1$. From this it will follow that if we fix some countable basis for the topology of M, and define $\mathcal{N}_m$ to be the intersection of the sets of continuity points of $\Gamma_{m,U}$ for all U in this basis, then $\mathcal{N}_m$ is residual. If $f \in \mathcal{N}_m$, then the union of all inverse images of the k-absorbing disks of f with $k \geq m$ is open and dense in M. The intersection $\mathcal{N} = \cap_m \mathcal{N}_m$ is also residual in $\mathcal{H}(M)$, and if $f \in \mathcal{N}_m$, then there is a residual subset of M consisting of points that are in the basins of k-absorbing subsets for f for arbitrarily large values of k.

Our proof of the claim is a version of one of the 'crushing arguments' from [12] and elaborated in [3]. For U and m as above and $\epsilon > 0$, it suffices to show that there is a homeomorphism g that is within ϵ of f, and with $\Gamma_{m,U}(g) = 1$. Fix $x \in U$. Since $\omega(x) \in CT(f)$, it follows from Theorem 10(2) that we can choose a point z from some terminal adding machine basic set and a nonnegative integer n such that $f^n(x)$ and z can be connected by a smooth arc J whose diameter is less than ϵ. We can assume that J is disjoint from the 'early points' $\{f^i(x) : 0 \leq i < n\}$. Because z lies in a terminal adding machine basic set we can choose a k-absorbing disk D with $z \in D$, $k \geq m$, and D disjoint from the early points (we can find D inside any pre-determined neighborhood of z). Moving along the arc J from $f^n(x)$ toward z, let z' be the first point encountered that lies on the forward orbit of D, and let J' be the subarc of J connecting $f^n(x)$ to z'. Note that z' is on the boundary of some iterate D' of D. Clearly J' is disjoint from $f^i(D)$ for all $i > 0$, and since D' is also k-absorbing, we can thicken J' to a small open set G that is disjoint from the early points and from $f^i(D')$ for all $i > 0$. Fix such a G such that $h(f^n(x)) = z'$. Setting $g = h \circ f$, we have that g is ϵ-close to f, D' is a k-absorbing disk for g, $g^i(x) = f^i(x)$ for $0 \leq i < n$, and $g^n(x) = h \circ f^n(x) = z' \in D'$, so that $\Gamma_{m,U}(g) = 1$. □

Remark. It is useful to note that there is a large difference in the conclusions of Theorems 14 and 17. Theorem 14 states that almost every basic set is both initial and terminal; such a basic set does not contain the ω-limit set or the α-limit set for any point not in the basic set. In other words, the union of the basins of these basic sets under both of f, f^{-1} is a subset of the closed, nowhere dense chain recurrent set. Thus there is a residual subset of M with the properties described in Theorem 17 that is contained in the union of the basic sets that are in the complement of the residual subset of $CT(f)$ that was described in Theorem 14. The next results serve to characterize which terminal basic sets correspond to each of these residual subsets.

Lemma 18. *A generic homeomorphism f has the following properties: each of its attractors is the closure of its interior. If A is an attractor that does not contain any connected components of M, then for each neighborhood U of the boundary* *of A there is an $\epsilon > 0$ such that if an ϵ-chain starts in the complement of A and ends in A, then it ends in $U \cap A$.*

Proof: This follows from parts (2) and (3) of Theorem 2. □

Theorem 19. *Let $\mathcal{A}$ denote the set of terminal basic sets Q each of which is contained in the boundary of some attractor A_Q, and let $\mathcal{NA}$ denote the set of terminal basic sets Q for which there is no such A_Q. For generic f, $\mathcal{NA}$ is residual in $CT(f)$, while the basins of the elements of $\mathcal{A}$ form a residual subset of M.*

Proof: By the previous remarks it is enough to show that $\mathcal{NA}$ is precisely the set of terminal basic sets that are also initial. It is easy to see that if Q lies on the boundary of some attractor A, then Q is not initial. On the other hand, if $Q \in \mathcal{NA}$, then because Q is a quasi-attractor there are attractors A_n with $Q = \cap_n A_n$ and Q contained in the interior of A_n for each n; there is no loss of generality in assuming that A_{n+1} is a proper subset of A_n for each n. Given any basic set $X \neq Q$, $X \cap A_n = \emptyset$ for sufficiently large n. By the previous lemma, there is an ϵ such that no ϵ-chain can start in X and end in Q, so Q is an initial basic set of f. □

Definition. The point x is called a *chain continuity point* of f if for each $\epsilon > 0$ there is a $\delta > 0$ with the property that the forward orbit of x ϵ-shadows any δ-chain that begins at a point within δ of x. (The forward orbit of x ϵ-shadows a δ-chain that begins at a point z within δ of x if each point on the orbit of z is within ϵ of the corresponding point of the δ-chain.) Chain continuity was defined and characterized in [1]; it is a stronger property than the property $CL(f;x) = \overline{\mathcal{O}^+(x;f)}$ (This was shown for residual subsets of M for generic f by Dobrynsky and Sharkovsky. See [15].)

Theorem 20. *Suppose that Q is totally disconnected and is a terminal basic set of f. If Q contains the ω-limit set of x, then x is a chain continuity point of f, and, in addition, there is a* *point y in its ω-limit set with $d(f^n(x), f^n(y)) \to 0$. It follows from this and Theorems 4, 5, and 16 that for generic f the set of chain continuity points is residual in M.*

Proof: To prove the first two assertions (which do not require the assumption of genericity), let Q and x be as in the statement of the theorem. Since Q is a quasi-attractor, if $\epsilon > 0$ is given then there is a trapping region U containing Q that has a finite union of connected components, each of which has diameter less than ϵ. There is an integer $n \geq 0$ with the property that $f^j(x) \in U$ for all $j \geq n$. By continuity there is a $\delta > 0$ such that (i) any δ-chain of length n beginning within δ of x will end in U; (ii) the points of such a length n δ-chain are within ϵ of the corresponding points on the orbit of x; (iii) no δ-chain beginning in U can ever leave U (since U is a trapping region); and (iv) δ is smaller than the minimum distance between any two connected components of U. It follows that for $j \geq n$ the j^{th} point on one of these δ-chains lies in the same connected component of U as $f^j(x)$, and so will be within ϵ of $f^j(x)$. This establishes the first assertion.

In particular, each point of Q is a chain continuity point. By the usual compactness argument, for each $\epsilon > 0$, we can choose $\delta > 0$ so that for any $z \in Q$ the orbit of z shadows any δ-chain that begins within δ of z.

Now we show that the orbit of x is asymptotic to the orbit of some point $y \in \omega(x)$. There is a monotonic sequence $n_k \to \infty$ of integers and a sequence of points $y_k \in Q$ with $d(f^{n_k}(x), y_k) < 1/k$ for each k. Let $z_k = f^{-n_k}(y_k) \in Q$; by compactness we may assume that $z_k \to z \in Q$. Given $\epsilon > 0$, let δ correspond to ϵ and Q as in the last paragraph. Choose k large enough that z_k is within δ of z and $1/k < \delta$. The orbit of z_k ϵ-shadows that of z and the orbit of $f^n(x)$ ϵ-shadows that of $y_k = f^{n_k}(z_k)$. By the triangle inequality, $d(f^n(x), f^n(y)) < 2\epsilon$ whenever $n \geq n_k$. $\square$

For more details and other results about chain continuity, see [1].

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